XIII
Applications of Paraconsistent Logic

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1. Introduction: the variety and types of applications

The most important application of paraconsistent logics is their application to possibly inconsistent theories. However one needs to interpret "theories" here fairly liberally, as any body of doctrine, statements, axioms etc. which can be thought of as inferentially closed. The theories can be historical, current, embryonic or merely entertained. Of course the formalization of such theories often requires much wider logical apparatus than the mere first order deductive logic discussed in the introduction to Part Two of the book. This may include probability, inductive logic, the logic of various modalities and other intentional notions such as belief, and so on. Such things, or at least some of them, have been considered by logicians. But, by and large, the logical theories produced have been tuned to classical or at least intuitionist logic. This is singularly inappropriate since as often as not, the material to which the logical apparatus is applied is inconsistent, as we shall see. Accordingly the ideas of paraconsistency need to be applied to the logical theories of modality, probability, etc. themselves to produce adequate logical machinery. In this essay we will consider first some interesting inconsistent theories, some of them in some detail, and then move on to consider the remodelling of various logical theories. It should be stressed that the studies of many applications mentioned are in their infancy, and we can often do no more than make suggestions for the directions of future research.

2. Historical and extant inconsistent theories.

There is a wide variety of inconsistent but non-trivial theories, some of them important. And some of them are true. Some of these important true theories, such as naive set theory, have been alluded to already in previous introductory sections. Of course paraconsistentists are not committed to the view that all contradictory theories are true—or even, if their position is weak enough, that *any* are true. Thus in many cases the formal reconstruction or paraconsistent representation of an inconsistent theory may be of mainly or merely a matter of historical interest. Formalizations of Bohr's theory of the atom, Spinoza's *Ethics* and Aristotle's theory of motion, inconsistencies and all, would be enterprises of this kind. These theories are no longer alive, or active. However, an important philosophical consequence of paraconsistency is that it allows the refloating of certain historical theories which had been pronounced dead prematurely. Meinong's theory of objects (which we will discuss in the introduction to the next part) is a theory of just this kind. (It went wrong because of imported features of the prevailing logical paradigm.) For some theories, such as the early theory of infinitesimals, it may not be clear which of these classes, the alive or the dead, they are in. Fortunately this is an issue we do not need to try to settle. We will start by giving a general overview of the range of inconsistent theories and then consider a few of the more interesting ones in detail.

Inconsistent theories are to be found in almost every discipline, but especially in:

2.1. Philosophy and theology

Among philosophical theories of this type we might mention the theories of Heraclitus, Hume, Hegel, Frege, Meinong and Wittgenstein, and some dialectical theories of change. We will not examine these theories here, since most of them are discussed elsewhere in the introductory chapters.[1] Some of these theories have parts which are of live interest for the development and elaboration of new theories.

Within this general category of philosophical theories we might also put certain theologies. In fact, most sophisticated theologies are inconsistent. Some, such as Christianity, run into inconsistency over issues such as the Trinity, the substantiality of God and the humanity of Christ. Other religions such as varieties of Buddhism, especially Zen, seem to court contradictions of the "mind is no-mind" type. In fact any religion which posits the existence of an all powerful God will run into the standard paradoxes of omnipotence (e.g. that He can invent a problem that He cannot solve, produce an immovable object, etc.); and in a similar way the assumption of an omniscient God is open to paradoxes of omniscience.[2]

In virtue of this sort of problem, some theologies, both medieval and modern, have bitten the bullet and allowed God to be an inconsistent object, though this solution may create more theological problems than it solves (e.g. how such an object can then exist, or be worthy of worship; how God can interact with the world at all). Undoubtedly the philosophically most sophisticated of these inconsistent theologies is Hegel's, where God is identified with the Absolute, with Self-Positing Spirit. Indeed inconsistency is part of the very essence of Hegel's Absolute.[3]

It is not merely specific philosophical or scientific theories that are inconsistent and call for applications of paraconsistent logic: the more general *theory of theories* is also of this sort.[4]

2.2. Natural and social sciences

As we have already begun to see, the sciences too have produced their share of inconsistent theories. For example: Bohr's theory of the atom, some versions of the Everett-Wheeler interpretation of quantum mechanics, and some other parts of quantum mechanics which involve causal anomalies or the Dirac δ function.[5] Some of these theories are certainly still of live interest. Travelling a bit further back into history it is quite arguable that Copernicus's joint theory of astronomy and dynamics was inconsistent, as were versions of the phlogiston theory, some theories of light, very likely Aristotle's theory of motion, and certainly earlier theories of motion which admitted Zeno's arguments.

The social sciences too have their share of contradictory theories. In particular, Freudian metapsychology is inconsistent. More generally any sociology or economics based on Marx's theory of alienation is inconsistent.[6] Similarly inconsistent is any theory incorporating conditions such as those unearthed by Arrow, for a general social or environmental theory.[7]

2.3. Logic and mathematics

A further class of theories, especially rich in contradiction, is that belonging to logic and mathematics. In this area we should cite, yet again, semantics, the theory of attributes (and of propositions), set theory, and the early theory of infinitesimals. It is almost certain that many other branches of mathematics—perhaps most—were inconsistent in their early versions. Recently research into the history of the growth of mathematics (as in Lakatos, 1976) seems to confirm this theme. However, the rewriting, and consequent transformation, of classical mathematics which has dominated this century, from *Principia Mathematica* to Bourbaki, has made the true historical situation more difficult to ascertain. But anyone who thinks that mathematics has always been done à la Bourbaki is guilty of serious historical anachronism.

3. A more detailed look at some of these theories

Obviously we cannot (for want both of space and of research time) discuss each of the above theories in detail. However, to give just a flavour of the investigation of inconsistent theories we will look at a few of them in more detail. In particular we will look at semantics, set theory, the infinitesimal calculus and some bits of quantum theory. We intend our discussion to be as neutral as we can make it with respect to the underlying paraconsistent logic we employ. But we will assume for the sake of definiteness that we are basing our theory on a suitable quantificational extension of a relevant logic such as that discussed in chapter V above (sect. 3.3). This is certainly the most versatile paraconsistent logic as well as the most philosophically adequate (as we argued in chapter V above). If another sort of paraconsistent logic is not suitable, we will mention this explicitly.

3.1. Naive semantics

Semantics is the theory of satisfaction, truth, denotation and other relationships between language and the world. No classical theory can adequately express its own semantics, on pain of inconsistency. However, an inconsistent theory obviously can be allowed to express its own semantics. And this is precisely what naive semantics does. Naive semantics is the theory of truth, satisfaction, denotation, definition, etc., which is capable of giving a semantics for itself.

This theory must be paraconsistently based because of the semantic paradoxes, and cannot be based on any paraconsistent logic which contains the absorption principle $A \rightarrow (A \rightarrow B)/A \rightarrow B$. This is because such a principle would trivialize the theory (see again ch. V, sect. 3.3). There remains however a good deal of scope for formalizing the theory in different ways: with one satisfaction predicate or many; with infinite sequences, finite sequences, and no sequences; as a many sorted theory or as a single sorted theory. We have chosen a way that seems particularly simple.

For every $n \geqslant 0$ the theory has an $n+1$ place predicate Sat_n such that $Sat_n y x_1 \ldots x_n$ is thought of as "$x_1 \ldots x_n$ satisfy y". In this context y is thought of as a formula with n free variables. We may suppose that if y has the wrong number of variables or indeed is not a formula at all, $Sat_n y x_1 \ldots x_n$ is false. The only other non-logical symbol is a functor $\ulcorner\ \urcorner$ satisfying the following formulation clause:

if φ is any formula or term, $\ulcorner\varphi\urcorner$ is a closed term.

