Solutions

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May 4, 2010

1. Fill in the details left as exercises in 10.3.6, 10.4.6, 10.4a.8, 10.5.5, 10.5.6, and 10.7.1.

10.3.6: Check that all formulas of the following form are logically valid in B:

(A1) $A \to A$

$$\begin{array}{c} A \to A, -0 \\ \text{r000, r00}^{\#}0^{\#}\text{, r011, r01}^{\#}1^{\#} \\ A, +1 \\ A, -1 \\ \otimes \end{array}$$

(A2) $A \to (A \vee B)$

$$\begin{array}{c} A \to (A \vee B), -0 \\ \text{r000, r00}^{\#}0^{\#}\text{, r011, r01}^{\#}1^{\#} \\ A, +1 \\ A \vee B, -1 \\ A, -1 \\ B, -1 \\ \otimes \end{array}$$

(A3) $(A \wedge B) \to A$

$$\begin{array}{c} (A \wedge B) \to A, -0 \\ \text{r000, r00}^{\#}0^{\#}\text{, r011, r01}^{\#}1^{\#} \\ A \wedge B, +1 \\ A, -1 \\ A, +1 \\ B, +1 \\ \otimes \end{array}$$

(A4) $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$

$$(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C)), -0$$
$$\mathrm{r}000,\ \mathrm{r}00^{\#}0^{\#},\ \mathrm{r}011,\ \mathrm{r}01^{\#}1^{\#}$$
$$A \wedge (B \vee C)), +1$$
$$(A \wedge B) \vee (A \wedge C), -1$$
$$A, +1$$
$$B \vee C, +1$$
$$A \wedge B, -1$$
$$A \wedge C, -1$$
$$B, +1 \qquad\qquad C, +1$$
$$A, -1 \quad B, -1 \qquad A, -1 \qquad B, -1$$
$$\otimes \qquad \otimes \qquad\quad \otimes \qquad\qquad\qquad$$
$$A, -1 \quad C, -1$$
$$\otimes \qquad \otimes$$

(A5) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

$$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C)), -0$$
$$\mathrm{r}000,\ \mathrm{r}00^{\#}0^{\#},\ \mathrm{r}011,\ \mathrm{r}01^{\#}1^{\#}$$
$$(A \to B) \wedge (A \to C), +1$$
$$A \to (B \wedge C), -1$$
$$A \to B, +1$$
$$A \to C, +1$$
$$\mathrm{r}123$$
$$\mathrm{r}022,\ \mathrm{r}02^{\#}2^{\#},\ \mathrm{r}033,\ \mathrm{r}03^{\#}3^{\#}$$
$$A, +2$$
$$B \wedge C, -3$$
$$B, -3 \qquad\qquad C, -3$$
$$A, -2 \quad B, +3 \quad A, -2 \quad C, +3$$
$$\otimes \qquad \otimes \qquad \otimes \qquad \otimes$$

(A6) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

$$
\begin{array}{c}
((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C), -0 \\
\mathrm{r}000,\ \mathrm{r}00^{\#}0^{\#},\ \mathrm{r}011,\ \mathrm{r}01^{\#}1^{\#} \\
(A \to C) \wedge (B \to C), +1 \\
(A \vee B) \to C, -1 \\
A \to C, +1 \\
B \to C, +1 \\
\mathrm{r}123 \\
\mathrm{r}022,\ \mathrm{r}02^{\#}2^{\#},\ \mathrm{r}033,\ \mathrm{r}03^{\#}3^{\#} \\
A \vee B, +2 \\
C, -3 \\
\begin{array}{cc}
\begin{array}{c} A, +2 \\ \begin{array}{cc} A, -2 & C, +3 \\ \otimes & \otimes \end{array} \end{array}
&
\begin{array}{c} B, +2 \\ \begin{array}{cc} B, -2 & C, +3 \\ \otimes & \otimes \end{array} \end{array}
\end{array}
\end{array}
$$

(A7) $\neg\neg A \to A$

$$
\begin{array}{c}
\neg\neg A \to A, -0 \\
\mathrm{r}000,\ \mathrm{r}00^{\#}0^{\#},\ \mathrm{r}011,\ \mathrm{r}01^{\#}1^{\#} \\
\neg\neg A, +1 \\
A, -1 \\
\neg A, -1^{\#} \\
A, +1 \\
\otimes
\end{array}
$$

(R1) $A, A \to B \vdash B$

$$
\begin{array}{c}
A, +0 \\
A \to B, +0 \\
B, -0 \\
\mathrm{r}000,\ \mathrm{r}00^{\#}0^{\#} \\
\begin{array}{cc} A, -0 & B, +0 \\ \otimes & \otimes \end{array}
\end{array}
$$

(R2) $A, B \vdash A \wedge B$

$$
\begin{array}{c}
A, +0 \\
B, +0 \\
A \wedge B, -0 \\
\mathrm{r}000,\ \mathrm{r}00^{\#}0^{\#} \\
\begin{array}{cc} A, -0 & B, +0 \\ \otimes & \otimes \end{array}
\end{array}
$$

(R3) $A \to B \vdash (C \to A) \to (C \to B)$

$$
\begin{array}{c}
A \to B, +0 \\
(C \to A) \to (C \to B), -0 \\
\text{r}000,\ \text{r}00^{\#}0^{\#},\ \text{r}011,\ \text{r}01^{\#}1^{\#} \\
C \to A, +1 \\
C \to B, -1 \\
\text{r}123 \\
\text{r}022,\ \text{r}02^{\#}2^{\#},\ \text{r}033,\ \text{r}03^{\#}3^{\#} \\
C, +2 \\
B, -3 \\
\begin{array}{cc}
\begin{array}{c}
A, -3 \\
\begin{array}{cc}
C, -2 & A, +3 \\
\otimes & \otimes
\end{array}
\end{array}
&
\begin{array}{c}
B, +3 \\
\otimes \\
\\
\end{array}
\end{array}
\end{array}
$$

(R4) $A \to B \vdash (B \to C) \to (A \to C)$

Shown in 10.3.2

(R5) $A \to \neg B \vdash B \to \neg A$

$$
\begin{array}{c}
A \to \neg B, +0 \\
B \to \neg A, -0 \\
\text{r}000,\ \text{r}00^{\#}0^{\#},\ \text{r}011,\ \text{r}01^{\#}1^{\#} \\
B, +1 \\
\neg A, -1 \\
A, +1^{\#} \\
\begin{array}{cc}
\begin{array}{c}
A, -1^{\#} \\
\otimes \\
\\
\end{array}
&
\begin{array}{c}
\neg B, +1^{\#} \\
B, -1 \\
\otimes
\end{array}
\end{array}
\end{array}
$$

Check that all of the above except A5, A6, R3 and R4 hold in N_*.

(A1) $A \to A$

$$
\begin{array}{c}
A \to A, -0 \\
A, +1 \\
A, -1 \\
\otimes
\end{array}
$$

(A2) $A \to (A \lor B)$

$$\begin{array}{c} A \to (A \lor B), -0 \\ A, +1 \\ A \lor B, -1 \\ A, -1 \\ B, -1 \\ \otimes \end{array}$$

(A3) $(A \land B) \to A$

$$\begin{array}{c} (A \land B) \to A, -0 \\ A \land B, +1 \\ A, -1 \\ A, +1 \\ B, +1 \\ \otimes \end{array}$$

(A4) $(A \land (B \lor C)) \to ((A \land B) \lor (A \land C))$

$$\begin{array}{c} (A \land (B \lor C)) \to ((A \land B) \lor (A \land C)), -0 \\ A \land (B \lor C)), +1 \\ (A \land B) \lor (A \land C), -1 \\ A, +1 \\ B \lor C, +1 \\ A \land B, -1 \\ A \land C, -1 \\ \begin{array}{cc} B, +1 & C, +1 \\ \begin{array}{cc} A, -1 & B, -1 \\ \otimes & \otimes \end{array} & \begin{array}{cc} A, -1 & B, -1 \\ \otimes & \begin{array}{cc} A, -1 & C, -1 \\ \otimes & \otimes \end{array} \end{array} \end{array} \end{array}$$

(A5) $((A \to B) \land (A \to C)) \to (A \to (B \land C))$ does not hold:

$$\begin{array}{c} ((p \to q) \land (p \to r)) \to (p \to (q \land r)), -0 \\ (p \to q) \land (p \to r), +1 \\ p \to (q \land r), -1 \\ p \to q, +1 \\ p \to r, +1 \end{array}$$

Counter-model such that

$$\begin{array}{ll} w_0 & w_1 \\ & +p \to q \\ & +p \to r \\ & -p \to (q \land r) \end{array}$$

(A6) $((A \to C) \land (B \to C)) \to ((A \lor B) \to C)$ does not hold:

$$\begin{array}{c}
((p \to r) \land (q \to r)) \to ((p \lor q) \to r), -0 \\
(p \to r) \land (q \to r), +1 \\
(p \lor q) \to r, -1 \\
p \to r, +1 \\
q \to r, +1
\end{array}$$

Counter-model such that

$$\begin{array}{ll}
w_0 & w_1 \\
 & +p \to r \\
 & +q \to r \\
 & -(p \lor q) \to r
\end{array}$$

(A7) $\neg\neg A \to A$

$$\begin{array}{c}
\neg\neg A \to A, -0 \\
\neg\neg A, +1 \\
A, -1 \\
\neg A, -1^{\#} \\
A, +1 \\
\otimes
\end{array}$$

(R1) $A, A \to B \vdash_{N_*} B$

$$\begin{array}{c}
A, +0 \\
A \to B, +0 \\
B, -0 \\
\begin{array}{cc} A, -0 & B, +0 \\ \otimes & \otimes \end{array}
\end{array}$$

