Solutions

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Note that the solutions given below are to the problems of chapter 11, as revised according to the corrections on this website.

1. Construct a sorites argument for each of the predicates mentioned in 11.2.3.

$\boxed{\text{‘is tall’}}$

M_0 = ‘Barry is tall at 200cm’. If someone is tall, they are still tall if they become 1 millimeter shorter. So, M_1 = ‘Barry is tall at 199cm 9mm.’ And so on. The argument is then of the same form as that of 11.2.4, and continues until Barry's hight is, for instance, 50cms, at which point he is certainly not tall.

$\boxed{\text{‘is drunk’}}$

M_0 = ‘Barry is drunk with a Blood Alcohol Content of 0.40%.’ (Assuming Barry is not dead from alcohol poisoning.) If someone is drunk, they are still drunk if their BAC reduces 0.001%. So, M_1 = ‘Barry is drunk with a BAC of 0.399%.’ And so on. The argument is then of the same form as that of 11.2.4, and continues until Barry's BAC is 0.001%, at which point he is, intuitively, not drunk.

$\boxed{\text{‘is red’}}$

M_0 = ‘Barry is red when the radiant energy reflected from him is of a wavelength of 700 nanometers.’ If something is red, than it is still red when the wavelength of the radiant energy is changed by 0.1 of a nanometer. So M_1 = ‘Barry is red when the radiant energy reflected from him is of a wavelength of 699.9 nanometers.’ And so on. The argument is then of the same form as that of 11.2.4 and continues until the wavelength of the light reflected from Barry is under 600 nanometers, at which point he is, intuitively, not red.

$\boxed{\text{‘appears red’}}$

M_0 = 'Barry appears red when he looks red to all human observers.' If something appears red, then it still appears red even if one observer disagrees. So, M_1 = 'Barry appears red when he looks red to all observers, except one'. The argument is then of the same form as that of 11.2.4, and continues until Barry doesn't look red to all but one observer, at which point he does not, intuitively, appear red. The argument could perhaps be strengthened by an appeal to all possible observers.

$\boxed{\text{'is a heap of sand'}}$

M_0 = 'x is a heap of sand when it contains 10,000 grains.' If something is a heap of sand, then it is still a heap of sand if a grain is removed. So M_1 = 'x is a heap of sand when it contains 9999 grains.' And so on. The argument is then of the same form as that of 11.2.4 and continues until x contains 1 grain, at which point it is, intuitively, not a heap of sand.

$\boxed{\text{'is dead'}}$

M_0 = 'Barry is not dead at t_1.' Dying takes longer than one nanosecond, so if someone is not dead at a time, they are not dead one nanosecond later. Thus M_1 = 'Barry is not dead at t_1 + one nanosecond.' And so on. The argument is then of the same form as that of 11.2.4, and continues until a point in time at which Barry is decomposing in his coffin, and so is, intuitively, dead.

2. Check the details omitted in 11.4.4, 11.4.5, 11.5.3, 11.5.9, 11.7.5, and 11.7.10.

11.4.4: Show that if $x \leq y$ then $z \ominus x \leq z \ominus y$

Suppose that $x \leq y$ (and so that $-y \leq -x$): If $z \leq y$ then $z \ominus y = 1$, so the result follows. If $x \leq y < z$, then $z \ominus x = 1 - (z - x) \leq 1 - (z - y) = z \ominus y$.

11.4.5 : Check that if we restrict ourselves to just the values 1, 0.5, and 0, then the truth functions are exactly the same as those of Ł$_3$, thinking of $\rightarrow$ as $\supset$ and 0.5 as i.

For this it will suffice to show that the restricted truth tables for the above continuum-valued logic match to the truth tables for L_3. Compare the following with the tables in 7.3.2 and 7.3.8.

$f_\neg$	
1	0
0.5	0.5
0	1

$f_\wedge$	1	0.5	0
1	1	0.5	0
0.5	0.5	0.5	0
0	0	0	0

$f_\vee$	1	0.5	0
1	1	1	1
0.5	1	0.5	0.5
0	1	0.5	0

$f_\rightarrow$	1	0.5	0
1	1	0.5	0
0.5	1	1	0.5
0	1	1	1

11.5.3 Show that in every interpretation, each of the axioms of CK takes the value 1 in Ł$_\aleph$, and that the rules preserve this property.

A2, R1, A5, A9 and A15 have already been shown.

A1: $A \rightarrow A$

$a \ominus a = 1$, so A1 takes the value 1.

A3: $(A \wedge B) \rightarrow A$ (and $(A \wedge B) \rightarrow B$)

$Min(a, b) \leq a$ (and $Min(a, b)) \leq b)$)

Hence, $\nu(A \wedge B) \leq \nu(\text{A})$, so $\nu((A \wedge B) \rightarrow A) = 1$. (Similarly for B.)

A4: $A \wedge (B \vee C) \rightarrow ((A \wedge B) \vee (A \wedge C))$

There are six cases: (i) $a \leq b \leq c$ (ii) $a \leq c \leq b$ (iii) $b \leq a \leq c$ (iv) $b \leq c \leq a$ (v) $c \leq a \leq b$ (vi) $c \leq b \leq a$. Since b and c are symmetrically placed in the formula, we can ignore cases (ii), (v) and (vi). Elementary checking of the other three cases shows that:

$Min(a, Max(b, c)) \leq Max(Min(a, b)(Min(a, c)))$

Hence, $\nu(A \wedge (B \vee C) \rightarrow ((A \wedge B) \vee (A \wedge C))) = 1$.