$\ulcorner\varphi\urcorner$ is thought of as the name of φ.

The theory has the family of axiom schemes:

$$Sat_n\ulcorner\varphi\urcorner x_1 \dots x_n \leftrightarrow \varphi(v_{i_1}/x_1 \dots v_{i_n}/x_n) \qquad \text{(SS)}$$

where $v_{i_1} \dots v_{i_n}$ are the free variables of φ in increasing order in some standard enumeration, and v/x denotes the substitution of 'x' for 'v'.

The satisfaction scheme represents the naively correct and analytic principle of satisfaction which generalizes claims of the kind

John and Mary satisfy 'x loves y' iff John loves Mary.

We could, at the cost of certain complications, simplify (in one sense) the axiomatization thus: if we restrict (SS) to the cases where φ is atomic, but add recursive clauses such as

$$Sat_n\ulcorner\phi \vee \varphi\urcorner x_1 \dots x_n \leftrightarrow Sat_n\ulcorner\phi\urcorner x_1 \dots x_n \vee Sat_n\ulcorner\varphi\urcorner x_1 \dots x_n$$

the more general scheme can be proved. We will leave this as a non-trivial exercise.

The theory gives satisfaction conditions for all the formulas in the language including those which contain the satisfaction relation. It therefore formulates its own semantics. Of course it is inconsistent. For if we let $n = 2$ and take φ to be $\sim Sat_2\, xx$ we get

$$Sat_2\ulcorner\sim Sat_2\, xx\urcorner y \leftrightarrow \sim Sat_2\, yy.$$

Now for y take $\ulcorner\sim Sat_2\, xx\urcorner$ to derive what is, in effect, the heterological paradox.

Other semantic notions are simply accommodated. In particular, the satisfaction scheme for $n = 0$ is

$$Sat_0\ulcorner\varphi\urcorner \leftrightarrow \varphi.$$

Hence 'Sat_0' is the truth predicate for the language. As for denotation Δ, if t is any closed term of the language, we simply take $\Delta\ulcorner t\urcorner x$ as $Sat_1\ulcorner y = t\urcorner x$. The satisfaction scheme for $n = 1$ then gives

$$\Delta\ulcorner t\urcorner x \leftrightarrow x = t$$

The paradoxes of truth and denotation characteristically depend on machinery that is not available in the very limited theory we have sketched. However, if the axioms of arithmetic were added, giving the theory of semantically closed arithmetic, the liar paradox, Berry's paradox, and so on, would be forthcoming in the usual way. The triviality of the theory specified has not yet been investigated (except indirectly through its representation in set theory). (On Berry's paradox, see Priest, 1983.)

Unlike naive set theory, naive semantics has not been much developed; investigation of its theorems has not been carried very far. Much waits to be done. However, the theory affords a clear axiomatization of the intuitive semantical notions, which are built into natural language.

3.2. Naive property theory, set theory and category theory

Naive semantics encodes intuitive (and correct) views about truth, satisfaction, etc. Naive property theory and set theory do the same for these notions. Neither theory can be formalized non-trivially without paraconsistent logic; neither can be formalized using a paraconsistent logic which admits absorption without trivialization occurring.

Let us take the theory of properties first. This uses a variable binding, term forming operator λ such that $\lambda x\varphi$ is thought of as the property expressed by the open sentence φ as a condition on x. The only additional predicate required is η, 'has the property'.

The axiom scheme for properties is the obvious abstraction principle

$$y\eta\lambda x\varphi \leftrightarrow \varphi(x/y) \qquad \text{(AP)}$$

A slight generalization of the theory is provided by allowing for n-place properties. For each $n \geqslant 0$ we now have an operator λ_n which binds n free variables, and an $n+1$ place predicate η_n (for which we will write only its last argument to the right) which satisfies

$$y_1 \ldots y_n \eta_n \lambda_n x_1 \ldots x_n \varphi \leftrightarrow \varphi(x_1/y_1 \ldots x_n/y_n)$$

When $n = 0$, this theory is just the theory of propositions, with $\lambda_0\varphi$ expressing the proposition that φ, and η_0 being, in effect, the truth predicate for propositions. However, for simplicity, we will restrict our further discussion to the fairly representative 1-place case.

Again, the theory is patently inconsistent since we have

$$y\eta\lambda x(\sim x\eta x) \leftrightarrow \sim y\eta y$$

Taking $\lambda x(\sim x\eta x)$ for y gives the expected contradiction, the impredicativity paradox. The non-triviality of this theory follows from that of naive set theory which we will discuss shortly.

Provided the language we are using contains modal functors, we can express the familiar identity condition for properties, namely that two properties are the same iff they are necessarily coinstantiated, i.e.

$$y = x \leftrightarrow L\forall z(z\eta y \leftrightarrow z\eta x).$$

At this point we should perhaps warn that the precise formulation of the axioms for = itself, is a sensitive business. The simple substitutivity principle

$$x = y \to (\varphi(z/x) \leftrightarrow \varphi(z/y))$$

leads to curiosities such as $x = y \to (\varphi \to \varphi)$, and to irrelevance. Moreover there are good reasons for supposing it to be invalid if the language in question contains more highly intensional functors concerning belief, etc.[8]

The formulation of naive set theory is now a simple business. For the only real difference between sets and properties as usually conceived, lies in their identity conditions. Thus if we write '$\in$' (is a member of the set) for 'η' and '$\{z|\varphi\}$' (the set of objects z which φ) for '$\lambda z\varphi$', the abstraction scheme for properties AP, becomes the abstraction scheme for sets AS:

$$y \in \{x|\varphi\} \leftrightarrow \varphi(x/y). \qquad \text{(AS)}$$

The identity condition for sets amounts to the familiar extensional one:

$$x = y \leftrightarrow \forall z(z \in y \leftrightarrow z \in y).$$

Naive set theory is the one inconsistent theory that has had its theorems investigated, at least to a certain extent. In particular, virtually the whole apparatus of basic set theory, Boolean operations, ordered pairs, functions, power sets, etc., can be developed in much the same way as normal, though some changes are necessary. For example, if we define the null set Λ in the usual way as $\{x/x \neq x)$ then we can no longer prove that $\Lambda \subseteq x$ since this trades on the paradox of material implication $A \to (\sim A \to B)$. However, if we define Λ as $\{x|\forall y\, x \in y\}$ this and the other usual properties of Λ are forthcoming. That there are infinite sets is also provable in a simple way. For example, let V be the universal set, defined as $\{x|\exists y\, x \in y\}$. Then V is mapped into a proper subset of itself by the map $x \mapsto \{x\}$. Hence V is infinite. Thus naive set theory appears to provide for the set theory required in all normal mathematics.[9] The extent to which classical set theory itself, including the theory of transfinite ordinals and cardinals, can be developed or represented is still an open problem, as is the problem of what interesting structure inconsistent sets such as $\{x|x \notin x\}$ have and yield.[10]

Let us return now to the abstraction scheme AS itself. If we formulate the abstraction scheme without set abstracts, it is the usual:

$$\exists z \forall y(y \in z \leftrightarrow \varphi)$$

where φ is arbitrary except that z may not occur in it.