(R2) $A, B \vdash_{N_*} A \land B$

$$\begin{array}{c}
A, +0 \\
B, +0 \\
A \land B, -0 \\
\begin{array}{cc} A, -0 & B, +0 \\ \otimes & \otimes \end{array}
\end{array}$$

(R3) $A \to B \nvdash_{N_*} (C \to A) \to (C \to B)$

$$\begin{array}{c}
p \to q, +0 \\
(r \to p) \to (r \to q), -0 \\
r \to p, +1 \\
r \to q, -1 \\
\begin{array}{cc}
\begin{array}{c} p, -0 \\ \begin{array}{cc} p, -1 & q, +1 \end{array} \end{array} &
\begin{array}{c} q, +0 \\ \begin{array}{cc} p, -1 & q, +1 \end{array} \end{array}
\end{array}
\end{array}$$

Counter-model from open left-most branch such that

$$\begin{array}{ll}
w_0 & w_1 \\
-p & +r \to p \\
 & -r \to q \\
 & -p
\end{array}$$

(R4) $A \to B \nvdash_{N_*} (B \to C) \to (A \to C)$

$$\begin{array}{c}
p \to q, +0 \\
(q \to r) \to (p \to r), -0 \\
q \to r, +1 \\
p \to r, -1 \\
\begin{array}{cc}
\begin{array}{c} p, -0 \\ \begin{array}{cc} p, -1 & q, +1 \end{array} \end{array} &
\begin{array}{c} q, +0 \\ \begin{array}{cc} p, -1 & q, +1 \end{array} \end{array}
\end{array}
\end{array}$$

Counter-model from open left-most branch such that

$$\begin{array}{ll}
w_0 & w_1 \\
-p & +q \to r \\
 & -p \to r \\
 & -p
\end{array}$$

(R5) $A \to \neg B \vdash_{N_*} B \to \neg A$

$$\begin{array}{c}
A \to \neg B, +0 \\
B \to \neg A, -0 \\
B, +1 \\
\neg A, -1 \\
A, +1^{\#} \\
\begin{array}{cc}
\begin{array}{c} A, -1^{\#} \\ \otimes \end{array} &
\begin{array}{c} \neg B, +1^{\#} \\ B, -1 \\ \otimes \end{array}
\end{array}
\end{array}$$

10.4.6 Show that A8 - A10 are valid in B if the appropriate constraints are added.

(A8) $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$

The inference is invalid in B without the constraint:

$$
\begin{array}{c}
(p \rightarrow \neg q) \rightarrow (q \rightarrow \neg p), -0 \\
\mathrm{r}000,\ \mathrm{r}00^{\#}0^{\#},\ \mathrm{r}011,\ \mathrm{r}01^{\#}1^{\#} \\
p \rightarrow \neg q, +1 \\
q \rightarrow \neg p, -1 \\
\mathrm{r}123 \\
\mathrm{r}022,\ \mathrm{r}02^{\#}2^{\#},\ \mathrm{r}033,\ \mathrm{r}03^{\#}3^{\#} \\
q, +2 \\
\neg p, -3 \\
p, +3^{\#} \\
\begin{array}{cc}
p, -2 & \neg q, +3 \\
 & q, -3^{\#}
\end{array}
\end{array}
$$

Counter-model from open left-most branch such that

$$
\begin{array}{ccccccc}
 & w_0 & & & w_0^* & & \\
 & & & & & & \\
 & w_1 & & & w_1^* & & \\
 & \angle & & & & & \\
-p, +q & w_2 & w_3 & w_2^* & w_3^* & +p
\end{array}
$$

$p \rightarrow \neg q$ is true at w_1, because p is false at w_2. However, $q \rightarrow \neg p$ is false at w_1, because q is true at w_2, but p is not false at w_3 - this can be seen by the fact that p is true in w_3^*. Thus $(p \rightarrow \neg q) \rightarrow (q \rightarrow \neg p)$ is false at w_0.

With the addition of the constraint, 'If $Rabc$ then Rac^*b^*', A8 is valid. This can be shown by repeating the tableau, with the addition of the following rule:

$$
\begin{array}{c}
rxyz \\
\downarrow \\
rx\bar{z}\bar{y}.
\end{array}
$$

$$(A \to \neg B) \to (B \to \neg A), -0$$
$$\text{r000, r00}^{\#}0^{\#}\text{, r011, r01}^{\#}1^{\#}$$
$$A \to \neg B, +1$$
$$B \to \neg A, -1$$
$$\text{r123, r13}^{\#}2^{\#}$$
$$\text{r022, r02}^{\#}2^{\#}\text{, r033, r03}^{\#}3^{\#}$$
$$B, +2$$
$$\neg A, -3$$
$$A, +3^{\#}$$
$$\begin{array}{cc} A, -3^{\#} & \neg B, +2^{\#} \\ \otimes & B, -2 \\ & \otimes \end{array}$$

In the below, I will be omitting the relations between world 0 and other worlds — these will be assumed.

(A9) $(A \to B) \to ((B \to C) \to (A \to C))$

The inference is invalid in B without the constraint:

$$(p \to q) \to ((q \to r) \to (p \to r)), -0$$
$$p \to q, +1$$
$$(q \to r) \to (p \to r), -1$$
$$\text{r123}$$
$$q \to r, +2$$
$$p \to r, -3$$
$$\text{r345}$$
$$p, +4$$
$$r, -5$$
$$\begin{array}{cc} p, -2 & q, +3 \end{array}$$

Counter-model from open left-most branch such that

$$\begin{array}{cccccccc} & & w_0 & & & & w_0^* & \\ & & & & & & & \\ & & w_1 & & & & w_1^* & \\ & & \angle & & & & & \\ -p & w_2 & & w_3 & & & w_2^* & w_3^* \\ & & \angle & & & & & \\ +p & w_4 & & w_5 & -r & & w_4^* & w_5^* \end{array}$$

However, with the addition of the constraint, 'If there is an $x \in W$ such that $Rabx$ and $Rxcd$, then there is a $y \in W$ such that $Racy$ and $Rbyd$', it is valid. This can be shown by repeating the tableau, with the addition of the following rule:

$$
\begin{array}{c}
rxyz \\
rzuv \\
\downarrow \\
rxuj,\ ryjv
\end{array}
$$

$$
\begin{array}{c}
(A \to B) \to ((B \to C) \to (A \to C)), -0 \\
A \to B, +1 \\
(B \to C) \to (A \to C), -1 \\
\text{r}123 \\
B \to C, +2 \\
A \to C, -3 \\
\text{r}345 \\
A, +4 \\
C, -5 \\
\text{r}146,\ \text{r}265 \\
\begin{array}{cc}
A, -4 & B, +6 \\
\otimes & \begin{array}{cc} B, -6 & C, +5 \\ \otimes & \otimes \end{array}
\end{array}
\end{array}
$$

(A10) $(A \to B) \to ((C \to A) \to (C \to B)$

The inference is invalid in B without the constraint:

$$
\begin{array}{c}
(p \to q) \to ((r \to p) \to (r \to q)), -0 \\
p \to q, +1 \\
(r \to p) \to (r \to q), -1 \\
\text{r}123 \\
r \to p, +2 \\
r \to q, -3 \\
\text{r}345 \\
r, +4 \\
q, -5 \\
\begin{array}{cc} p, -2 & q, +3 \end{array}
\end{array}
$$

Counter-model from open left-most branch such that

$$
\begin{array}{cccccccc}
 & & w_0 & & & & w_0^* & \\
 & & & & & & & \\
 & & w_1 & & & & w_1^* & \\
 & & \angle & & & & & \\
-p & w_2 & & w_3 & & w_2^* & & w_3^* \\
 & & \angle & & & & & \\
+r & w_4 & & w_5 & -q & w_4^* & & w_5^*
\end{array}
$$

However, with the addition of the constraint, 'If there is an $x \in W$ such that $Rabx$ and $Rxcd$, then there is a $y \in W$ such that $Rbcy$ and $Rayd$', it is valid. This can be shown by repeating the tableau, with the addition of the rule (T10):

$$\begin{array}{c} rxyz \\ rzuv \\ \downarrow \\ ryuj,\ rxjv \end{array}$$

$$\begin{array}{c} (A \to B) \to ((C \to A) \to (C \to B)), -0 \\ A \to B, +1 \\ (C \to A) \to (C \to B), -1 \\ \text{r}123 \\ C \to A, +2 \\ C \to B, -3 \\ \text{r}345 \\ C, +4 \\ B, -5 \\ \text{r}246,\ \text{r}165 \end{array}$$

$$\begin{array}{ccc} A, -6 & & B, +5 \\ & & \otimes \end{array}$$

$$\begin{array}{cc} C, -4 & A, +6 \\ \otimes & \otimes \end{array}$$

10.4a.8 Show that A13 - A16 are valid in B provided the appropriate constraint is added.

(A13) $A \vee \neg A$

To show that (A13) is not valid in B, the following counter-model will do:

$$-p \quad w_0 \qquad\qquad w_0^* \quad +p$$

p is false at w_0, and since p is true at w_0^*, $\neg p$ is false at w_0, so $p \vee \neg p$ is false at w_0.

A13 is valid given C13 (the condition 'If $a \in N$, $a^* \sqsubseteq a$). We can show this by proving that it cannot be the case that both A is false, and $\neg A$ is false at w_0 on any interpretation.

Take any interpretation such that $w_0 \in N$. Suppose that this interpretation makes A false at w_0. Then it makes $\neg A$ true at w_0^*. But by the condition, since $w_0 \in N$, $w_0^* \sqsubseteq w_0$. So, because $v_{w_0^*}(\neg A) = 1$, $v_{w_0}(\neg A) = 1$. In other words, $\neg A$ is true at w_0.