A6: $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

There are six cases: (i) $a \leq b \leq c$ (ii) $a \leq c \leq b$ (iii) $b \leq a \leq c$ (iv) $b \leq c \leq a$ (v) $c \leq a \leq b$ (vi) $c \leq b \leq a$. Since a and b are symmetrically placed in the formula, we can ignore cases (iii), (iv) and (vi). Checking the other three cases shows that:

$Min(a \circleddash c, b \circleddash c) \leq (Max(a,b) \circleddash c)$

Case (i) is trivial. For case (ii): $a \ominus c = 1$. Hence $Min(a \circleddash c, b \circleddash c) = b \ominus c$. This is the same as $Max(a,b) \circleddash c$. For case (v): $a \leq b$, so $1 - b \leq 1 - a$ and $1 - b + c \leq 1 - a + c$. So $b \ominus c \leq a \ominus c$ and $Min(a \circleddash c, b \circleddash c) = b \ominus c$. This is the same as $Max(a,b) \circleddash c$.

A7: $\neg\neg A \to A$

$1 - (1 - a) = a$

The result follows.

A8: $(A \to \neg B) \to (B \to \neg A)$

There are two cases (i) $a + b \leq 1$ and $a + b \geq 1$. We wish to show that, in each case, $a \circleddash (1 - b) \leq b \circleddash (1 - a)$. The result follows.

In case (i), $b \leq 1 - a$, so $b \ominus (1 - a) = 1$, and the result follows. In the second case, $a \geq 1 - b$. So $a \ominus (1 - b) = 1 - a + 1 - b = 2 - a - b$. Similarly, $b \ominus (1 - a) = 1 - b + 1 - a = 2 - a - b$. So $a \circleddash (1 - b) \leq b \circleddash (1 - a)$.

A12: $A \to ((A \to B) \to B)$

Consider $a \circleddash ((a \circleddash b) \circleddash b)$. This has value 1. For:

If $a \leq b$, then $a \circleddash b = 1$, $(a \circleddash b) \circleddash b = 1 - 1 + b = b$ so $a \circleddash ((a \circleddash b) \circleddash b) = a \circleddash b = 1$.

If, on the other hand, $b \leq a$, then $a \circleddash b = 1 - a + b$, which is greater than b. So $((a \circleddash b) \circleddash b) = 1 - (1 - a + b) + b = a$. Thus, $a \circleddash ((a \circleddash b) \circleddash b) = a \circleddash a = 1$.

The result follows.

R2 $A, B \vdash A \wedge B$

If $a = 1$ and $b = 1$ then $Min(a, b) = 1$. The result follows.

11.5.9 Prove that $(A \vee B) \to ((A \to B) \to B)$ from the axioms of 11.5.2

(1)	$A \to ((A \to B) \to B)$	A12
(2)	$B \to ((A \to B) \to B)$	A15
(3)	$(A \to ((A \to B) \to B)) \wedge (B \to ((A \to B) \to B))$	(1), (2), R2
(4)	$((A \to ((A \to B) \to B)) \wedge (B \to ((A \to B) \to B))) \to$ $((A \vee B) \to ((A \to B) \to B))$	A6
(5)	$(A \vee B) \to ((A \to B) \to B)$	(3), (4), R1

11.6.5 Check that the following hold in Ł:

$\boxed{A \vDash B \to A}$

We show that $\nu(A \to (B \to A)) = 1$. Suppose that $a \ominus (b \ominus a) \neq 1$. Then $a > (b \ominus a)$. So, $b > a$, and $a > 1 - b + a$, which is impossible.

$\boxed{\neg A \vDash A \to B}$

The proof is similar.

$\boxed{(A \wedge B) \to C \vDash (A \to C) \vee (B \to C)}$

We show that $Min(a, b) \ominus c) \leq Max(a \ominus c, b \ominus c)$. The result follows.

If $a \leq b$, then $Min(a, b) \ominus c = a \ominus c$. By 11.4.4 $b \ominus c \leq a \ominus c$; so $Max(a \ominus c, b \ominus c) = a \ominus c$, as well.

If $b \leq a$ the argument is symmetric.

$\boxed{\neg(A \to B) \vDash A}$

We show that $1 - (a \ominus b) \leq a$. The result follows.

If $a \leq b$ then $a \ominus b = 1$, and $1 - (a \ominus b) = 0 \leq a$.

If $b \leq a$ then $1 - (a \ominus b) = 1 - (1 - a + b) = a - b \leq a$.

11.7.5 Show that the axioms of B are logically true in FB, and all the rules of B preserve this property.

A2, A5, R1 and R4 have already been shown.

A1: $A \to A$

At any world, x, of an interpretation, $a_x \ominus a_x = 1$. So for every normal world, w, $Glb\{a_x \ominus a_x : Rwxx\} = 1$.

A3: $(A \wedge B) \to A$ (and $(A \wedge B) \to B$).