The qualification is required in consistent set theories since if it is violated inconsistencies are soon forthcoming. However this is no reason for keeping it in naive set theory, and the condition can be dropped. If we do this we can prove the existence of some more interesting sets. For example, consider the set defined thus:

$$x \in f \leftrightarrow \exists uv(u \in y \wedge v \in u \wedge x = \langle uv \rangle) \wedge \forall uv_1v_2(\langle uv_1 \rangle \in f \wedge \langle uv_2 \rangle \in f \rightarrow v_1 = v_2)$$

It is not difficult to check that f is a function and that for all $u \in y$, $f(u) \in u$. Thus f is a choice function on y, and we have proved the axiom of choice[11]. In fact we can take y to be V. Hence we have the global axiom of choice. Obviously this raises the important question of whether the continuum hypothesis or generalized continuum hypothesis can be settled by naive set theory. The answer to this is as yet unknown. However it *is* known that naive set theory is non-trivial even without the restriction on z in its formulation.[12] We will discuss the significance of this in the introduction to the next part.[13]

Before we leave the topic of set theory we should mention the situation with category theory. If we take ZF set theory and define the notion of category in the standard way, a category has to be a set. Thus we are precluded from considering such categories as the category of all sets. Alternatively if we allow categories to be proper classes, we are not able to consider the category of all proper classes or even all groups, since some of these are proper classes. These are well known difficulties. Standard solutions to them, such as the Gröthendeick hierarchy are not very successful.[14] However, if category theory is developed in naive set theory, we can define such categories as the category of all groups and be sure that *all* groups are in it. We can define the category of all sets, and since this is a set it will be a member of itself. Similarly we can consider the category of all categories. This not only frees the category theorist, whose hands are chained by ZF, but also introduces exciting new possibilities within naive inconsistent category theory. But to what extent these further non-well-founded categories exhibit interesting category theoretic properties remains to be seen.

In this last part we have been concerned with inconsistencies involving very large objects, such as certain infinite sets and categories. We turn next to inconsistencies involving the very small: infinitesimals and microphysical objects.[15]

3.3. The infinitesimal calculus

The third theory we should mention is the theory of infinitesimals. It is often suggested that the reworking by Robinson of the infinitesimal calculus in terms of non-standard analysis shows that the theory was not really inconsistent. But however elegant and useful the non-standard analysis theory of "infinitesimals" is, it is a gross anachronism to suggest that it is the original theory.[16] For infinitesimals had to be genuine inconsistent objects: in the calculation of a derivative, at different points, it had to be assumed that an infinitesimal was both zero and non-zero. Thus the theory is highly suitable for a paraconsistent formalization. Exactly how this is to be done is, however, a subject which requires a good deal more research. For the present we consider only the following suggestion for an absolutely naive infinitesimal theory, and some of its features.

First, the theory is based on the second-order theory of reals, which, we may suppose, is formulated to allow for specification of functions by λ-abstraction. Division is to be taken as a primitive symbol satisfying the condition

1) $x \neq 0 \rightarrow x \cdot y/x = y.$

The theory has one additional function symbol 'd', 'an infinitesimal part of', satisfying the two extra axioms

2) $dx = 0.$
3) $dx \neq 0.$

The derivative Df, of a function, f, can be defined in the usual way:

$$Df = \lambda x\left(\frac{f(x+dx)-f(x)}{dx}\right).$$

Thus the derivative of f at x is the ratio of the change in fx produced by an infinitesimal change in x. The calculation of a derivative can now proceed in the absolutely obvious way. For example, let f be $\lambda y\, y^2$. Then

$$Df = \lambda x\left(\frac{\lambda y\, y^2(x+dx) - \lambda y\, y^2(x)}{dx}\right)$$
$$= \lambda x\left(\frac{(x+dx)^2 - x^2)}{dx}\right) = \lambda x\left(\frac{2xdx + dx^2}{dx}\right)$$
$$= \lambda x\left(\frac{dx(2x+dx)}{dx}\right).$$

But by 3), 1) and the properties of λ,

$$Df = \lambda x(2x + dx).$$

And by 2) and the properties of λ,

$Df = \lambda x 2x.$

To prove various further properties of derivatives, extra axioms are required, such as $dfx = f(x + dx)$. However this will suffice to indicate something of the general shape of the theory. Perhaps the nicest thing about the theory is the way it allows d to be what it was originally thought to be, namely, an infinitesimal forming functor—where an infinitesimal is now thought of as an infinitely small inconsistent object.

Regrettably, a nasty thing about the theory is that it is trivial, or at least so close to trivial as to make no real difference (as observed by Dunn). For $0 = 1$ can be proved, and thus applied to prove every equation. Since $0 = 0 + 0$, (by 2), $dx = dx + dx$. Hence using 3) and 1), $dx/dx = dx/dx + dx/dx$. Therefore $1 = 1 + 1$, whence $0 = 1$, and disaster. A less naive, genuinely paraconsistent theory, which should be a conservative extension of arithmetic, will have to proceed more circumspectly. There are various possibilities to be explored. One proposal is that arithmetical operations on infinitesimals be limited. Another, suggested by the practice of the pristine theory, is that axiom 2) is contextually qualified so that, e.g., it only applies in certain d-contexts. The idea here is that while dx is *not* strictly zero, it is so close to zero that suitably placed d-terms elsewhere can absorb it.

3.4. Quantum mechanics

There are many parts of quantum theory which suggest paraconsistent formalization, because on the face of it they yield contradictions. Areas of especial sensitivity as regards consistency are those concerning the collapse of wave packets upon measurement, and in particular the matter of the exact determination of operators such as those of position and momentum. We will look briefly at some of these areas, beginning with the formalization of the Dirac delta function.

Very commonly, quantum mechanics is formulated in terms of Hilbert spaces. Thus the state description of a system is a member of the Hilbert space, H, which is the set of total functions from the reals $\mathbb{R}$ to the complex plane $\mathbb{C}$, with suitable operations defined. There is no insuperable problem in axiomatizing such a theory and we will leave it as an exercise for the diligent reader.[17] The important point at present is that it will imply that

$\forall\psi \in H, \forall x \in \mathbb{R}, \exists y \in \mathbb{C}, \psi(x) = y.$

Now to solve many problems it is necessary to invoke the Dirac δ-functions and to suppose them to be elements of the Hilbert space. The δ-functions are characterized by the axioms

(i) $\delta_x \in H \wedge \delta_x = \delta_x^*$
(ii) $\forall y \neq x\ \delta_x(y) = 0$
(iii) $\int_{-\infty}^{+\infty} \delta_x(y)dy = 1.$

If we add these to the axioms for the Hilbert space we can quickly derive an inconsistency. For since $\delta_x \in H$, $\forall y \in \mathbb{R}$, $\exists z \in \mathbb{C}$, $\delta_x(y) = z$. Thus $\exists z \in \mathbb{C}$ $\delta_x(x) = z$. But since $\delta_x = \delta_x^*$, z is real. Hence $\int_{-\infty}^{+\infty} \delta_x(y)dy = 0$. Contradiction.

Thus the theory as it stands requires paraconsistent formalization. That the theory is in deep classical trouble was observed by von Neumann.

> The method of Dirac ... in no way satisfies the requirements of mathematical rigour—not even if these are reduced in a natural and proper fashion to the extent common elsewhere in theoretical physics. For example, the method adheres to the fiction that each self-adjoint operator can be put in diagonal form. In the case of those operators for which this is not actually the case, this requires the introduction of "improper" functions with self-contradicting properties.[18]

Of course a paraconsistent reformulation is only one way of surmounting the problem, and we are certainly not claiming that it is the best. The theory of the Dirac δ-function can for instance be formalized using the theory of distributions, though the adequacy of this formalization is another question. All we are claiming is that this is one not unreasonable formalization and one, moreover whose consequences it might well be fruitful to investigate.