(A14) $(A \to \neg A) \to \neg A$

To show that A14 is not valid in B, the following counter-model will do:

$$\begin{array}{cccccccc}
 & & w_0 & & & w_0^* & & \\
 & & & & & & & \\
 & & w_1 & & & w_1^* & & +p \\
 & & \angle & & & & & \\
-p & w_2 & & w_3 & w_2^* & & w_3^* &
\end{array}$$

$p \to \neg p$ is true at w_1, because p is false at w_2. However, $\neg p$ is not true at w_1 because p is true at w_1^*.

To establish that A14 is valid given C14 (the condition 'If $a \in N$, $a^* \sqsubseteq a$; and if $a \in W - N$, Raa^*a'), suppose that in an interpretation $w_0 \in N$ and Rw_0aa. We need to show that if $A \to \neg A$ is true at a, so is $\neg A$.

Suppose that $a \in N$, and that $A \to \neg A$ is true at a. If A is true at a, then $\neg A$ is true at a. If A is not true at a, then $\neg A$ is true at a^*. Since $a^* \sqsubseteq a$ $\neg A$ is true at a. So, in both cases we have the result.

Suppose that $a \in W - N$. If $A \to \neg A$ is true at a, then for all b and c such that $Rabc$, either A is false at b or $\neg A$ is true at c. By the condition, Raa^*a. So, either A is false at a^* or $\neg A$ is true at a. In either case, $\neg A$ is true at a, as required.

(A15) $A \to (B \to A)$

To show that A15 is not valid in B, the following will do, where w_0 is the only normal world, and $\sqsubseteq$ is $=$.

$$\begin{array}{ccccccccc}
 & & w_0 & & & & w_0^* & & \\
 & & & & & & & & \\
+p & & w_1 & & & & w_1^* & & \\
 & & \angle & & & & & & \\
+q & w_2 & & w_3 & -p & w_2^* & & w_3^* &
\end{array}$$

p is true at w_1. But $Rw_1w_2w_3$, q is true at w_2, and p is false at w_3, hence $q \to p$ is false at w_1 and $p \to (q \to p)$ is false at w_0.

To show that A15 is valid given C15 (the condition 'If $Rabc$ then $a \sqsubseteq c$'), suppose that in an interpretation $w_0 \in N$ and Rw_0aa. We must show that if A is true at a, then so is $B \to A$. Let us take A to be true at a. If $B \to A$ were not true at a then there would be two worlds b and c such that $Rabc$, B is true at b, and A is false at c. But, by the condition, since $Rabc$, $a \sqsubseteq c$. A is true at

a, so A is true at c.

(A16) $A \to (A \to A)$

To show that A16 is not valid in B, the following will do, where w_0 is the only normal world, and $\sqsubseteq$ is $=$.

$$\begin{array}{cccccccc}
 & & w_0 & & & & w_0^* & \\
 & & & & & & & \\
+p & & w_1 & & & & w_1^* & \\
 & & \angle & & & & & \\
+p & w_2 & & w_3 & -p & w_2^* & & w_3^*
\end{array}$$

p is true at w_1. However, $Rw_1w_2w_3$, and p is true at w_2, but p is false at w_3, so $p \to p$ is false at w_1.

To show that A16 is valid given C16 (the condition 'If $Rabc$ then $a \sqsubseteq c$ or $b \sqsubseteq c$), suppose that in an interpretation $w_0 \in N$ and Rw_0aa. We must show that if A is true at a, then so is $A \to A$. Suppose that A is true at a. If $A \to A$ were not true at a then there must be two worlds b and c such that $Rabc$, and A is true at b, but A is not true at c. By the condition, since $Rabc$, $a \sqsubseteq c$ or $b \sqsubseteq c$. Since A is true at both a and b, in either case, A is true at c, as required.

10.5.5 Check that every axiom of R takes a designated value in RM_3.

RM_3: $\mathcal{D} = \{1, i\}$

$f_\supset$	1	i	0
1	1	0	0
i	1	i	0
0	1	1	1

I will substitute $\supset$ for $\to$ in the following. I will take R to be axiomatised by A1 - A12 plus R1 and R2.

(A1) $A \supset A$

If the values of A and B are the same, then the value of $A \supset B$ is designated. Hence $A \supset A$ is always designated.

(A2) $A \supset (A \vee B)$

Suppose this were undesignated in RM_3. Then the truth values of the antecedent and consequent are (1, i), (i,0) or (1, 0). If (1, i) the truth-value of $A \vee B$ is i. Looking at the truth-table for $\vee$ we can see that this means the truth value of A (B) cannot be 1. But this is contradictory. If (i, 0) then the truth value of $A \vee B$ is 0 so the truth-value of A (B) is 0. But this is contradictory. Likewise, if (1, 0), the truth-value of $A \vee B$ is 0, so the truth value of A (B) is 0. But this is contradictory.

(A3) $(A \wedge B) \supset A$

Suppose this were undesignated in RM_3. Then the truth values of the antecedent and consequent are (1, i), (i,0) or (1, 0). If (1, i) then the truth-value of $A \wedge B$ is 1, so the truth value of A (B) is 1. But this is contradictory. If (i, 0) then looking at the truth-table for $\wedge$ we see that the truth value of A (B) cannot be 0. But this is contradictory. If (1, 0), the truth-value of $A \wedge B$ is 1, so the truth value of A (B) is 1. But this is contradictory.

(A4) $A \wedge (B \vee C) \supset ((A \wedge B) \vee (A \wedge C))$

Suppose this were undesignated in RM_3. Then the truth values of the antecedent and consequent are (1, i), (i,0) or (1, 0). If (1, i) then the truth-value of $A \wedge (B \vee C)$ is 1, so the truth value of A is 1, and the truth value of B or C is 1. The truth value of $(A \wedge B) \vee (A \wedge C)$ is i, but if the truth value of A and B (or C) is 1, this cannot be the case. If (i, 0) then the truth value of $A \wedge (B \vee C)$ is i, so the truth value of A is i or 1.But the truth value of the consequent is 0, so the truth value of $A \wedge B$, and $A \wedge C$, is also. But this is contradictory. If (1, 0), similarly, the antecedent tells us that the truth-value of A is 1. But the consequent tells us that the truth-value of A is 0.

(A5) $((A \supset B) \wedge (A \supset C)) \supset (A \supset (B \wedge C))$

Suppose this were undesignated in RM_3. Then the truth values of the antecedent and consequent are (1, i), (i,0) or (1, 0). If (1, i) then the truth-value of $(A \supset B) \wedge (A \supset C)$ is 1, so the truth value of $A \supset B$ is 1, and the truth-value of A is 0, or the truth-value of B is 1 or i, and from the second conjunct, if A is not 0, C is 1 or i. So, either A is 0, or B and C are 1 or i. If A is 0, then the consequent is not 0. And if B and C are 1 or i then $B \wedge C$ is 1 or i, so the value of the consequent is not 0. If (i, 0) then the truth value of $A \supset (B \wedge C)$ is 0, so again there are three possibilities for the consequent: (1, i), (i,0) or (1, 0). In the cases where A is 1, $B \wedge C$ is i or 0. So the values of one of B or C are i or 0. Hence one of the conjuncts in the antecedent must take the value 0, and hence the antecedent must take the value 0. In the case where A is i, B and C are 0, so the antecedent is 0. If the value of the inference is (1,0) then again there are three possibilities for the consequent, (1, i), (i,0) or (1, 0). The same reasoning as above shows the inference to always take a designated value.

(A6) $((A \supset C) \wedge (B \supset C)) \supset ((A \vee B) \supset C)$

Suppose this were undesignated in RM_3. Then the truth values of the antecedent and consequent are (1, i) (i,0) or (1, 0). If (1, i) then the truth-value of $(A \vee B) \supset C)$ is i, so the truth value of $A \vee B$ is i, meaning it is not the case that both A and B is 0. If C is i, the only way in which the antecedent can take the value 1 is for both A and B to take the value 0. But this is not the case. If (i, 0) then there are three possibilities for the consequent: (1, i) (i,0) or (1, 0). In the case where C is i, $A \vee B$ is 1, so A or B is 1. But then one of the conjuncts in the antecedent takes the value 0, and so does the conjunct. In the cases where C is 0, $A \vee B$ is either 1, or i, so it is not the case that both A and B take the value 0. But both (i, 0) and (1, 0) are assign conditionals the value 0, so the value of the antecedent is 0. If the inference takes the values (1, 0), then again the consequent takes the value 0, and the same reasoning shows that the antecedent cannot be 1.

(A7) $\neg\neg A \supset A$

Suppose this were undesignated in RM_3. Then the truth values of the antecedent and consequent are (1, i) (i,0) or (1, 0). Looking at the truth table for $\neg A$ in RM_3, it is clear that $\neg\neg A$ will always take the same value as$A$, so all three possibilities are contradictory.

(A8) $(A \supset \neg B) \supset (B \supset \neg A)$

Suppose this were undesignated in RM_3. Then the truth values of the antecedent and consequent are (1, i) (i,0) or (1, 0). In the first case, from the consequent, the truth values of B and $\neg A$ are both i. Hence the value of A is i, and the value of $\neg B$ is i. But then the value of the antecedent is i. In the second case, the same reasoning can be applied to the antecedent. In the third case, there are three possibilities for the consequent: (1, i) (i,0) or (1, 0). In the case where B is 1, and $\neg A$ is i, $\neg B$ is 0 and A is i. So $A \supset \neg B$ is 0. In the case where B is i and $\neg A$ is 0, $\neg B$ is i and A is 1. So the value of $A \supset \neg B$ is 0. In the case where B is 1 and $\neg A$ is 0, $\neg B$ is 0, and A is 1. So the truth value of $A \supset \neg B$ is 0.