At any world of an interpretation, x, $(a \wedge b)_x \leq a_x$; so $(a \wedge b)_x \ominus a_x = 1$. Thus, for every normal world, w, $Glb\{(a \wedge b)_x \ominus a_x; Rwxx\} = 1$.

A4: $A \wedge (B \vee C) \to ((A \wedge B) \vee (A \wedge C))$

For A4 in 11.5.3, we saw that: $Min(a, Max(b, c)) \leq Max(Min(a, b)(Min(a, c)))$. Hence, for every normal world w, $Glb\{a_x \wedge (b \vee c)_x \ominus (a \wedge b)_x \vee (a \wedge c)_x : Rwxx\} = 1$.

A6: $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

Suppose that in an interpretation $Rxyz$. Suppose that $a_y \leq b_y$. Then by 11.4.4 $b_y \ominus c_z \leq a_y \ominus c_z$. Moreover, $(a \vee b)_y = b_y$, so $(a \vee b)_y \ominus c_z = b_y \ominus c_z = Min\{a_y \ominus c_z, b_y \ominus c_z\}$. If $b_y \leq a_y$, on the other hand, the same result holds by a similar argument. Hence (cf. 11.7.6):

$$\begin{aligned}
((a \to c) \wedge (b \to c))_x \quad & = Min((a \to c)_x, (b \to c)_x) \\
& = Min(Glb\{a_y \ominus c_z : Rxyz\}, Glb\{b_y \ominus c_z : Rxyz\}) \\
& \leq Glb\{Min(a_y \ominus c_z, b_y \ominus c_z : Rxyz\} \\
& = Glb\{(a \vee b)_y \ominus c_z : Rxyz\} \\
& = ((a \vee b) \to c)_x
\end{aligned}$$

Hence for normal w, $Glb\{((a \to c) \wedge (b \to c))_x \ominus ((a \vee b) \to c)_x : Rwxx\} \leq 1$ as required.

A7: $\neg\neg A \to A$

At any world of an interpretation, $1 - (1 - a_x) = a_x$. So if w is any mormal world, $(\neg\neg a \to a)_w = Glb\{a_x \ominus a_x : Rwxx\} = 1$, as required.

R2 $A, B \vdash A \wedge B$

Suppose that w is normal, and that a_w and b_w are both 1. Then, $Min(a_w, b_w) = 1$.

R3 $A \to B \vdash (C \to A) \to (C \to B)$

Suppose that w is normal, and that $(a \to b)_w = 1$. Then for all y, $a_y \leq b_y$. It follows by 11.4.4 that $c_z \ominus a_y \leq c_z \ominus b_y$. Hence, $Glb\{c_z \ominus a_y : Rxyz\} \leq Glb\{c_z \ominus b_y : Rxyz\}$. That is, $(c \to a)_x \leq (c \to b)_x$. Hence, $Glb\{(c \to a)_x \ominus (c \to b)_x : Rwxx\} = 1$, as required.

R5 $A \to \neg B \vdash B \to \neg A$

Suppose that w is a normal world in an interpretation. In A8 of 11.5.3, we saw that $a \ominus (1-b) \leq b \ominus (1-a)$. So for any world x, $a_x \ominus (1-b)_x \leq b_x \ominus (1-a)_x$. Hence, $Glb\{a_x \ominus (1-b)_x : Rwxx\} \leq Glb\{b \ominus (1-a)_x : Rwxx\}$, as required.

11.7.10 Show that $p \to q \models \neg q \to \neg p$ holds in FB.

Take any interpretation and any world x of that interpretation. If $p_x \leq q_x$, then $1 - q_x \leq 1 - p_x$, so if $Glb\{p_x \ominus q_x : Rwxx\} = 1$, $Glb\{1 - q_x \ominus 1 - p_x : Rwxx\} = 1$.

3. Show the following in $L_{\aleph}$ (either by giving a deduction or by showing that whenever the premises have the value 1, so does the conclusion):

(a) $\models (A \to B) \vee (B \to A)$

In any interpretation, either the value of A is greater than or equal to the value of B or vice versa. Hence, one of the disjuncts must take the value 1, and the value of the whole sentence is 1 also.

(b) $\models (A \to (B \to C)) \to (B \to (A \to C))$

(i) Suppose that $a \leq c$. Then the value of the consequent is 1, as is the value of the whole conditional.

(ii) So suppose that $c \leq a$. Then the value of $A \to C$ is $1 - a + c$.

(iia) Suppose that $b \leq 1 - a + c$. Then the value of the consequent is 1, as is the value of the whole conditional.

(iib) So suppose that (α) $b \geq 1 - a + c$. Then the value of the consequent is $1 - b + (1 - a + c) = 2 - a - b + c$. By (α), $b \geq c$, so the value of $B \to C$ is $1 - b + c$. And by (α) again, $a \geq 1 - b + c$. Hence, the value of the antecedent is $1 - a + 1 - b + c = 2 - a - b - c$. So the antecedent and consequent have the same value, and the conditonal has value 1.

(d) $A \to B \models \neg B \to \neg A$

Suppose that in an interpretation $\nu(A \to B) = 1$. Then $\nu(A) \leq \nu(B)$. Hence, $1 - \nu(B) \leq 1 - \nu(A)$. That is, $\nu(\neg B \to \neg A) = 1$.