More immediate than the problems of the δ-function or general wave packet reduction are, what underlie these problems, the causal anomalies of quantum mechanics. Perhaps the simplest example is provided by the famous two-slit experiment: Suppose we fire a beam of light through a screen with two slits, A and B, in it. Having passed through the slits the light hits a screen. We wish to make sense of this in particular terms. If one slit is open a certain characteristic pattern of light is observed on the screen. If the other slit is open a similar pattern is observed. It would seem that if both slits are open the pattern obtained *should be* the simple superposition of these two patterns; but it is not. Consider first the proof that it should be, before the weak points in it are assessed. The intensity of light at a certain point x on the screen is determined by the probability of a photon hitting it. Let us write r for 'a photon hits x', a for 'a photon passes through A' and b for 'a photon passes through B'. We are interested in $Pr(r/a \vee b)$. This can be calculated as follows:

$Pr(r/a \vee b) = Pr(r \wedge (a \vee b))/Pr(a \vee b).$
(*) $\quad = Pr((r \wedge a) \vee (r \wedge b))/Pr(a \vee b).$
(**) $\quad = Pr(r \wedge a)/Pr(a \vee b) + Pr(r \wedge b)/Pr(a \vee b),$
since $\sim((r \wedge b) \wedge (r \wedge a)).$

But

(***) $\sim(a \wedge b)$. Hence $Pr(a \vee b) = Pr\,a + Pr\,b$, and by symmetry $Pr\,a = Pr\,b$.

Thus

$$Pr(r/a \vee b) = Pr(r \wedge a)/2\,Pr\,a + Pr(r \wedge b)/2\,Pr\,b = \tfrac{1}{2}(Pr(r/a) + Pr(r/b)).$$

Orthodox quantum logic tries to block the proof of stage (*) by rejecting distribution. While this does what is required in the given two-slit example, there is good reason to doubt that the strategy succeeds in the larger quantum mechanical context in which it has eventually to be set. For, firstly, the damaging superposition proof can be adapted to work with what orthodox quantum logic appears to allow; and secondly, when combined with arithmetic, which is essential to any larger venture, orthodox quantum logic permits the proof of distribution.[19] Paraconsistent strategies are different. Leading paraconsistent options are to reject steps (**) and/or (***). According to the stronger, dialethic option, even though $\sim(a \wedge b)$ is true, $a \wedge b$ is not thereby ruled out. Hence both those steps of the argument fail.[20] What this means in qualitative terms is that the particle which obviously cannot pass through both slits, actually does so. Far fetched as this may seem, once the idea that some inconsistencies are true is taken seriously, who is to say that some inconsistencies are not realized at the micro-level? (That micro-particles are not also waves?) It would be very strange; but we already know that strange things happen in this domain. In fact, given that the other steps of the argument are acceptable, the experimental evidence shows that sometimes a particle must (as a wave) pass through both slits,[21] even though this is impossible!

Once we have our eyes attuned to the possibility of particles doing the impossible, several other phenomena in quantum physics spring to mind, for example, the penetration of a potential barrier by a particle with (classically) insufficient energy. However, we need not pursue this issue further.

On the face of it the quantum-theoretical account of measurement is also inconsistent. For 'the result of a measurement is a superposition of vectors, each representing the quantity being observed as having one of its possible values': yet 'in practice we only observe one value', not many.[22] The predicament of Schrödinger's cat provides a celebrated example of the problem: the wave function for the system has 'a form in which the living cat and the dead cat are mixed in equal proportions' (De Witt, p. 31), but only one cat, a living or else a dead, is observed. There are several well known attempts—none particularly convincing—to resolve the matter, to consistencize in a coherent way; in particular, the Copenhagen collapse of the wave-packet, the hidden variable interpretation, Wigner's conscious interference proposal (see De Witt, p. 32). A different attempt—very much in the tradition of Jainist pluralism and discursive logic—is the EWG (Everett-Wheeler/Graham) interpretation, according to which all the possible values are realized in different worlds, to a distinguished one of which we observers are confined. The semantical framework of discursive logic is ready-made for the interpretation. But despite its paraconsistent connections the EWG theory is not really a paraconsistent one at all. For the branching tree of alternative worlds fits in an evolving Hilbert space (that of the nested superposition) which conforms to classical logic.[23]

4. Paraconsistency and wider logical notions

Let us now consider the application of paraconsistency to the theories of logic themselves. What an adequate paraconsistent logic does, at bottom, is to provide canons of good reasoning that can be used in all situations—including the many that misbehave classically.[24] However, it is but a limited basis for this. For it will account at best for deductive reasoning concerning a very restrictive class of logical notions. Beyond that there are other types of reasoning, such as inductive methods, and—not unrelated—there are many other notions we use in our reasoning, such as probability, various modalities, and so on; and each of these notions and types of reasoning must be properly tuned to the paraconsistent. We will make a beginning on showing how some of the adjusting of notions is to be accomplished in subsequent subsections. But first let us consider a little more generally the matter of the construction of adequate logics.

4.1. Reason, inference, fallacies, and the inconsistent

As should by now be very clear, reason and inference do not break down in inconsistent situations (whatever the friends of consistency in logic and artificial intelligence may say). If one finds an inconsistency in one's reasoning one certainly does not invoke *ex falso quodlibet* and conclude that one *ought* to accept everything; nor does one grind to a complete halt. Of course it is common, once one finds a contradiction, to take evasive action, to modify one's views until they are consistent. But common enough though this is, it is by no means rationally obligatory. The rational thing to do may well be to accept the contradiction, or at least to see what emerges from it. We will discuss this further in the introduction to the next part. The important point for now is just that a theory of reason certainly must be paraconsistent. The attempt of much recent literature to provide an account of rational human reason based on classical logic and probability theory, from which human inferential practice frequently and perversely deviates—is misguided. The classical theory is no ideal, but is itself defective.

Reasoning (whether artificial or natural, human or otherwise) is of several types: deductive, inductive, analogical, dialectical.[25] A logic for each one of these kinds of reasoning should tell us which principles of inference to accept. Moreover, each kind of reasoning will have associated fallacies, inferences that it is the business of logic to reject. Formal systems such as those of paraconsistent sentential logic codify only a small part of reasoning practice, namely some accepted principles of deductive reasoning.

Such systems can be expanded however, in the fashion of Łukasiewicz, to encompass rejected rules of deductive reasoning too, through rules of rejection. In this way they can reflect and help codify classes of fallacies. But even this enterprise has not yet been undertaken for intensional logics, though it would have interesting features. For example, central principles of classical logic would be rejected as fallacious in paraconsistent logic. In particular such obvious fallacies of relevance as the Lewis paradoxes would be rejected: i.e., $\dashv p \wedge \sim p \rightarrow q, \dashv q \rightarrow p \vee \sim p$. Given the expected linkage between assertions and rejections, e.g.

$$(*) \quad \vdash (A \rightarrow B) \rightarrow (\vdash A \rightarrow \vdash B)$$
$$(**) \quad \vdash (A \rightarrow B) \rightarrow (\dashv B \rightarrow \dashv A),$$

several other rejections would follow, perhaps most controversially, $\dashv p \wedge (\sim p \vee q) \rightarrow q$. For since, $\vdash p \wedge \sim p \rightarrow p \wedge (\sim p \vee q)$, $\vdash (p \wedge (\sim p \vee q) \rightarrow q) \rightarrow ((p \wedge \sim p) \rightarrow q)$, whence, using (**), instances of the disjunctive syllogism are rejected. (In a similar way, given the set abstraction scheme, it can be shown that instances of $A \wedge (A \rightarrow B) \rightarrow B$ are rejected in depth relevant logics.)