(A9) $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$

Suppose this were undesignated in RM_3. Then the truth values of the antecedent and consequent are (1, i) (i,0) or (1, 0). In the first case, the value of $B \supset C$ and $A \supset C$ is i, thus the values of A, B and C are i. But then the value of the antecedent is i. In the second case, from the antecedent, the values of A and B are i. But then the two halves of the conditional in the consequent take the same truth value, and so the value of the consequent is not 0. In the third case, there are three possibilities for the consequent: (1, i) (i,0) or (1, 0). In the case where $B \supset C$ is 1, and $A \supset C$ is i, both A and C are i, so then B must be 0. But this means that the antecedent $A \supset B$ is 0. In the case where $B \supset C$ is i and $A \supset C$ is 0, both B and C are i, so A must be 1. But then $A \supset B$ is 0. In the case where $B \supset C$ is 1 and $A \supset C$ is 0, A and C take either (1, i) (i,0) or (1, 0). In every case, B must take 0, so $A \supset B$ takes 0.

(A10) $(A \supset B) \supset ((C \supset A) \supset (C \supset B))$

Suppose this were undesignated in RM_3. Then the truth values of the antecedent and consequent are (1, i) (i,0) or (1, 0). In the first case, the value of $C \supset A$ and $C \supset B$ is i, thus the values of A, B and C are i. But then the value of the antecedent is i. In the second case, from the antecedent, the values of A and B are i. But then the two halves of the conditional in the consequent take the same truth value, and so the value of the consequent is not 0. In the third case, there are three possibilities for the consequent: (1, i) (i,0) or (1, 0). In the case where $C \supset A$ is 1, and $C \supset B$ is i, both B and C are i, so then A must be 1. But this means that the antecedent $A \supset B$ is 0.In the case where $C \supset A$ is i and $C \supset B$ is 0, both A and C are i, so A must be 1. But then $A \supset B$ is 0. In the case where $C \supset A$ is 1 and $C \supset B$ is 0, C and B take either (1, i) (i,0) or (1, 0). In every case, A must take 0, so $A \supset B$ takes 0.

(A11) $(A \supset (A \supset B)) \supset (A \supset B)$

Suppose this were undesignated in RM_3. Then the truth values of the antecedent and consequent are (1, i) (i,0) or (1, 0). In the first case, the value of $A \supset B$ is i, thus the values of A and B i. But then the value of the antecedent is i. In the second case, from the antecedent, the values of A and B are i. But then the value of the consequent is i. In the third case, there are three possibilities for the consequent: (1, i) (i,0) or (1, 0). In all three cases, the value of the antecedent comes out as 0.

(A12) $A \supset ((A \supset B) \supset B)$

Suppose this were undesignated in RM_3. Then the truth values of the antecedent and consequent are (1, i) (i,0) or (1, 0). In the first case, the value of $(A \supset B) \supset B$ is i, hence the value of A is i. In the second case the value of A is i. If B is 1, the value of the consequent is 1. If B is i, the value of the consequent is i. If B is 0, the value of the consequent is 0. In the third case, the value of A is 1. If B is 1, the value of the consequent is 1. If B is i, the value of the consequent is 1. If B is 0, the value of the consequent is 1.

(R1) $A, A \supset B \vdash B$

Suppose the premises were designated, and conclusion undesignated in RM_3. Then the value of B is 0, and the value of each premise is 1 or i. However, if A is 1 or i, the value of $A \supset B$ is 0.

(R2) $A, B \vdash A \wedge B$

Suppose the premises were designated, and conclusion undesignated in RM_3. Then the value of $A \wedge B$ is 0, so the value of either A or B is 0. Hence one of the premises is undesignated.

10.5.6 Check that all the axioms of R are valid in the logic given in this section.

Unfortunately the author of these solutions is not enough of a masochist to attempt this exercise. Get some large pieces of paper, and check the 8, 64, or 512 cases in the truth table for each axiom!

10.7.1 Check that the inferences of 5.2.1 are all valid in N_*.

Antecedent strengthening: $A \rightarrow B \vDash_{N_*} (A \wedge C) \rightarrow B$

$$\begin{array}{c}
A \rightarrow B, +0 \\
(A \wedge C) \rightarrow B, -0 \\
A \wedge C, +1 \\
B, -1 \\
A, +1 \\
C, +1 \\
\begin{array}{cc}
A, -1 & B, +1 \\
\otimes & \otimes
\end{array}
\end{array}$$

Transitivity: $A \rightarrow B, B \rightarrow C \vDash_{N_*} A \rightarrow C$

$$\begin{array}{c}
A \rightarrow B, +0 \\
B \rightarrow C, +0 \\
A \rightarrow C, -0 \\
A, +1 \\
C, -1 \\
\begin{array}{cc}
A, -1 & B, +1 \\
\otimes & \begin{array}{cc} B, -1 & C, +1 \\ \otimes & \otimes \end{array}
\end{array}
\end{array}$$

Contraposition: $A \rightarrow B \vDash_{N_*} \neg B \rightarrow \neg A$

$$\begin{array}{c}
A \rightarrow B, +0 \\
\neg B \rightarrow \neg A, -0 \\
\neg B, +1 \\
\neg A, -1 \\
B, -1^{\#} \\
A, +1^{\#} \\
\begin{array}{cc}
A, -1^{\#} & B, +1^{\#} \\
\otimes & \otimes
\end{array}
\end{array}$$

2. Show that the following fail in B:

(a) $(p \wedge q) \to r \vdash p \to (q \to r)$

$$
\begin{array}{c}
(p \wedge q) \to r, +0 \\
p \to (q \to r), -0 \\
\mathrm{r}000,\ \mathrm{r}00^{\#}0^{\#},\ \mathrm{r}011,\ \mathrm{r}01^{\#}1^{\#} \\
p, +1 \\
q \to r, -1 \\
\mathrm{r}123 \\
q, +2 \\
r, -3 \\
\mathrm{r}022,\ \mathrm{r}02^{\#}2^{\#},\ \mathrm{r}033,\ \mathrm{r}03^{\#}3^{\#}
\end{array}
$$

- $p \wedge q, -3$
  - $p, -3$
    - $p \wedge q, -2$
      - $p, -2$
        - $p \wedge q, -1$
          - $p, -1$ $\otimes$
          - $q, -1$
        - $r, +1$
      - $q, -2$ $\otimes$
    - $r, +2$
      - $p \wedge q, -1$
        - $p, -1$ $\otimes$
        - $q, -1$
      - $r, +1$
  - $q, -3$
    - $p \wedge q, -2$
      - $p, -2$
      - $q, -2$ $\otimes$
    - $r, +2$
      - $p \wedge q, -1$
        - $p, -1$ $\otimes$
        - $q, -1$
      - $r, +1$
- $r, +3$ $\otimes$

The rule for $A \to B, +i$ is applied to line 1 for world 0 and all the star-worlds as well, but since negation does not appear in the non-star worlds, no contradictions will arise. I have ommitted this step for space reasons.

A countermodel can be read off the left-most open branch of the tableau as below:

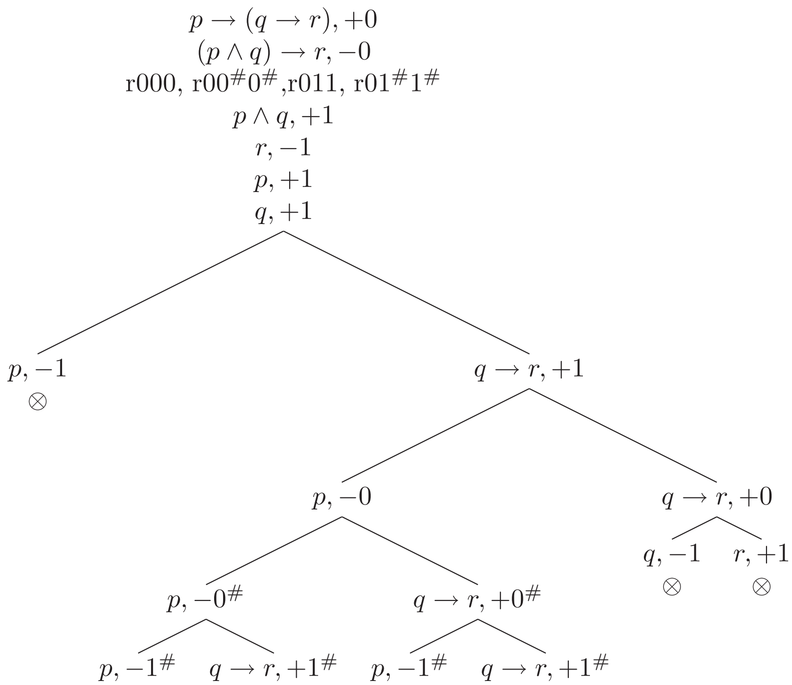
$$
\begin{array}{cccccccc}
 & & w_0 & -p & & & w_0^* & -p \\
 & & w_1 & -q, +p & & & w_1^* & -p \\
 & & \angle & & & & & \\
-p, +q & w_2 & w_3 & -p, -r & -p & w_2^* & w_3^* & -p
\end{array}
$$

The normality relations are assumed, as in the text.

Let us check that this interpretation works:

There is no world where $p \wedge q$, therefore the premise is true. However, at w_1, p is true, so the conclusion states that $q \rightarrow r$ must be true there as well. $Rw_1w_2w_3$. and q is true at w_2 but r is false at w_3, so the conclusion is false.