4. By constructing appropriate counter-models, show the following in $L_{\aleph}$:

(a) $\not\models p \vee \neg p$

Suppose $v(p) = 0.5$, then the the value of the formula is also 0.5.

(b) $\not\models (p \wedge (\neg p \vee q)) \to q$

Suppose $v(p) = 0.5$ and $v(q) = 0.3$. Then $v(\neg p \vee q) = 0.5$, and $v(p \wedge (\neg p \vee q)) = 0.5$, but $v((p \wedge (\neg p \vee q)) \to q) = 0.8$.

(c) $\not\models ((p \to q) \to q) \to q$

Suppose $v(p) = 0.5$ and $v(q) = 0.4$. Then $v(p \to q) = 0.9$, $v((p \to q) \to q) = 0.5$ and $v(((p \to q) \to q) \to q) = 0.9$.

(d) $\not\models ((p \to q) \wedge (q \to r)) \to (p \to r)$

Suppose $v(p) = 1$, $v(q) = 0.8$ and $v(r) = 0.5$. Then $v(p \to q) = 0.8$, $v(q \to r) = 0.7$, and the value of the conjunction of the two is 0.7. However $v(p \to r) = 0.5$, so the value of the conditional is 0.8.

(e) $\not\models (p \to \neg p) \to \neg p$

Suppose $v(p) = 0.7$, then $v(\neg p) = 0.3$, $v(p \to \neg p) = 0.6$, and $v((p \to \neg p) \to \neg p) = 0.7$.

5. Show the following in FB:

(a) $A \to B, A \to C \models A \to (B \wedge C)$

Consider any interpretation, and normal world, w. As proved in 11.7.6:

$Min\{(a \to b)_x, (a \to c)_x\} \leq ((a \to (b \wedge c))_x$.

Hence:

$$\begin{array}{rcl}
Glb\{Min\{(a \to b)_x, (a \to c)_x : Rwxx\} & \leq & Glb\{((a \to (b \wedge c))_x : Rwxx\} \\
Min\{Glb\{(a \to b)_x : Rwxx\}, Glb\{(a \to c)_x : Rwxx\} & \leq & Glb\{((a \to (b \wedge c))_x : Rwxx\} \\
Min((a \to b)_w, (a \to c)_w) & \leq & (a \to (b \wedge c))_w
\end{array}$$

As required.

(b) $A \to C, B \to C \models (A \vee B) \to C$

Consider any interpretation, and normal world, w. As we saw in connection with A6 of 11.7.5:

$Min((a \to c)_x, (b \to c)_x) \leq ((a \vee b) \to c)_w$

Hence:

$$\begin{array}{rcl} Glb\{Min((a \to b)_x, (a \to c)_x) : Rwxx\} & \leq & Glb\{((a \lor b) \to c)_x : Rwxx\} \\ Min(Glb\{(a \to b)_x : Rwxx\}, Glb\{(a \to c)_x : Rwxx\}) & \leq & Glb\{((a \lor b) \to c)) : Rwxx\} \\ Min((a \to b)_w, (a \to c)_w) & \leq & ((a \lor b) \to c)_w) \end{array}$$

As required.

(c) $p \to q, q \to r \not\models p \to r$

Consider an interpretation with one (normal) world, w, such that $v_w(p) = 0.7$, $v_w(q) = 0.5$, and $v_w(r) = 0.3$ Then $v_w(p \to q) = 0.8$, $v_w(q \to r) = 0.8$, so the *Glb* of the premises is 0.8. However $v_w(p \to r) = 0.6$..

6. Give the semantics of the *ceteris paribus* clause for fuzzy relevant logic (see 11.7.11), and investigate the properties of enthymematic conditionals.

We explain how to formulate the semantics for FB. Stronger relevant logics are obtained by adding constraints on the ternary R. Details are left for the reader.

For the most basic fuzzy relevant *ceteris paribus* logic $FB^{>}$, we add a *ceteris paribus* conditional $>$ to the language. Let I be an interpretation for FB. To obtain a semantics for the extended language, we add a collection of accessibility relations, $\{R_A : A$ is a formula of the (extended) language$\}$ to I. The relations are arbitrary. The truth conditions for the old connectives are as for FB. The conditions for $>$ are:

- $v_w(A > B) = Glb\{\nu_{w'}(B) : wR_Aw'\}$

The definition of validity is as before.

If all interpretations are two-valued, these semantics are those for the relevant *ceteris paribus* logic C_B of 10.7. Hence, the logic is a sub-logic of C_B. (It is a proper sub-logic for exactly the same reason that FB is a proper sub-logic of B (11.7.8).) In particular, we have the following:

- $p > q, q > r \not\models p > r$
- $p > r \not\models (p \land q) > r$
- $p > q \not\models \neg q > \neg r$

In FB the first of these also fails when '$>$' is replaced by '$\to$'. (See Ex. 5(a).) But the other two hold. Details are left as an exercise.

One may strengthen the logic by adding constraints on the family of accessability relations. Further details are left to the reader.