To provide an adequate norm for reasoning, the reach of relevant/paraconsistent logic has to be expanded beyond the state to which it has currently advanced. Even when this has been accomplished, there remains the question of the relationship of norm to practice. Analytically, the norm is a guide to *correct* practice and correct practice must bear some suitable relation to practice (though teaching and brainwashing can seriously affect the relation). Anyway this question is a little premature at the moment since there has been little unbiased testing of the way people (and other creatures) actually reason, especially in inconsistent situations. It is enough for the present purposes, however, that clear cases of irrelevance, such as the Lewis paradoxes exhibited, are widely recognized in preanalytic thought as fallacious or mistaken, and that reasoning continues, in non-classical fashion, in incomplete, inconsistent and paradoxical situations. For this indicates that the normative theory of reasoning adopted should be a relevant/paraconsistent one.

4.2. Extending paraconsistent logic by intensional functors: modal and tense operators

With this firmly in mind let us revert to the question of the tuning of notions used extensively in reasoning to the paraconsistent, beginning with, what is relatively straightforward, modal and tense notions.

Tense and modal operators can be added to a paraconsistent logic in standard syntactical and semantical ways. Consider, for example, logical modality from a semantical slant. Let K be a class of evaluations suitable for a paraconsistent logic. (These might be da Costa evaluations for C_1, a set of Routley/Meyer worlds, a set of truth filters on a De Morgan lattice, etc.; see the introduction to Part Two.) Let R be some binary relation, considered as relative possibility, with domain K. Then in the usual way, we can define:

(1) $L\varphi$ holds at $w \in K$ iff for all $v \in K$ such that wRv, φ holds at v.
(2) $M\varphi$ holds at $w \in K$ iff for some $v \in K$ such that wRv, φ holds at v.

What modal principles hold for the logic so defined depends, of course, not only on relation R but on the underlying paraconsistent logic. However, for most paraconsistent logics it is easy enough to produce paraconsistent versions of such systems as T, S4, S5, etc. Completeness proofs are usually forthcoming from a fusion of the completeness proofs for standard modal logics and those of extant paraconsistent logics.[26]

Such paraconsistent bases allow for the elaboration of essentially novel modal logics. Little has as yet been done in this area, but we might mention as an example systems formed by adding Nietzsche's theme: $M\varphi$—everything is possible.[27]

Any normal modal logic which contains Nietzsche's theme is inconsistent, but if the underlying logic is paraconsistent, there is no reason why it should be trivial. In fact, all we need to realize Nietzsche's theme in a modal model structure is that the structure contain the trivial evaluation/world (where everything is true). Of course, though this is sufficient it is not necessary.

The addition of tense operators to a paraconsistent logic is also fairly straightforward. The truth conditions for G (it will always be the case that) and F (it will be the case that) are given by (1) and (2) with 'G' replacing 'L' and 'F' replacing 'M'. The truth condition of H (it has always been the case that) and P (it was the case that) are given in the same way except that R is replaced by its converse $\breve{R}$. R is now thought of as temporal order.[28]

Paraconsistent tense logic is important in connection with the dialectical theory of change. It is sometimes said that the postulation that time is real is necessary to restore the contradictions produced by change. What is meant by this is that if a thing changes it is both P and not P. The apparent contradiction is resolved when we admit that it is P at one time and not P at another. (So Idealists such as Bradley who denied the reality of time were therefore forced to admit that change involves contradiction. Because of this they relegated change to the realm of appearances, which, unlike reality, could be inconsistent.) Even *if* the postulation of time is necessary, it is certainly not sufficient. For there may be instants of time at which a contradiction holds. This is precisely what those who accepted a dialectical account of change believed. Something which is moving from here to somewhere else is, at one and the same time, both here and not here.[29] And more generally if something is changing from P to not P at a certain instant it is both P and not P. Let us call this view *Zeno's principle* since it was he who first argued it. Then within a paraconsistent tense logic this view results in principles such as

$$(A \wedge F{\sim}A) \rightarrow F((A \wedge {\sim}A) \wedge F{\sim}A \wedge PA) \qquad \text{(Z)}$$

and its past/future dual. The completeness of (Z) with respect to Zeno's principle is obviously a question which depends upon many details concerning the underlying logic, so we will not pursue it here. However it is clear that to make much sense of Zeno's principle a paraconsistent logic is required. For classically we can prove ${\sim}F(A \wedge {\sim}A)$ and hence ${\sim}(A \wedge F{\sim}A)$ i.e. there is no change. Thus (Z) plus classical logic produces Parmenides' position, that there is no change. Hence paraconsistent tense logic is an important logical theory for investigation of the dialectical account of change. But principle (Z) and its further logical consequences remain to be investigated.

4.3. Moral dilemmas: deontic logic

Another philosophically significant extension of paraconsistent logic is that to deontic logic. The deontic operators 'O' (it is obligatory that) and 'P' (it is permissible/permitted that) can be added in much the same way as the modal operators of the previous section were. We can now think of the relation R as one of "moral accessibility" (i.e. xRy means that y is obtained from x by the performance of some morally permissible acts) or alternatively as affording access to ideal worlds (i.e. y is ideal as seen from x). The truth conditions for 'O' and 'P' are obtained simply by replacing 'L' with 'O' and 'M' with 'P' in (1) and (2) of the previous section. As usual a range of deontic logics is obtained by allowing R to have various properties. Details of sound and complete axiom systems depend, of course, also on the underlying paraconsistent logic, as does the range of deontic systems encompassed.

Paraconsistent deontic logics are particularly important; for they rectify the gross distortions of the concepts of obligation that obtain in classically based deontic logics. For example, it is notoriously the case that we may incur inconsistent obligations. Examples of moral dilemmas abound in the literature. Not all of these are cases of inconsistent obligations. Some moral dilemmas are just situations where it is difficult to decide what the right thing to do is. Others are cases where there is no inconsistency since it is clear that one prima facie obligation is overridden by a more important one. For example the obligation produced by my promising to arrive home at a certain time disappears if I delay to save someone's life. However there are cases where we do end up with genuinely inconsistent obligations.[30] I promise to go to dinner with x the next time he is in town. I promise to go to dinner with y, and x announces that he will be in town during that time (and only that time). Let us write X for 'I go to dinner with x' and Y for 'I go to dinner with y'. Thus we have both OX and OY. But assuming that I cannot go to dinner with one if I go to dinner with the other (maybe because they will be in different parts of town), we have $X \rightarrow \sim Y$. And since OX, we obtain $O \sim Y$. (If this inference be doubted, just consider the case where I promise x I will do something and, forgetfully, I promise y I will not do it.) Hence we have $OY \wedge O \sim Y$, which is equivalent to $O(Y \wedge \sim Y)$.[31]

This last formula describes a realizable state of affairs. Yet it cannot be true in any world of standard deontic logic. However, worse is to come. For classically $\vdash Y \wedge \sim Y \rightarrow A$. Hence, distributing O, $\vdash O(Y \wedge \sim Y) \rightarrow OA$. Thus according to classical deontic theory if I incur inconsistent obligations I ought to do everything. This is ridiculous. In the situation described it is obviously not permissible, let alone obligatory, to go and shoot one of the friends or some innocent bystander. However, even worse is to come. Assuming (as for usual deontic logics) that every world has some accessible extension, we have $\vdash \sim (OY \wedge O \sim Y)$, and thus $\vdash (OY \wedge O \sim Y) \rightarrow B$. In other words, by making inconsistent promises I bring it about that Paris is the capital of China. In fact I make the world trivial!