(b) $p \rightarrow (q \rightarrow r) \vdash (p \wedge q) \rightarrow r$



A counter-model can be read off the left-most open branch of the tableau as below:

w_0	$-p$	w_0^*	$-p$
w_1	$+p, +q, -r$	w_1^*	$-p$

The normality relations are assumed, as in the text.

Let us check that this interpretation works:

w_1, where p is true, is not related to any other world, so $q \rightarrow r$ is trivially true there, and the premise is true at w_0. However, at w_1 where $p \wedge q$ is true, r is not true, so the conclusion is false.

(c) $\vdash ((p \to q) \wedge (q \to r)) \to (p \to r)$

$$((p \to q) \wedge (q \to r)) \to (p \to r), -0$$
$$\text{r000, r00}^{\#}0^{\#}\text{,r011, r01}^{\#}1^{\#}$$
$$(p \to q) \wedge (q \to r), +1$$
$$p \to r, -1$$
$$p \to q, +1$$
$$q \to r, +1$$
$$\text{r123}$$
$$p, +2$$
$$r, -3$$
$$\text{r022, r02}^{\#}2^{\#}\text{, r033, r03}^{\#}3^{\#}$$

$$\begin{array}{ccc} p, -2 & & q, +3 \\ \otimes & q, -2 & \quad r, +3 \\ & & \quad \otimes \end{array}$$

A counter-model can be read off the open branch of the tableau as below:

$$\begin{array}{cccccccc} & & w_0 & & & & w_0^* & \\ & & & & & & & \\ & & w_1 & & & & w_1^* & \\ & & \angle & & & & & \\ +p, -q & w_2 & & w_3 & +q, -r & & w_2^* & w_3^* \end{array}$$

The normality relations are assumed, as in the text.

Let us check that this interpretation works:

p is true at w_2 and q is true at w_3 making $p \to q$ true at w_1. q is false at w_2 making $q \to r$ true at w_1. Thus the antecedent $(p \to q) \wedge (q \to r)$ is true at w_1. However, p is true at w_2, and false at w_3 where $Rw_1w_2w_3$, meaning the consequent $p \to r$ is false at w_1, and the whole inference is false at w_0.

(d) $\vdash (p \to q) \to ((p \wedge r) \to (q \wedge r))$

$$
\begin{array}{c}
(p \to q) \to ((p \wedge r) \to (q \wedge r)), -0 \\
\mathrm{r000},\ \mathrm{r00^{\#}0^{\#}}, \mathrm{r011},\ \mathrm{r01^{\#}1^{\#}} \\
p \to q, +1 \\
(p \wedge r) \to (q \wedge r), -1 \\
\mathrm{r123} \\
p \wedge r, +2 \\
q \wedge r, -3 \\
\mathrm{r022},\ \mathrm{r02^{\#}2^{\#}},\ \mathrm{r033},\ \mathrm{r03^{\#}3^{\#}} \\
p, +2 \\
r, +2 \\
\begin{array}{cc}
\begin{array}{c} q, -3 \\ \begin{array}{cc} p, -2 & q, +3 \\ \otimes & \otimes \end{array} \end{array}
&
\begin{array}{c} r, -3 \\ \begin{array}{cc} p, -2 & q, +3 \\ \otimes & \end{array} \end{array}
\end{array}
\end{array}
$$

A counter-model can be read off the only open branch of the tableau as below:

$$
\begin{array}{cccccc}
 & \multicolumn{2}{c}{w_0} & & \multicolumn{2}{c}{w_0^*} \\
 & \multicolumn{2}{c}{w_1} & & \multicolumn{2}{c}{w_1^*} \\
 & \multicolumn{2}{c}{\angle} & & & \\
+p, +r & w_2 & w_3 & +q, -r & w_2^* & w_3^*
\end{array}
$$

The normality relations are assumed, as in the text.

Let us check that this interpretation works:

The antecedent is true at w_1 because at p is true at w_2 and q is true at w_3. The consequent is false at w_1 because $Rw_1w_2w_3$, and while both p and r are true at w_1, r is false at w_3.

(e) $(p \wedge q) \to r \vdash (p \wedge \neg r) \to \neg q$

$$(p \wedge q) \to r, +0$$
$$(p \wedge \neg r) \to \neg q, -0$$
$$\text{r000, r00}^{\#}0^{\#}\text{,r011, r01}^{\#}1^{\#}$$
$$p \wedge \neg r, +1$$
$$\neg q, -1$$
$$p, +1$$
$$\neg r, +1$$
$$r, -1^{\#}$$
$$q, +1^{\#}$$

- $p \wedge q, -1$
  - $p, -1$
    - $\otimes$
  - $q, -1$
    - $p \wedge q, -1^{\#}$
      - $p, -1^{\#}$
      - $q, -1^{\#}$
        - $\otimes$
    - $r, +1^{\#}$
      - $\otimes$
- $r, +1$
  - $p \wedge q, -1^{\#}$
    - $p, -1^{\#}$
    - $q, -1^{\#}$
      - $\otimes$
  - $r, +1^{\#}$
    - $\otimes$

The rule for $A \to B, +i$ is applied to line 1 for w_0 and the star world of w_0 as well, but since all other rules have been applied, and no parameters appear in those worlds, no contradictions will arise. I have omitted these steps for space reasons.

A counter-model can be read off the left-most open branch of the tableau as below:

w_0	$-p$	w_0^*	$-p$
w_1	$+p, -q$	w_1^*	$-p, +q, -r$

The normality relations are assumed, as in the text.

Let us check that this interpretation works:

There are no worlds where $p \wedge q$ so the premise is true at w_0. However, at w_1, p and $\neg r$ obtain (because r is false at w_1^*), but $\neg q$ does not (because q is true at w_1^*, making the conclusion false at w_0.

3. Show that $(p \wedge (p \rightarrow q)) \rightarrow q$ is not logically valid in B. Show that it is if we require every world, w, of every interpretation to meet the condition $Rwww$.

$\nvdash_B (p \wedge (p \rightarrow q)) \rightarrow q$

$$(p \wedge (p \rightarrow q)) \rightarrow q, -0$$
$$\text{r}000,\ \text{r}00^{\#}0^{\#}, \text{r}011,\ \text{r}01^{\#}1^{\#}$$
$$p \wedge (p \rightarrow q), +1$$
$$q, -1$$
$$p, +1$$
$$p \rightarrow q, +1$$

A counter-model can be read off the open branch of the tableau as below:

w_0		w_0^*
w_1	$+p, -q$	w_1^*

The normality relations are assumed, as in the text.

Let us check that this interpretation works:

At w_1, both p and $p \rightarrow q$ are true (because w_1 is not related to other worlds as the first of a triple), but q is false, therefore the formula is false at w_0.

If we require every world of every interpretation to meet the condition $Rwww$, then $\vdash_B (p \wedge (p \rightarrow q)) \rightarrow q$

This can be shown by a short semantic proof:

Suppose the inference is false. Then there is an interpretation such that $w_0 \in N$, Rw_0aa, $p \wedge (p \rightarrow q)$ is true at a, and q is not true at a. So p is true at a, and $p \rightarrow q$ is true at a. However, by the condition above, $Raaa$, so q is true at a.

4. Give deductions for the following in R:

(a) $\vdash \neg A \to \neg(A \wedge B)$

(1)	$(A \wedge B) \to A$	A3
(2)	$\neg A \to \neg A$	A1
(3)	$(\neg A \to \neg A) \to (A \to \neg\neg A)$	A8
(4)	$A \to \neg\neg A$	(2), (3) and R1
(5)	$((A \wedge B) \to A) \to ((A \to \neg\neg A) \to ((A \wedge B) \to \neg\neg A))$	A9
(6)	$(A \to \neg\neg A) \to ((A \wedge B) \to \neg\neg A)$	(1), (5) and R1
(7)	$(A \wedge B) \to \neg\neg A$	(4), (6) and R1
(8)	$((A \wedge B) \to \neg\neg A) \to (\neg A \to \neg(A \wedge B))$	A8
(9)	$\neg A \to \neg(A \wedge B)$	(7), (8) and R1

(b) $\vdash \neg(A \wedge \neg A)$

(1)	$A \to \neg\neg A$	Shown above 4(a)
(2)	$(A \wedge \neg A) \to A$	A3
(3)	$(A \to \neg\neg A) \to ((A \wedge \neg A) \to \neg\neg A)$	(2) and R4
(4)	$(A \wedge \neg A) \to \neg\neg A$	(1), (3) and R1
(5)	$((A \wedge \neg A) \to \neg\neg A) \to (\neg A \to \neg(A \wedge \neg A))$	A8
(6)	$\neg A \to \neg(A \wedge \neg A)$	(4), (5) and R1
(7)	$(A \wedge \neg A) \to \neg A$	A3
(8)	$((A \wedge \neg A) \to \neg A) \to (A \to \neg(A \wedge \neg A))$	A8
(9)	$A \to \neg(A \wedge \neg A)$	(7), (8) and R1
(10)	$(A \to \neg(A \wedge \neg A)) \wedge (\neg A \to \neg(A \wedge \neg A))$	(6), (9) and R2
(11)	$((A \to \neg(A \wedge \neg A)) \wedge (\neg A \to \neg(A \wedge \neg A))) \to ((A \vee \neg A) \to \neg(A \wedge \neg A))$	A6
(12)	$(A \vee \neg A) \to \neg(A \wedge \neg A)$	(10), (11) and R1
(13)	$A \vee \neg A$	(10.5.4)
(14)	$\neg(A \wedge \neg A)$	(13), (12) and R1