8. A notion of semantic consequence, $\models$, is said to be *compact* just if whenever $\Sigma \models A$ there is some finite $\Sigma' \subseteq \Sigma$ such that $\Sigma' \models A$. Let $\vdash$ be the deducibility

relationship of any axiom system. Since proofs are finite, then whenever $\Sigma \vdash A$ there is some finite $\Sigma^{'} \subseteq \Sigma$ such that $\Sigma^{'} \vdash A$. Show that if $\vdash$ is sound and complete with respect to $\models$, $\models$ is compact.

If $\vdash$ is sound and complete with respect to $\models$, then $\Sigma \vdash A$ iff $\Sigma \models A$. Suppose that $\models$ were not compact. Then there is some formula such that $\Sigma \models A$ and it is not the case that there is a finite $\Sigma^{'} \subseteq \Sigma$ such that $\Sigma' \models A$. But since $\Sigma \models A$, $\Sigma \vdash A$. Thus there is some finite $\Sigma^{'} \subseteq \Sigma$ such that $\Sigma^{'} \vdash A$, and so $\Sigma^{'} \models A$. Hence, $\models$ is compact.

9. Let $A * B$ be $\neg A \rightarrow B$. Show that (α) given any interpretation of $Ł_{\aleph}$, $v(A * B) = Min(1, v(A) + v(B))$. Let A^1 be A, and A^{n+1} be $A^n * A$. Show that (β) $\nu(A^n) = Min(1, n.v(A))$. Let $\Sigma = \{p^n \rightarrow q : n \geq 1\}$. Show that (γ) in $Ł_{\aleph}, \Sigma \models q$. (Hint: If $v(p) > 0$, then we can make $n.v(p) > 1$ by taking n to be large enough.) Show that (δ) if $\Sigma^{'}$ is any finite subset of Σ, $\Sigma^{'} \not\models q$. (Hint: there must be a largest n such that $p^n \rightarrow q$ is in $\Sigma^{'}$. Choose a v such that $v(p) < 1/n$.) Infer, from the last question that (ϵ) $Ł_{\aleph}$ has no axiom system that is sound and complete (with respect to arbitrary sets of premises).

(α) $v(A * B) = Min(1, v(A) + v(B))$.

Proof: $v(\neg A \rightarrow B) = (1 - a) \ominus b$. If $1 - a \leq b$ then $\nu(\neg A \rightarrow B) = 1$, and $1 \leq a + b$. Hence, $v(A * B) = Min(1, a + b)$. If $1 - a \geq b$ then $\nu(\neg A \rightarrow B) = 1 - (1 - a) + b = a + b$, and $1 \geq a + b$. Hence, $v(A * B) = Min(1, a + b)$.

(β) $v(A^n) = Min(1, n.v(A))$.

Proof: The proof is by induction on n. If $n = 1$ the result is trivial. Suppose that the result holds for n. Then by part (α), $\nu(A^{n+1}) = Min(1, Min(1, n.\nu(A)) + \nu(A))) = Min(1, (n + 1).\nu(A))$.

(γ) $\Sigma \models q$.

Proof: Suppose that ν gives every member of Σ the value 1. Suppose that $\nu(p) = 0$. Then the value of the every antecedent of a formula in Σ is 1 (by part (β)). Hence, the value of every member of Σ is $\nu(q)$. Hence, $\nu(q) = 1$. If $\nu(p) > 0$ then for some n, $\nu(p^n) = 1$ (by part (β)). Since $p^n \rightarrow q \in \Sigma$, $\nu(q) = 1$. Hence $\Sigma \models q$.

(δ) For no finite $\Sigma' \subseteq \Sigma$, $\Sigma' \models q$.

Proof: If Σ' is finite, there must be a greatest n such that $p^n \rightarrow q \in \Sigma'$. Take an interpretation ν such that $\nu(p) = \varepsilon < 1/n$. Let $\nu(q) = \delta$, where $n\varepsilon < \delta < 1$. Then for every $m \leq n$, $\nu(p^m) < \delta$ (by part (β)). Then for every $p^m \rightarrow q \in \Sigma'$, $\nu(p^m \rightarrow q) = 1$. Hence, $\Sigma' \not\models q$.

(ϵ) $L_{\aleph}$ is not compact.

Proof: The result follows from parts (δ) and (ϵ) by question 8.

10. *Check the details omitted in 11.7a.4, 11.7a.5, 11.7a.7, 11.7a.9, 11.7a.10 (soundness only), 11.7a.11, 11.7a.12 (soundness only), 11.7a.13 and 11.7a.14 (soundness only).

Prefix notation is not easy to read. We will therefore write, $f_{\rightarrow}(x, y)$ as $x \rightarrow y$, and so on. We will also write $\circ$ as $\bullet$, and $\underline{0}$ as 0. This makes all connectives and logical constants ambiguous. The style of the variable disambiguates.

$x \rightarrow y$ is, by definition, $Lub\{z\colon x \bullet z \leq y\}$. Since $\bullet$ is continuous, the least upper bound is achieved (11.7a.4). That is, $x \rightarrow y = Max\{z\colon x \bullet z \leq y\}$. Hence, $x \bullet (x \rightarrow y) \leq y$.