None of these ridiculous consequences follow from a paraconsistent deontic logic of the relevant sort: there are worlds in which $O(A \wedge \sim A)$ is true and (therefore) $(OA \wedge O \sim A) \rightarrow OB$ (and *a fortiori* $(OA \wedge O \sim A) \rightarrow B$)) fails.

One important consequence of this approach is that the Kantian dictum, Ought implies Can, i.e. 'everything that is obligatory is possible' (or conversely 'if something is impossible it is not obligatory'), needs to be rejected. On the semantics given it is of course true that $OA \rightarrow MA$ (at least when the deontic R is a subrelation of the altheic R, and $\forall x \exists y xRy$ for the deontic R). But presumably if $O(A \wedge \sim A)$ is true at w, all the worlds accessible to w will be "impossible" worlds. But the Kantian moral maxim presumably means that everything that is obligatory is true in some *possible* world. So let C be the set of consistent (classical) worlds or evaluations. If we characterize a connective M′ thus:

M′A holds at w iff, for some x in C such that wRx, A holds at x, then the intended Kantian maxim is expressed by $OA \rightarrow M'A$. But this obviously fails in general.

We have taken our examples of inconsistent obligation from the area of morals. But the sphere of obligation is obviously much wider, and it would be easy to produce inconsistent obligations by considering political situations, legal systems, contracts, constitutions, games and so on.[32] Thus the application of paraconsistent deontic logic has broad scope.

Virtually all the points we have made concerning inconsistent obligations can be repeated with respect to inconsistent orders (which may be produced both intentionally or by accident). Hence our discussion applies *mutatis mutandis* to imperatival logics. Satisfactory imperatival logics will be paraconsistent.

4.4. Belief systems: doxastic logic

Thought is more comprehensive than reasoning. While reasoning is included in thought, thought also involves assumption and (as the term is commonly used) belief. Reasoning proceeds in accord with principles; thought may involve the adoption of the principles, reflection upon them, and much else, *Reflection* alone, before assumption and beliefs are brought in, may include more than reasoning: it may include such things as sorting or assembling and comparison of things so sorted or assembled. The operations included in reflection beyond those of reasoning have been little investigated in modern logic, though they were included, at a time when psychology was in a much more primitive state, in traditional logic and are sometimes said to have an important place in Hegel's logic, and so in dialectical logic. With *belief*, and to a lesser extent *assumption*, the situation is somewhat better: elements of doxastic logic have been furnished, though mostly on an inadequate classical basis, by direct analogy with weaker modal logics.

A key feature of belief, as of many psychological functors, in contrast to reason, is that it is not deductively closed. A creature may perfectly well believe A but not believe B though B is deducible from A. Thus the following theses of some doxastic logics should be rejected:

$$(A \rightarrow B) \rightarrow (x\mathrm{Bel}A \rightarrow x\mathrm{Bel}B)$$
$$((A \rightarrow B) \wedge x\mathrm{Bel}A) \rightarrow x\mathrm{Bel}B.$$

Other principles that need to be rejected are various consistency postulates, in particular

$$\sim x\mathrm{Bel}(A \wedge \sim A)$$
$$x\mathrm{Bel} \sim A \rightarrow \sim x\mathrm{Bel}A.$$

It is clear that an agent may well believe a contradiction either wittingly or unwittingly. Moreover (as we will argue in the introduction to the next part), the agent may *rationally* believe a contradiction. The rejection of consistency postulates would cause serious problems if we were to try to base a doxastic logic on classical possible-world theory. However, the more general worlds-theory of paraconsistent logics enables the rejection of these principles without any trouble.

In contrast to standard modal logics doxastic logic is very weak. (Indeed this is little more than a corollary of belief not being closed under entailment.) For this reason it is characterized as much by the principles it rejects as by those it accepts. But one might well wonder whether one should accept any principle that involves belief essentially. In fact, one should. An example is the conjunction principle

$$\text{xBel}(A \wedge B) \leftrightarrow . \text{xBel}A \wedge \text{xBel}B.$$

While simplification, $\text{xBel}(A \wedge B) \rightarrow . \text{xBel}A \wedge \text{xBel}B$, is not in much dispute, its converse, $\text{xBel}A \wedge \text{xBel}B \rightarrow \text{xBel}(A \wedge B)$, is moot. But it can be persuasively argued that it correctly characterizes belief.[33] This converse principle is important. For amongst those who duly concede that our beliefs may well be inconsistent, it is common to propose a non-adjunctive paraconsistent logic,[34] on the grounds that though one may believe A and believe $\sim A$ one will not believe $A \wedge \sim A$. Obviously if the conjunction principle for belief is correct, this defense—one of the main defenses—of non-adjunctive systems, fails.

The logic of *rational* belief, while certainly closed at least under more elementary logical operations (such as adjunction no doubt), is, like the logic of belief, not encumbered by consistency postulates. Hence it too is not satisfactorily based on classical logic. Nor therefore, since probability assignments are so intricately tied to rational belief assessments, is probability theory—which leads to another major newer application (though one with older roots).

4.5. Probability and inductive reasoning

The standard approaches to probability theory are squarely based on classical logic. However, they can alternatively, and easily, be based on a paraconsistent logic and, as we shall see, doing so produces a number of advantages. There are many different approaches to classical probability theory. One of the easiest to adapt to paraconsistency is Carnap's. Let C be a class of paraconsistent worlds/evaluations suitable for some paraconsistent logic. Let m be a normal measure function defined on C. In particular then $m(C) = 1$. The probability of a formula A, Pr(A), may now be defined as $m(\{x \in C | A \text{ holds at } x\})$. It is easy to check the following:

(i) $0 \leq Pr(A) \leq 1$
(ii) if A entails B in the logic (i.e. if every valuation at which A holds, B holds), $Pr(A) \leq Pr(B)$
(iii) $Pr(A \vee B) = Pr(A) + Pr(B) - Pr(A \wedge B)$
(assuming that conjunction and disjunction behave in the normal way).
(iv) if A is a logical truth (i.e. holds in all evaluations), $Pr(A) = 1$.

In general, all the principles of probability theory that do not concern negation will carry over straightforwardly into paraconsistent probability theory. Typically, where paraconsistent probability theory diverges from the classical theory is in the vicinity of negation. In particular, it will not in general be true that $Pr(A) + Pr(\sim A) = 1$. If $(A \vee \sim A)$ is a logical truth it will certainly be the case that

$$1 = Pr(A) + Pr(\sim A) - Pr(A \wedge \sim A),$$

by (iii) and (iv), but of course $Pr(A \wedge \sim A) \neq 0$ in general.