(c) $A \to B, A \to \neg B \vdash \neg A$

(1)	$A \to B$	Assumption
(2)	$A \to \neg B$	Assumption
(3)	$(A \to B) \wedge (A \to \neg B)$	(1), (2) and R2
(4)	$((A \to B) \wedge (A \to \neg B)) \to (A \to (B \wedge \neg B))$	A5
(5)	$A \to (B \wedge \neg B)$	(3), (4) and R1
(6)	$(B \wedge \neg B) \to \neg\neg(B \wedge \neg B)$	Shown above 4(a)
(7)	$(A \to (B \wedge \neg B)) \to (((B \wedge \neg B) \to \neg\neg(B \wedge \neg B)) \to (A \to \neg\neg(B \wedge \neg B)))$	A9
(8)	$((B \wedge \neg B) \to \neg\neg(B \wedge \neg B)) \to (A \to \neg\neg(B \wedge \neg B)$	(5), (7) and R1
(9)	$A \to \neg\neg(B \wedge \neg B)$	(6), (8) and R1
(10)	$(A \to \neg\neg(B \wedge \neg B)) \to (\neg(B \wedge \neg B) \to \neg A)$	A8
(11)	$\neg(B \wedge \neg B) \to \neg A$	(9), (10) and R1
(12)	$\neg(B \wedge \neg B)$	Shown above 4(b)
(13)	$\neg A$	(11), (12) and R1

5. Show that in R, A12 may be replaced by permutation: $(A \to (B \to C)) \to (B \to (A \to C))$. Show that in R, A11 may be replaced by A14. (Hint for the second: take $(A \to \neg A) \to \neg A$, and prefix the antecedent and consequent with $\neg B$. Then use permutation on the antecedent.)

A12: $A \to ((A \to B) \to B)$

Permutation: $(A \to (B \to C)) \to (B \to (A \to C))$

I will show that A12 may be replaced by permutation by showing that the former may be deduced from permutation plus the other R assumptions. The textbook already shows that permutation is a theorem of the axiom system of R with the axiom $A \to ((A \to B) \to B))$

(1)	$(A \to B) \to (A \to B)$	A1
(2)	$((A \to B) \to (A \to B)) \to (A \to ((A \to B) \to B))$	Permutation
(3)	$A \to ((A \to B) \to B)$	(1), (2) and R1

A11: $(A \to (A \to B)) \to (A \to B)$

A14: $(A \to \neg A) \to \neg A$

I will show that A11 may be replaced by A14 by showing that the former may be deduced from the latter plus the other R assumptions.

(1)	$(A \to \neg A) \to \neg A$	A14
(2)	$(\neg B \to (A \to \neg A)) \to (\neg B \to \neg A)$	(1), R3
(3)	$(A \to (\neg B \to \neg A)) \to (\neg B \to (A \to \neg A))$	Permutation
(4)	$((\neg B \to (A \to \neg A)) \to (\neg B \to \neg A)) \to ((A \to (\neg B \to \neg A)) \to (\neg B \to \neg A))$	(3), R4
(5)	$(A \to (\neg B \to \neg A)) \to (\neg B \to \neg A)$	(2), (4), R1
(6)	$(\neg B \to \neg A) \to (A \to \neg\neg B)$	A8
(7)	$\neg\neg B \to B$	Shown above
(8)	$(A \to \neg\neg B) \to (A \to B)$	(7), R3
(9)	$((A \to \neg\neg B) \to (A \to B)) \to ((\neg B \to \neg A) \to (A \to B))$	(6) and R4
(10)	$(\neg B \to \neg A) \to (A \to B)$	(8), (9), R1
(11)	$((\neg B \to \neg A) \to (A \to B)) \to ((A \to (\neg B \to \neg A)) \to (A \to B))$	(5), and R4
(12)	$(A \to (\neg B \to \neg A)) \to (A \to B)$	(10), (11), R1
(13)	$(A \to \neg\neg B) \to (\neg B \to \neg A)$	A8
(14)	$B \to \neg\neg B$	Shown above
(15)	$(B \to \neg\neg B) \to ((A \to B) \to (A \to \neg\neg B))$	A10
(16)	$(A \to B) \to (A \to \neg\neg B)$	(14), (15), R1
(17)	$((A \to \neg\neg B) \to (\neg B \to \neg A)) \to ((A \to B) \to (\neg B \to \neg A))$	(16), R4
(18)	$(A \to B) \to (\neg B \to \neg A)$	(13), R1
(19)	$((A \to B) \to (\neg B \to \neg A)) \to ((A \to (A \to B)) \to (A \to (\neg B \to \neg A)))$	A10
(20)	$(A \to (A \to B)) \to (A \to (\neg B \to \neg A))$	(18), (19), R1
(21)	$((A \to (\neg B \to \neg A)) \to (A \to B)) \to ((A \to (A \to B)) \to (A \to B))$	(20), R4
(22)	$(A \to (A \to B)) \to (A \to B)$	(12), (21), R1

6. Show that $\vdash_{RM} (p \wedge \neg p) \to \neg(q \wedge \neg q)$. This is non-trivial. Start by showing that in $R \vdash (A \vee \neg A) \leftrightarrow \neg(A \wedge \neg A)$, $\vdash (A \wedge \neg A) \leftrightarrow \neg(A \vee \neg A)$, and $\vdash (A \to B) \to (\neg B \to \neg A)$ (contraposition). (Use any appropriate method) Now formalise the following deduction. Let A be $(p \wedge \neg p) \vee (q \wedge \neg q)$. A16 gives $A \to (A \to A)$; so by contraposition and permutation $\neg A \to (A \to \neg A)$. Substituting for A, we have:

$$\neg((p \wedge \neg p) \vee (q \wedge \neg q)) \to (((p \wedge \neg p) \vee (q \wedge \neg q)) \to \neg((p \wedge \neg p) \vee (q \wedge \neg q)))$$

But the antecedent is equivalent to the conjunction of two instances of Excluded Middle. Hence we can detach the consequent. This is equivalent to $((p \wedge \neg p) \vee (q \wedge \neg q)) \to (\neg(p \wedge \neg p) \wedge \neg(q \wedge \neg q))$. $(p \wedge \neg p) \to \neg(q \wedge \neg q)$ follows.

"Start by showing that in $R \vdash (A \vee \neg A) \leftrightarrow \neg(A \wedge \neg A)$, $\vdash (A \wedge \neg A) \leftrightarrow \neg(A \vee \neg A)$, and $\vdash (A \to B) \to (\neg B \to \neg A)$ (contraposition)."
$\vdash_R (A \vee \neg A) \leftrightarrow \neg(A \wedge \neg A)$

To show this, we can show each half of the biconditional using a tree diagram.

$$\begin{array}{c}
(A \vee \neg A) \to \neg(A \wedge \neg A), -0 \\
A \vee \neg A, +1 \\
\neg(A \wedge \neg A), -1 \\
A \wedge \neg A, +1^{\#} \\
A, +1^{\#} \\
\neg A, +1^{\#} \\
A, -1
\end{array}$$

$$\begin{array}{cc}
A, +1 & \neg A, +1 \\
\otimes & A, -1^{\#} \\
 & \otimes
\end{array}$$

$$\begin{array}{c}
\neg(A \wedge \neg A) \to (A \vee \neg A), -0 \\
\neg(A \wedge \neg A), +1 \\
A \vee \neg A, -1 \\
A, -1 \\
\neg A, -1 \\
A, +1^{\#} \\
A \wedge \neg A, -1^{\#}
\end{array}$$

$$\begin{array}{cc}
A, -1^{\#} & \neg A, -1^{\#} \\
\otimes & A, +1 \\
 & \otimes
\end{array}$$

$\vdash_R (A \wedge \neg A) \leftrightarrow \neg(A \vee \neg A)$

We can show this also by showing that each half of the biconditional holds using tree diagrams:

$$
\begin{array}{c}
(A \wedge \neg A) \rightarrow \neg(A \vee \neg A), -0 \\
A \wedge \neg A, +1 \\
\neg(A \vee \neg A), -1 \\
A, +1 \\
\neg A, +1 \\
A, -1^{\#} \\
A \vee \neg A, +1^{\#} \\
\begin{array}{cc}
A, +1^{\#} & \neg A, +1^{\#} \\
\otimes & A, -1 \\
 & \otimes
\end{array}
\end{array}
$$

$$
\begin{array}{c}
\neg(A \vee \neg A) \rightarrow (A \wedge \neg A), -0 \\
\neg(A \vee \neg A), +1 \\
A \wedge \neg A, -1 \\
A \vee \neg A, -1^{\#} \\
A, -1^{\#} \\
\neg A, -1^{\#} \\
A, +1 \\
\begin{array}{cc}
A, -1 & \neg A, -1 \\
\otimes & A, +1^{\#} \\
 & \otimes
\end{array}
\end{array}
$$

$\vdash_R (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$

This can be shown by a short deduction:

(1)	$(A \rightarrow \neg\neg B) \rightarrow (\neg B \rightarrow \neg A)$	A8
(2)	$B \rightarrow \neg\neg B$	p.193
(3)	$(A \rightarrow B) \rightarrow ((B \rightarrow \neg\neg B) \rightarrow (A \rightarrow \neg\neg B))$	A9
(4)	$((A \rightarrow B) \rightarrow ((B \rightarrow \neg\neg B) \rightarrow (A \rightarrow \neg\neg B))) \rightarrow$	
(4)	$(((B \rightarrow \neg\neg B)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow \neg\neg B)))$	(3), Permutation
(5)	$((B \rightarrow \neg\neg B)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow \neg\neg B))$	3,4,R1
(6)	$(A \rightarrow B) \rightarrow (A \rightarrow \neg\neg B)$	(2), (5) and R1
(7)	$((A \rightarrow \neg\neg B) \rightarrow (\neg B \rightarrow \neg A)) \rightarrow ((A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A))$	6, R4
(8)	$(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$	(1), (7) and R1

"Now formalize the following deduction. Let A be $(p \wedge \neg p) \vee (q \wedge \neg q)$. A16 gives $A \rightarrow (A \rightarrow A)$; so by contraposition and permutation $\neg A \rightarrow (A \rightarrow \neg A)$. Substituting for A, we have:

$$\neg((p \wedge \neg p) \vee (q \wedge \neg q)) \rightarrow (((p \wedge \neg p) \vee (q \wedge \neg q)) \rightarrow \neg((p \wedge \neg p) \vee (q \wedge \neg q)))$$

But the antecedent is equivalent to the conjunction of two instances of Excluded Middle. Hence we can detach the consequent. This is equivalent to $((p \wedge \neg p) \vee (q \wedge \neg q)) \rightarrow (\neg(p \wedge \neg p) \wedge \neg(q \wedge \neg q))$. $(p \wedge \neg p) \rightarrow \neg(q \wedge \neg q)$ follows."