11.7a.4 Show that the definition $x \rightarrow y = Lub\{z\colon x \bullet z \leq y\}$ implies: (1) $x \bullet y \leq z$ iff $y \leq x \rightarrow z$. (2) $x \rightarrow y = 1$ iff $x \leq y$.

1. L to R: If $x \bullet y \leq z$ then $y \leq Lub\{y\colon x \bullet y \leq z\} = x \rightarrow z$
1. R to L: If $y \leq x \rightarrow z$ then $y \leq Lub\{y : x \bullet y \leq z\}$. So $x \bullet y \leq z$.

2. L to R: If $x \rightarrow y = 1$ then $1 = Max\{z : x \bullet z \leq y\}$. So $x \bullet 1 \leq y$, i. e., $x \leq y$.

2. R to L: If $x \leq y$ then $x \bullet 1 \leq y$. Hence $1 = Lub\{z : x \bullet z \leq y\}$, i. e., $x \rightarrow y = 1$.

We note a useful corollary: $1 \rightarrow y = y$. *Proof*: Since $1 \bullet (1 \rightarrow y) \leq y$, $1 \rightarrow y \leq y$. But $y \bullet 1 \leq y$. So by (1), $y \leq 1 \rightarrow y$.

11.7a.5 Show that 1: $x \wedge y = Min(x, y)$, and 2: $x \vee y = Max(x, y)$.

1: $x \bullet (x \rightarrow y) = Min(x, y)$. If $x \leq y$ then $Min(x, y) = x$, and $x \rightarrow y = 1$. The result follows. Suppose that $y \leq x$. Then $Min(x, y) = y$. Consider $z \bullet x$. This is a continuous function (of both arguments). $0 \bullet x = 0$ and $1z \bullet x = x$. Hence (by the Mean-Value Theorem) for some z, $z \bullet x = y$. The maximal such z is $x \rightarrow y$. Hence $(x \rightarrow y) \bullet x = x \bullet (x \rightarrow y) = x$.

2: We have to show that $Max(x, y) = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$. Suppose that $x \leq y$. (The case for $y \leq x$ is symmetric.) Then $x \rightarrow y = 1$. So $((x \rightarrow y) \rightarrow y) = 1 \rightarrow y = y$. Moreover, since $y \bullet (y \rightarrow x) \leq x$, $y \leq (y \rightarrow x) \bullet x$. Hence $((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x) = y$.

11.7a.7 Show that any theorem of BL is logically true in all $L(\bullet)$.

We show that all the axioms of BL take the value 1, and all the rules preserve this property. Given an evaluation, let a be $\nu(A)$, etc. There is only one rule, *modus ponens*: since $a \bullet (a \to b) \leq b$, if $a = 1$ and $a \to b = 1$ then $b = 1$. Now the axioms:

1. $(A \to B) \to ((B \to C) \to (A \to C)$.

$a \bullet (a \to b) \leq b$
$a \bullet (a \to b) \bullet (b \to c) \leq b \bullet (b \to c) \leq c$
$(a \to b) \bullet (b \to c) \leq a \to c$
$(a \to b) \leq (b \to c) \to (a \to c)$
$1 \leq (a \to b) \to ((b \to c) \to (a \to c))$

2. $(A \bullet B) \to A$.

$a \leq 1$
$a \bullet b \leq 1 \bullet b = b$
$1 \leq a \bullet b \to b$

3. $(A \bullet B) \to (B \bullet A)$.

$a \bullet b \leq b \bullet a$
$1 \leq (a \bullet b) \to (b \bullet a)$

4. $(A \bullet (A \to B)) \to (B \bullet (B \to A))$.

$a \bullet (a \to b)) = a \wedge b = Min(a, b) = Min(b, a) = b \wedge a = b \bullet (a \to b)$
$1 \leq (a \bullet (a \to b)) \to (b \bullet (b \to a))$

5. $(A \to (B \to C)) \to ((A \bullet B) \to C))$.

Let x be $a \to (b \to c)$. Then:
$x \leq a \to (b \to c)$
$x \bullet a \leq (b \to c)$
$x \bullet a \bullet b \leq c$
$x \leq (a \bullet b) \to c$
$1 \leq x \to ((a \bullet b) \to c)$

6. $((A \bullet B) \to C)) \to (A \to (B \to C))$

Let x be $(a \bullet b) \to c$. Then:
$x \leq (a \bullet b) \to c$
$x \bullet a \bullet b \leq c$
$x \bullet a \leq b \to c$

$x \leq a \to (b \to c)$
$1 \leq x \to (a \to (b \to c))$

7. $((A \to B) \to C) \to (((B \to A) \to C) \to C)$

Either $a \leq b$ or $b \leq a$. That is, either $1 = a \to b$ or $1 = b \to a$. In the first case, $(a \to b) \to c = 1 \to c = c$. But $c \bullet ((b \to a) \to c) \leq c$. Hence $c \leq ((b \to a) \to c) \to c$, as required. In the second case, $((b \to a) \to c) \to c = (1 \to c) \to c = c \to c = 1$. (Since $1 \bullet c \leq c$, $1 \leq c \to c$.) Hence, $(a \to b) \to c \leq ((b \to a) \to c) \to c$, as required.