An especial advantage of this approach is the following. The standard definition of conditional probability, Pr(A/B), is $Pr(A \wedge B)/Pr(B)$, which of course makes sense only if $Pr(B) \neq 0$. It follows from what we have already said therefore that $Pr(A/B \wedge \sim B)$ may be well defined. Thus we can have sensible evaluations of the probabilities of statements relative to inconsistent data. Preanalytically, this is something we do all the time. For example we estimate what is happening in various countries (with degrees of probability) given the inconsistent newspaper reports we read. Yet according to the classical theory this is impossible. More generally, it is a feature of many paraconsistent logics (once again, the relevant and positive-plus systems, but *not* the non-adjunctive ones) that any formula holds under some evaluation, or more correctly, in some non-trivial class of evaluations. Hence if we choose our measure with care, we can ensure that $Pr(A) \neq 0$ for all A. This means that conditional probability is *always* defined—a very pleasing feature. It might be said that if paraconsistently $Pr(A) \neq 0$, this shows that a paraconsistentist is crazy enough to countenance anything. However one could put it in a slightly less biased way thus: there is nothing that a paraconsistentist will dogmatically and with a closed mind rule out a *a priori.*

The non-classical behaviour of negation means that some parts of classical probability have to be reworked slightly. For example consider Bayes' theorem. In the usual way we can show that $Pr(h/e) = Pr(e/h) \cdot Pr(h)/Pr(e)$. Now suppose we have two hypotheses h and h_1 (the application to an arbitrary finite number is routine) and h and h_1 are exhaustive and exclusive in sense that $h_1 \vee h$ and $\sim(h_1 \wedge h)$, are logical truths. Then

$$\begin{aligned} Pr(e) &= Pr(e \wedge (h \vee h_1)) = Pr((e \wedge h) \vee (e \wedge h_1)) \\ &= Pr(e \wedge h) + Pr(e \wedge h_1) - Pr(e \wedge h \wedge e \wedge h_1) \\ &= Pr(h)Pr(e/h) + Pr(h_1)Pr(e/h_1) - Pr(h \wedge h_1)Pr(e/h \wedge h_1) \end{aligned}$$

$$\text{Thus } Pr(h/e) = \frac{Pr(e/h) \cdot Pr(h)}{Pr(h)Pr(e/h) + Pr(h_1)Pr(e/h_1) - Pr(h_1 \wedge h)Pr(e/h_1 \wedge h)}$$

This is the paraconsistent two-hypotheses case of Bayes' theorem. In the classical case the last summand in the denominator can be dropped since $Pr(h_1 \wedge h) = 0$. However $Pr \sim (h_1 \wedge h) = 1$ is no longer a guarantee of this.[35]

Probability plays a role in many other logical theories, and often a paraconsistent probability theory has a distinct advantage over a classical one. For example in confirmation theory it allows for the high probability/confirmation of contradictory hypotheses. For Pr(e/h) and Pr(~e/h) may both be $> \frac{1}{2}$. (Indeed $Pr(p/p \wedge \sim p) = Pr(\sim p/p \wedge \sim p) = 1$.) This has obvious connections with the grue paradox, where contrary hypotheses are both confirmed by the evidence.

As another example, consider the theory of rational acceptance. It is frequently mooted, and plausibly so, that a claim should be rationally accepted just if it has a high enough probability. In obvious notation:

(1) $Pr(A) \geq 1 - \varepsilon$ iff Rat(A) $(\varepsilon \ll \frac{1}{2})$

Since, as we have seen, we may well have $Pr(A \wedge \sim A) \geq 1 - \varepsilon$, $Rat(A \wedge \sim A)$, i.e. there are some contradictions that are rationally acceptable.

A standard problem with (1) is illustrated by the lottery paradox. Consider a fair lottery with n tickets. Let us write A_n for 'Ticket n wins'. Then obviously if we choose n large enough we have $Pr(\sim A_i) \geq 1 - \varepsilon$, $1 \leq i \leq n$, whilst $Pr(A_1 \vee \ldots \vee A_n) = 1$. Thus the set of rationally accepted beliefs includes $\{A_1 \vee \ldots \vee A_n, \sim A_1, \ldots, \sim A_n\}$ which is obviously inconsistent. This will not bother a paraconsistentist. Similar remarks apply as regards the paradox of the preface.[36]

It is often suggested in connection with rational acceptability, that the set of things rationally accepted should be deductively closed. This is obviously a serious problem for any classical logician who accepts the conclusion of a paradox since this move would trivialize rational belief. Plainly it is not a similar problem for a paraconsistentist. Despite this, we think that the suggestion is incorrect. Indeed, it is easily proved that this suggestion is incompatible with (1). For it is easy enough to produce situations where logical consequence is probability *decreasing*. (Just consider

$\{A, \sim A\} \vdash A \wedge \sim A$.) Hence if we accept both (1) and the deductive closure of the rationally accepted, we could prove that there is an A such that $Pr(A) \geq 1 - \varepsilon$ and $Pr(A) < 1 - \varepsilon$. This is a contradiction there is no sufficient reason for accepting.

4.6. Information content and data processing

Just as standard probability theory is based on classical logic but may be reworked paraconsistently to give a more satisfactory theory, so also can other classically-based theories; for example, the same is true of classical accounts of information content. Again, there are many possible approaches to content, but one that is most easily generalized to paraconsistent logic is due to Carnap and Bar-Hillel.

Let C be a class of worlds (evaluations) suitable for some paraconsistent logic. Then the information content of A, Con(A), is just $C - \{x \in C\text{: A holds at } x\}$. Assuming that conjunction and disjunction behave normally then usual results about the contents of conjunctions and disjunctions are forthcoming, e.g.

(i) A entails B in the logic iff $\mathrm{Con}(B) \subseteq \mathrm{Con}(A)$.
(ii) $\mathrm{Con}(A \vee B) = \mathrm{Con}(A) \cap \mathrm{Con}(B)$
(iii) $\mathrm{Con}(A \wedge B) = \mathrm{Con}(A) \cup \mathrm{Con}(B)$

However, as is to be expected, results concerning formulas containing negation differ. In particular we may have $\mathrm{Con}(A \wedge \sim A) \neq \Lambda$. A contradiction may therefore have determinate, non-trivial content, and different contradictions different contents. Thus, claims that contradictions have no, or trivial, content can be sustained only by insisting that the valuations or worlds, over which content is defined, are consistent. Any justification of this is liable to beg the original question.

If a numerical measure of content, c, is required we can take a suitable measure function, m, defined on C and take

$$c(A) = m\,\mathrm{Con}(A).$$

Standard results about the numerical content of conjunctions and disjunction are then forthcoming. Relative content can be defined, etc[37].