$\vdash_R \neg A \rightarrow (A \rightarrow \neg A)$

(1)	$A \to (A \to A)$	A16
(2)	$(A \to A) \to (\neg A \to \neg A)$	Contraposition
(3)	$(A \to (A \to A)) \to (((A \to A) \to (\neg A \to \neg A)) \to (A \to (\neg A \to \neg A)))$	A9
(4)	$((A \to A) \to (\neg A \to \neg A)) \to (A \to (\neg A \to \neg A))$	(1), (3) and R1
(5)	$A \to (\neg A \to \neg A)$	(2), (4) and R1
(6)	$(A \to (\neg A \to \neg A)) \to (\neg A \to (A \to \neg A))$	Permutation.
(7)	$\neg A \to (A \to \neg A)$	(5), (6) and R1

Now we have established that the above is valid, we can substitute $(p \wedge \neg p) \vee (q \wedge \neg q)$ for A:

$$\neg((p \wedge \neg p) \vee (q \wedge \neg q)) \to (((p \wedge \neg p) \vee (q \wedge \neg q)) \to \neg((p \wedge \neg p) \vee (q \wedge \neg q)))$$

By De Morgan's law, the antecedent is equivalent to $\neg(p \wedge \neg p) \wedge \neg(q \wedge \neg q)$. Both conjuncts of this formula are logical truths, hence the conjunction is also. The antecedent is a logical truth, so by detachment,

$$\vdash \neg((p \wedge \neg p) \vee (q \wedge \neg q)) \to (((p \wedge \neg p) \vee (q \wedge \neg q)) \to \neg((p \wedge \neg p) \vee (q \wedge \neg q)))$$

iff

$$\vdash ((p \wedge \neg p) \vee (q \wedge \neg q)) \to \neg((p \wedge \neg p) \vee (q \wedge \neg q))$$

The whole inference now reads:

$$\vdash ((p \wedge \neg p) \vee (q \wedge \neg q)) \to (\neg(p \wedge \neg p) \wedge \neg(q \wedge \neg q))$$

If either A or B entail C, then A entails C, and B entails C (and possibly both entail C), so we can detach one half of the disjunct in the antecedent:

$$\vdash (p \wedge \neg p) \to \neg(p \wedge \neg p) \wedge \neg(q \wedge \neg q)$$

And if A entails B and C, then A entails B — so we can take away half of the conjunct in the consequent:

$$\vdash (p \wedge \neg p) \to \neg(q \wedge \neg q)$$

Leaving us with the conclusion.

7. Show that if all the worlds of an interpretation are normal, the constraints C8-C11 hold. Infer that any logic obtained by adding to B any of A8-A11 is a sub-logic of K_*. Show that the same is not true of A12. Is it true of A13?

If all the worlds of an interpretation are normal, then for any worlds w_i, w_j, $Rw_iw_jw_j$.

(C8) If $Rabc$ then Rac^*b^*

Suppose that $Rabc$. Since a is normal, $b = c$. So $b^* = c^*$, and Rac^*b^*.

(C9) If there is an $x \in W$ such that $Rabx$ and $Rxcd$, then there is a $y \in W$ such that $Rbcy$ and $Rayd$.

Since all worlds in the interpretation are normal, x is normal and a is normal. So $Rabx$ means that $x = b$, and $Rxcd$ means that $c = d$. $Radd$ because a is normal; $Racd$ because $c = d$. $Rbcc$ because b is normal. Thus c is a y such that $Rbcy$ and $Rayd$.

(C10) If there is an $x \in W$ such that $Rabx$ and $Rxcd$ then there is a $y \in W$ such that $Rbcy$ and $Rayd$.

Assume that there is an $x \in W$ such that $Rabx$ and $Rxcd$. $b = x$ because a is normal. $c = d$ because x is normal. $Rbcc$ because b is normal, and $Radd$ because a is normal. Then, because $c = d$, $Racd$. Thus c is a $y \in W$ such that $Rbcy$ and $Rayd$.

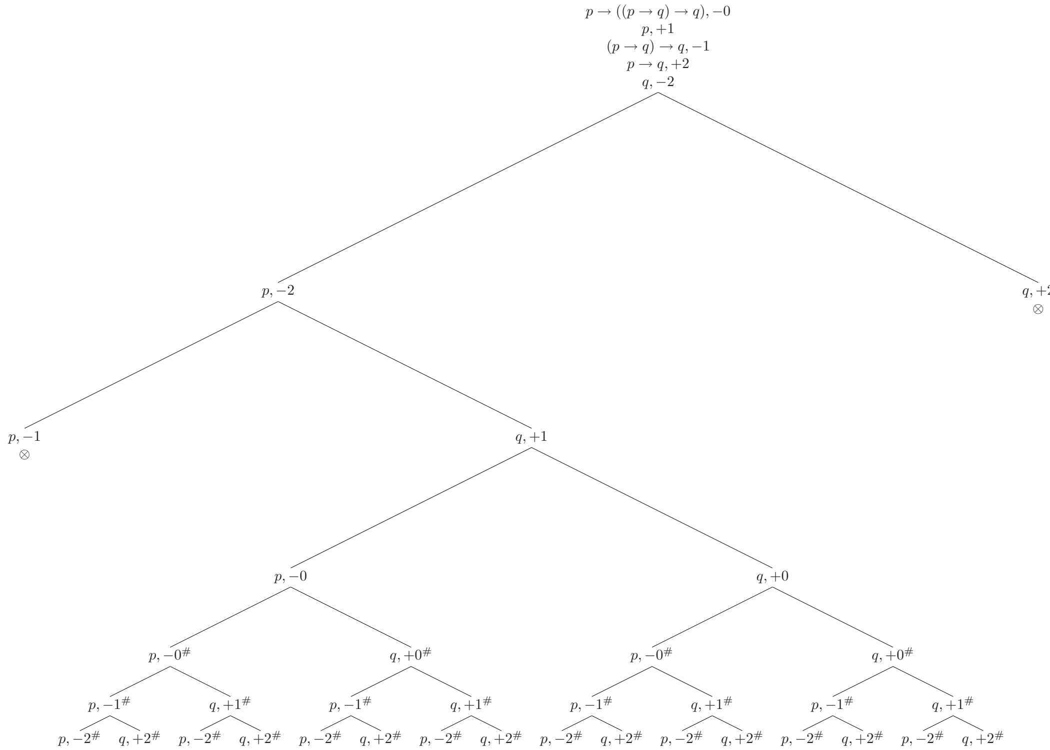
(C11) If $Rabc$ then for some $x \in W$, $Rabx$ and $Rxbc$.

If $Rabc$ then, because a is normal, $b = c$. Because a is normal, $Rabb$; because $b = c$, $Rabc$. Because c is normal, $Rccc$. Because $b = c$, $Rcbc$. Thus c is an $x \in W$ such that $Rabx$ and $Rxbc$.

In K_*, all worlds of all interpretations are normal. Thus $C8 - C11$ hold. By 10.4.6, this means that $A8 - A11$ hold in K_*. B is a sub-logic of K_*, so B with the addition of any of $A8 - A11$ is still a sub-logic of K_*.

$A12$ does not hold in K_*:

$\nvdash_{K_*} p \to ((p \to q) \to q)$



Countermodel taken from the open left-most branch as below:

$$\begin{array}{cccccc} w_0 & w_1 & w_2 & w_0^* & w_1^* & w_2^* \\ -p & +p & -p & -p & -p & -p \\ & +q & -q & & & \end{array}$$

Let us check that this interpretation works:

$p \to q$ is true at all worlds, but q is false at w_2, so $(p \to q) \to q$ is false at w_1. p is true at w_1, so the whole sentence is false at w_0.

$A13$ also does not hold in K_*:

$\nvdash_{K_*} p \vee \neg p$

$$\begin{array}{c} p \vee \neg p, -0 \\ p, -0 \\ \neg p, -0 \\ p, +0^{\#} \end{array}$$

Counter-model taken from the open left-most branch as below :

$$\begin{array}{cc} w_0 & w_0^* \\ -p & +p \end{array}$$

Let us check that this interpretation works:

p is not true at w_0, but since p is true at $w_0^{\#}$, $\neg p$ is not true at w_0 either.