8. $0 \to A$

$0 \leq a$
$1 \leq 0 \to a$

11.7a.9 The Łukasiewz t-norm, $x \bullet y$, is $Max(0, x + y - 1)$. Show that (1) if $x \leq y$ then $x \to y = 1$; (2) if $x \geq y$ then $x \to y = 1 - x + y$; and (3) $\neg x = 1 - x$.

1. $x \to y = Max\{z : z \bullet x \leq y\} = Max\{z : Max(0, z + x - 1) \leq y\}$. If $x \leq y$ this is 1.
2. Suppose, on the other hand, that $x \geq y$. $x + z - 1 \leq y$ iff $z \leq 1 - x + y$. Hence, $Max\{z : Max(0, z + x - 1) \leq y\} = 1 - x + y$.
3. $\neg x = x \to 0 = 1 - x + 0 = 1 - x$.

11.7a.10 Show that the axiom system for BL plus $\neg\neg A \to A$ is sound and with respect to the Łukasiewicz t-norm.

Since we know that the axiom system for BL is sound with respect to all t-norms, to check the soundness of the augmented system, all we have to show is that the new axiom schema is sound for the Łukasiewicz semantics. This we have already seen in Question 2, 11.5.3, A7.

11.7a.11 The product t-norm, $x \bullet y$, is the product $x.y$. Show (1) if $x \leq y$ then $x \to y = 1$; (2) if $x > y$ then $x \to y = y/x$; (3) if $x = 0$ then $\neg x = 1$; and (4) if $x > 0$ then $\neg x = 0$.

1. $x \to y = Lub\{z : x.z \leq y\}$. If $x \leq y$, this is obviously 1.
2. If $y < x$ then $Lub\{z : x.z \leq y\}$ is clearly y/x.
3. $\neg x = x \to 0$. If $x = 0$, this is 1 (by part 1).
4. If $x > 0$, this is $0/x$ (by part 2). This is 0.

11.7a.12 Show that the axiom system for BL plus (1) $(A \wedge \neg A) \to 0$ and (2) $\neg\neg C \to (((A \bullet C) \to (B \bullet C)) \to (A \to B))$ is sound with respect to the product t-norm.

Since we know that the axiom system for BL is sound with respect to all t-norms, to check the soundness of the augmented system, all we have to show is that the new axiom schemata are sound for the product semantics.

For 1: either $a = 0$ or $\neg a = 0$ (by parts 3 and 4 of the previous question). Hence $a \wedge \neg a = 0$. The result follows.

For 2: If $c = 0$, $\neg\neg c = 0$. The result follows. So suppose that $c > 0$. If $a \leq b$ then $c.a < c.b$. Hence, $a \bullet c \rightarrow b \bullet c = a \rightarrow b = 1$. The result follows. If $b < a$ then $a \rightarrow b = b/a$. Moreover, $b.c < a.c$. Hence, $a \bullet c \rightarrow b \bullet c = b.c/a.c = b/a$ as well. Hence $(a \bullet c \rightarrow b \bullet c) \rightarrow (a \rightarrow c) = 1$.

11.7a.13 The Goedel t-norm, $x \bullet y$ is $Min(x, y)$. Show (1) if $x \leq y$ then $x \rightarrow y = 1$; (2) if $x > y$ then $x \rightarrow y = y$; (3) if $x = 0$ then $\neg x = 1$; and (4) if $x > 0$ then $\neg x = 0$.

1. $x \rightarrow y = Lub\{z : Min(x, z) \leq y\}$. If $x \leq y$, this is obviously 1.
2. If $y < x$ then $Lub\{z : Min(x, z) \leq y\}$ is clearly y.
3. $\neg x = x \rightarrow 0$. If $x = 0$, this is 1 (by part 1).
4. If $x > 0$, this is 0 (by part 2).

11.7a.14 Show that the axiom system for BL plus $A \rightarrow (A \bullet A)$ and is sound with respect to the product t-norm.

Since we know that the axiom system for BL is sound with respect to all t-norms, to check the soundness of the augmented system, all we have to show is that the new axiom schema is sound for the product semantics.

Since $Min(a, a) = a$, $a \rightarrow (a \bullet a) = a \rightarrow a = 1$.

11. *Show that the Łukasiewicz, product, and Goedel t-norms *are* t-norms; that is, that they satisfy the conditions of 11.7a.2.

For the product and Goedel t-norms, this is entirely trivial.

The Łukasiewicz norm, $x \bullet y$, is $Max(0, x + y - 1)$. All the conditions are trivial apart from associativity. The argument for this is as follows: $(x \bullet y) \bullet z = Max(0, Max(0, x+y-1)+z-1)$. This is $Max(0, x+y+z-2)$. For if $x+y \leq 1$, both are 0. If $x + y > 1$ then $Max(0, x + y - 1) + z - 1 = x + y + z - 2$. A similar argument shows that $x \bullet (y \bullet z) = Max(0, x + y + z - 2)$ as well.

12. *Show that with the Łukasiewicz t-norm $x \bullet y$ may be defined as $\neg(x \rightarrow \neg y)$.