The question of information naturally suggests a further application of paraconsistent logic: data processing. We wish to store data in a computer and be able to retrieve it. However we usually want to do more than that:

we want our computer to be able to make inferences from its data and to give us the conclusions. Not only is this a question of efficiency (it is quicker to program 'John has brown hair and no one else has' than 'John has brown hair; Fred has not got brown hair; Steve has not got brown hair, ...') but we want to be able to determine the consequences of our data when it is too large to handle humanly. Now notoriously, data collected from various sources is liable to be inconsistent. Conceivably, one might want to test the data before feeding it into the computer but even if an inconsistency is found (and of course there is no decision procedure in general for inconsistency) we are faced with the problem of how to consistentize it without throwing out too much data. In many ways it is much more sensible to let the machine have it all. But now it is obvious that the computer must be programmed with a paraconsistent logic. One that resulted in the computer answering 'yes' to every question including 'Is there a God?' when fed the speeches of virtually any politician would be useless. The question of how best to construct such a practical implementation of a paraconsistent logic in the computing field is an important one—one that will again depend, of course, on the logic in question. It is very tempting to think that the logic should be a relevant one.[38]

4.7. Vagueness

Finally, it is worth noting the role of paraconsistent logic as the underlying logic for a language with vague predicates. It is frequently suggested that what characterizes a vague predicate is that in a certain range of application objects satisfy neither the predicate nor its negation.[39] However, what intuition says is that the predicate and its negation are just as true of the borderline object as they are false. Hence an alternative treatment, consonant with this intuition, is that the object satisfies both the predicate and its negation, and hence that the situation is paraconsistent.[40] Moreover, there are reasons why this approach may be preferable, at least in particular cases and perhaps in general. First, consider a colour transition from red to blue through magenta. At the borderline area between red and blue, it seems much more plausible to suppose the colour to be both red and blue, than neither red nor blue. An argument that the paraconsistent approach is better in general, is that truth-value-gap approaches characteristically produce a failure of the law of excluded middle at the borderline area. Yet as Dummett and others have pointed out[41], this is not so plausible. In a borderline case between orange and red, we would be inclined to say that the colour is either orange or red, and it follows from this that it is either orange or not orange.

Actually, no standard three-valued (or by parity of reasoning, finite-valued) approach to vagueness is satisfactory, be it inconsistent or incomplete. This is because the area between definiteness and vagueness is itself vague. This has suggested that a continuum (or at least a dense sequence) of semantic values be used. As with the three-valued case, this can result in either incompleteness or inconsistency, depending on the truth conditions of negation and the set of designated values.[42] As with the three-valued case, there are reasons for supposing the inconsistent variant to be preferable.

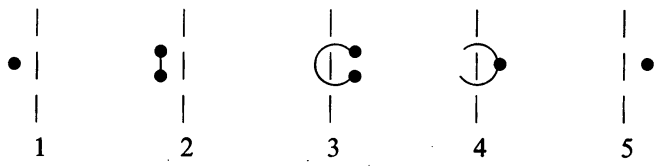
5. Conclusion

We have now given an overview of some of the applications of paraconsistent logic. The view has concentrated on breadth rather than depth. (Even so it can hardly claim to be comprehensive). What will be clear is that little more than a start—if that—has been made on most of the topics we have introduced. For a subject, the serious study of whose theory is little more than 20 years old, and which got away to a slow beginning, this is hardly surprising. However, it will also be clear that the investigation of these areas promises to be a fascinating task for paraconsistent theorists. The only thing that would really surprise us about future work in these areas would be its failure to produce surprises.

Notes

[1] See especially chapter XVIII, below.
[2] See e.g. Routley, 1981.
[3] See section 5 of chapter II of this book. For a further discussion of contradictory theology, see section 3 of chapter I of this book, and also Peña, 1981.
[4] The semantics for relevant logic of Routley and Meyer, 1973, furnishes a fairly general framework for the static elaboration of the theory of theories. The static development is taken much further in Meyer's unpublished work on the theory of theories.
[5] See section 3 below.
[6] See chapter II, section 6.
[7] For fuller explanation and proof see Routley, 1980.
[8] However, we will not discuss these issues here. Details can be found in Routley, 1977, §7, and in EMJB.
[9] Further details can be found in Routley, 1977, §8.

[10] For initial investigations of the latter issues, in somewhat stronger paraconsistent settings, see Arruda and Batens, 1982.
[11] See Routley, 1977, §8.
[12] See Brady, 1989.
[13] See chapter XVIII, sect 3.3.3.
[14] See Fraenkel, Bar-Hillel, and Levy, 1973, pp. 143ff.
[15] Symmetry of treatment would perhaps suggest a section on relativity theory, indicating parts of the theory ripe or fit for paraconsistent reformulation and investigation. Certainly relativity theory has generated its share of paradoxes.
[16] "At the time (circa 1740) ... mathematicians still felt that the calculus must be interpreted in terms of what is intuitively reasonable, rather than of that which is logically consistent", Boyer, 1949, p. 232.
[17] Elementary quantum theory is axiomatized in this way by J. von Neumann, 1953. As was customary with mathematical axiomatizations, the precise logical base is not specified.
[18] Von Neumann, 1953. Von Neumann's point has been adumbrated by Feyerabend, 1978, p. 157, and then set down in Mortensen, 1982, which we have made use of.
[19] For the second, see Dunn, 1980. On the first see Gibbins, 1981.
[20] Strictly this calls for a paraconsistent probability theory, an issue taken up in sect. 4.5 below. For elaboration of a non-dialethic paraconsistent approach to quantum theory see Routley, 1977, where a relevant position is advanced.
[21] We can indicate—in a way—how it can happen, through the following sequence of snapshots:



Nothing stops visualizing the impossible or diagramatic representation of the impossible: see EMJB, and also Canadian Education Program, 1982.
[22] De Witt, 1970, p. 30.
[23] Rescher and Brandom say that 'the systematisation of quantum-physics by the Everett-Wheeler approach *invites* (though it does not irrevocably demand) a logical apparatus that is inconsistency-tolerant', 1980, p. 60 (italics added). But they do not say how. Moreover on the basis of what they do say the possible world theories of Democritus and David Lewis would equally invite "inconsistency-tolerant apparatus"—which they decidedly do not. (Accounting for the 'maverick worlds' of De Witt, p. 34, *would* require a larger apparatus: the semantics for relevant logic would suffice.)
[24] This theme is much elaborated at the beginning of Routley, 1977.
[25] We are certainly not claiming that these types are either exclusive or exhaustive.
[26] For details in the case of the most difficult of these, relevant logics, see RLR, chapters 8 and 9, where multiply intensional relevant logics are much more fully treated.
[27] Relevant logics of this kind are investigated in RLR, ch. 8, and the philosophical issues involved are discussed in Routley, 1983. Nietzsche's nihilistic theme also has a notorious deontic analogue, Dostoievsky's Axiom: Everything is permissible (also investigated in the above sources).
[28] Further details can be found in RLR, ch. 8 and in Priest, 1982.
[29] Thus, e.g., Hegel, 1812, vol. II, bk. 2, ch. 2, §3.

[30] Many examples of such moral dilemmas are given in Routley and Plumwood, 1989.
[31] In non-adjunctive paraconsistent logics this principle is rejected (in our view erroneously). In such theories there are moral dilemmas of the form $OA \wedge O \sim A$ but not of the form $O(A \wedge \sim A)$.
[32] As we discuss, in the legal case, in the introductions to Parts Two and Four of the book.
[33] Arguments to this end are assembled in R. and V. Routley, 1975. The logic of belief there outlined is improved upon and elaborated in EMJB 8.11.
[34] See e.g. Lewis, 1982, Rescher and Brandom, 1980, Schotch and Jennings, 1989, Ellis, 1979.
[35] The development of the rest of paraconsistent probability theory lies beyond the scope of this introduction. For a fuller discussion of the developed theory and discussion of some other aspects of relevant probability theory, see Routley 1977.
[36] On this "paradox" see Makinson, 1964-65. The "lottery paradox" is further considered in R. and V. Routley, 1975.
[37] Details can be found in Routley, 1977.
[38] As to why, and for interesting suggestions as to how the elaboration should go see Belnap, 1977.
[39] See e.g. Haack, 1974, p. 109 ff.
[40] This approach to vagueness is taken in Pinter, 1980.
[41] Haack, *op. cit.*, p. 114.
[42] A version of the inconsistent variant is found in Peña, 1989.

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