We prove this in three steps. In Łukasiewicz logic:

1. $\neg\neg x = x$
2. $x \rightarrow y = \neg y \rightarrow \neg x$
3. $x \bullet y = \neg(x \rightarrow \neg y)$

The proofs are as follows:

For1: $1 - (1 - x) = x$

For 2: $x \to y \leq (x \to 0) \to (y \to 0)$ (See proof of Question 10, 11.7a.1, axiom 1). So $x \to y \leq \neg x \to \neg y$. Similarly, $\neg x \to \neg y \leq \neg\neg x \to \neg\neg y = x \to y$, by part 1.

For 3: We prove the two halves.

$x \bullet (x \to \neg y) \leq \neg y$
$x \leq (x \to \neg y) \to \neg y$
$x \leq \neg\neg y \to \neg(x \to \neg y)$ (Part 2)
$x \leq y \to \neg(x \to \neg y)$ (Part 1)
$x \bullet y \leq \neg(x \to \neg y)$

$x \leq y \to (x \bullet y)$
$x \leq \neg(x \bullet y) \to \neg y$ (Part 2)
$x \bullet \neg(x \bullet y) \leq \neg y$
$\neg(x \bullet y) \leq x \to \neg y$
$1 \leq \neg(x \bullet y) \to (x \to \neg y)$
$1 \leq \neg(x \to \neg y) \to \neg\neg(x \bullet y)$ (Part 2)
$1 \leq \neg(x \to \neg y) \to (x \bullet y)$ (Part 1)
$\neg(x \to \neg y) \leq x \bullet y$

13. *Show that the axiom system for BL plus $\neg\neg A \to A$ proves everything that the axiom system for Ł of 11.5.1 can prove. (Hint: Use the previous question to formulate BL without $\bullet$. You will need to prove the substitutivity of equivalents for BL. This is relatively simple, since the only connective is then $\to$.) Since that is theoremwise complete, it follows that this axiom system is theoremwise complete too. (Soundness was already proved in Question 10, 11.7a.10.)

First, given the previous question, we can simply define $A \bullet B$ as $\neg(A \to \neg B)$. So done, the two axiom systems have the same vocabulary. Next, in the axiom system of 11.5.1, the last two axioms can be dispensed with, since $\vee$ and $\wedge$ be taken as defined. If we can prove the other four axioms, the result follows, since *modus ponens* is a rule of BL as well. The axioms are:

1. $(A \to B) \to ((B \to C) \to (A \to C))$
2. $A \to (B \to A)$
3. $(A \to \neg B) \to (B \to \neg A)$
4. $((A \to B) \to B) \to ((B \to A) \to A)$

1 is an axiom 1 of BL.

For 2: $(A \bullet B) \to A$ is axiom 2 of BL.The result follows by axiom 6.

For 3 and 4, we need the substitutivity of equivalents: if in BL (plus $\neg\neg A \to A$) $\vdash A \to B$ and $\vdash B \to A$ then $\vdash C(A) \to C(B)$ and $\vdash C(B) \to C(A)$. This is proved by induction on the complexity of C. The basis case is trivial. Given that $\bullet$ is defined in terms of $\neg$ and $\to$, the only connective in the language is $\to$. Hence, assuming that $\vdash C(A) \to C(B)$ and $\vdash C(B) \to C(A)$ we have to show that $\vdash (D \to C(A)) \to (D \to C(B))$ and $\vdash (C(B) \to D) \to (C(A) \to D)$. The second follows from axiom 1. Using axioms 1, 2, and 3, it is easy to establish that $(B \to C) \to ((A \to B) \to (A \to C))$. The first follows from that. Note that it follows by *modus ponens* that if $\vdash A \to B$ and $\vdash B \to A$ then $C(A) \vdash C(B)$ and $C(B) \vdash C(A)$. Call this form of substitutivity *SE*.

Two more lemmas are useful:

1. $\vdash A \to \neg\neg A$ and $\vdash \neg\neg A \to A$
2. $\vdash (A \to B) \to (\neg B \to \neg A)$ and $\vdash (\neg A \to \neg B) \to (B \to A)$

For 1: the second conjunct is the extra negation axiom. For the first, $A \bullet (A \to 0) \to 0$ (axioms 2 and 3). Hence $A \to ((A \to 0) \to 0)$ (axiom 6). For 2: the first conjunct is an instance of axiom 1, given the definition of $\neg$. The second conjunct follows from another instance, $\vdash (\neg A \to \neg B) \to (\neg\neg A \to \neg\neg B)$, given SE and 1.

Now, for 3: an instance of the first conjunct of 2 is $\vdash (A \to \neg B) \to (\neg\neg B \to \neg A)$. The result follows by SE and 1.

Finally, for 4:

$(\neg A \bullet (\neg A \to \neg B)) \to (\neg B \bullet (\neg B \to \neg A))$ (axiom 4)
$(\neg A \bullet (B \to A)) \to (\neg B \bullet (A \to B))$ (SE and 2)
$\neg(\neg B \bullet (A \to B)) \to \neg(\neg A \bullet (B \to A))$ (first conjunct of 2)
$\neg((A \to B) \bullet \neg B) \to \neg((B \to A) \bullet \neg A)$ (axiom 3 and SE)
$\neg\neg((A \to B) \to B) \to \neg\neg((B \to A) \to A)$ (Definition of $\bullet$)
$((A \to B) \to B) \to ((B \to A) \to A)$ (SE and 1)