8. (Another exercise for masochists.) Show that all the axioms of R are valid in the following many-valued logic, and that all the rules of R preserve validity; hence, that R is a sub-logic of the logic. The values of the logic are the integers, together with a new object ∞. All but 0 are designated. The logical operators are defined as follows:

$\neg 0 = \infty$; $\neg\infty = 0$; $\neg a = -a$ otherwise
$0 \wedge a = a \wedge 0 = 0; \infty \wedge a = a \wedge \infty = a$
$0 \vee a = a \vee 0 = a; \infty \vee a = a \vee \infty = \infty$
$0 \to a = a \to \infty = \infty$; if $a \neq 0, a \to 0 = 0$, if $a \neq \infty, \infty \to a = 0$

if a and b are positive integers then:

if a divides b, $a \to b = b/a$;otherwise, $a \to b = 0$
$a \wedge b$ is the greatest common divisor of a and b
$a \vee b$ is the least common multiple of a and b

if a and b are negative integers then:

$a \wedge b = -(-a \vee -b)$
$a \vee b = (-a \wedge -b)$
$a \to b = -b \to -a$

if a is a negative integer and b is a positive integer, then:

$a \to b = 0; b \to a = b.a$
$a \wedge b = b \wedge a = b$
$a \vee b = b \vee a = a$

I will go through the axioms of R (A1 - A12, R1 and R2) to show that they are valid in this many valued logic.

It will be helpful to note first some facts about the Greatest Common Divisor (GCD) and the Lowest Common Multiple (LCM):

The GCD of any two numbers is calculated by expressing both as multiples of their prime factors, and multiplying together the common prime factors.

The LCM of any two numbers is calculated by expressing both as multiples of their prime factors, and multiplying one by the non-common prime factors of the other.

Further, the GCD and LCM are closely related:

(1) $GCD(A, B) \cdot LCM(A, B) = A \cdot B$

Thus, by distributivity,

(2) $GCD(A, LCM(B, C)) = LCM(GCD(A, B), GCD(A, C))$

and

(3) $LCM(A, GCD(B, C)) = GCD(LCM(A, B), LCM(A, C))$

Finally, If m is a nonzero common divisor of A and B, then

$$GCD(\frac{A}{m}, \frac{B}{m}) = \frac{GCD(A, B)}{m}$$

These facts will be referred to in the proofs below.

(A1) $A \to A$

Suppose this were undesignated. Then it takes the value 0. For $\to$, this happens in one of four cases: (i) $A \neq 0$ and $B = 0$ (ii) $B \neq \infty$ and $A = \infty$ (iii) A and B are both positive, and A does not divide B (iv) A is negative and B is positive. All of these possibilities imply that the value of the consequent is different to the value of the antecedent, but since they are both A, this cannot be.

(A2) $A \to (A \vee B)$

Suppose this were undesignated. Then it takes the value 0. For $\to$, this happens in one of four cases: (i) $A \neq 0$ and $B = 0$ (ii) $B \neq \infty$ and $A = \infty$ (iii) A and B are both positive, and A does not divide B (iv) A is negative and B is positive.

In the case where $A \neq 0$ and $A \vee B = 0$, the consequent is $0 \vee 0$, so $A = 0$.

In the case where $A = \infty$, $A \vee B \neq \infty$, the consequent is $\infty \vee B$, so its value is ∞.

In the case where A and $A \vee B$ are positive and A does not divide $A \vee B$, $A \vee B$ is the LCM of A and B, so A must divide $A \vee B$.

In the case where A is negative, and $A \vee B$ is positive, either B is positive or negative. If B is positive, then $(A \vee B) = A$, so $A \vee B$ is negative. If B is negative, then $(A \vee B) = -(-A \wedge -B)$. So, it is the negation of the LCM of positive A and B, hence it is negative.

(A3) $(A \wedge B) \to A$

Suppose this were undesignated. Then it takes the value 0. For $\to$, this happens in one of four cases: (i) $A \neq 0$ and $B = 0$ (ii) $B \neq \infty$ and $A = \infty$ (iii) A and B are both positive, and A does not divide B (iv) A is negative and B is positive.

In the case where $A \wedge B \neq 0$ and $A = 0$, the antecedent is $0 \wedge B$, so its value is 0

In the case where $A \wedge B = \infty$, $A \neq \infty$, the antecedent is $\infty \wedge \infty$, so $A = \infty$.

In the case where $A \wedge B$ and A are positive and $A \wedge B$ does not divide A, $A \wedge B$ is the GCD of A and B, so $A \wedge B$ must divide A.

In the case where $A \wedge B$ is negative, and A is positive, B is either positive or negative. If B is positive, then $A \wedge B$ is the LCM of positive A and B, so $A \wedge B$ is positive. If B is negative, then $(A \wedge B) = A$, hence $A \wedge B$ is positive.

(A4) $A \wedge (B \vee C) \to ((A \wedge B) \vee (A \wedge C))$

Suppose this were undesignated. Then it takes the value 0. For $\to$, this happens in one of four cases: (i) $A \neq 0$ and $B = 0$ (ii) $B \neq \infty$ and $A = \infty$ (iii) A and B are both positive, and A does not divide B (iv) A is negative and B is positive.

In the case where $(A \wedge (B \vee C)) \neq 0$ and $((A \wedge B) \vee (A \wedge C)) = 0$, $(A \wedge B) = 0$ and $(A \wedge C)) = 0$, so A or (B and C) $= 0$. If $A = 0$ then the antecedent is 0. If (B and C) $= 0$, then $(B \vee C) = 0$, so again the antecedent is 0.

In the case where $(A \wedge (B \vee C)) = \infty$, $((A \wedge B) \vee (A \wedge C)) \neq \infty$, both A and $B \vee C$ are ∞. So either B or C is ∞. If B is ∞, then $A \wedge B$ is ∞ so the consequent is ∞. If C is ∞ then $A \wedge C$ is ∞ so the consequent is ∞.

In the case where the antecedent and consequent are positive, and the antecedent does not divide the consequent, the antecedent is the GCD of A and (the LCM of B and C). While the consequent is the LCM of (the GCD of A and B) and (the GCD of A and C).

Clearly then, A4 is an instance of (2) above: $GCD(A, LCM(B, C)) = LCM(GCD(A, B), GCD(A, C))$. The antecedent and consequent are equal to one another, therefore the antecedent divides the consequent.

In the case where $A \wedge (B \vee C)$ is negative, and $(A \wedge B) \vee (A \wedge C)$ is positive, the consequent is GCD($A \wedge B$, $A \wedge C$), and from the antecedent, A and $B \vee C$ are negative. Hence one or both of B and C are negative. But then one or both of $A \wedge B$ and $A \wedge C$ are negative, so the consequent is not a GCD.

(A5) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

Suppose this were undesignated. Then it takes the value 0. For $\to$, this happens in one of four cases: (i) $A \neq 0$ and $B = 0$ (ii) $B \neq \infty$ and $A = \infty$ (iii) A and B are both positive, and A does not divide B (iv) A is negative and B is positive.

In the case where $((A \to B) \wedge (A \to C)) \neq 0$ and $(A \to (B \wedge C)) = 0$, the consequent is a conditional, so there are four further possibilities:

$A \neq 0, (B \wedge C) = 0$, $(B \wedge C) \neq \infty, A = \infty$, A and $B \wedge C$ are positive, and A does not divide $B \wedge C$, or A is negative, and $B \wedge C$ is positive.

If $A \neq 0, (B \wedge C) = 0$, then B or C is equal to 0. Since $A \neq 0$, $(A \rightarrow B) = 0$ or $(A \rightarrow C) = 0$. In either case, one of the conjuncts in the antecedent is equal to 0, therefore the antecedent is equal to 0.

If $(B \wedge C) \neq \infty, A = \infty$, then $B \neq \infty$ and $C \neq \infty$. But then $(A \rightarrow B) = 0$, and so does the antecedent.

If $B \wedge C$ and A are positive, and A does not divide $B \wedge C$, then either one or both of B and C are positive. If both are positive, $(B \wedge C) = GCD(B, C)$. So, A does not divide GCD(B,C). So A does not divide either B or C. But then $(A \rightarrow B) = 0$ or $(A \rightarrow C) = 0$, and so does the antecedent. If only one of B and C is positive, for instance, B, $(B \wedge C) = B$. So A does not divide B. But then $(A \rightarrow B) = 0$. The case is similar for C.

If A is negative and $B \wedge C$ is positive, then one or more of B and C is positive. If either are positive, then one of the conditionals in the antecedent is equal to 0, and so is the antecedent.

In the case where $(A \rightarrow (B \wedge C)) = \infty$, $((A \rightarrow B) \wedge (A \rightarrow C)) \neq \infty$, either $A = 0$ or $(B \wedge C) = \infty$. In the first case, $(A \rightarrow B) = \infty$ and $(A \rightarrow C) = \infty$, so $((A \rightarrow B) \wedge (A \rightarrow C)) = \infty$. In the second case B and C are equal to ∞, therefore again both conjuncts of the antecedent, and the antecedent itself, are equal to ∞.

In the case where antecedent and consequent are positive, and $((A \rightarrow B) \wedge (A \rightarrow C))$ does not divide $(A \rightarrow (B \wedge C))$, there are three propositional parameters, and hence eight ways in which the parameters could be assigned positive or negative values:

A	B	C
$+$	$+$	$+$
$+$	$+$	$-$
$+$	$-$	$+$
$+$	$-$	$-$
$-$	$+$	$+$
$-$	$+$	$-$
$-$	$-$	$+$
$-$	$-$	$-$

$\boxed{(+,+,+)}$

The antecedent is $GCD((A \rightarrow B), (A \rightarrow C))$. Since all parameters are positive, and the antecedent is as well, A divides B, and A divides C. So the antecedent is $GCD(\frac{B}{A}, \frac{C}{A})$. Since the consequent is positive, the A divides the GCD of B and C, and the consequent becomes $\frac{GCD(B,C)}{A}$. A is a non-zero common divisor of B and C, so $GCD(\frac{B}{A}, \frac{C}{A}) = \frac{GCD(B,C)}{A}$. So the antecedent divides the consequent.

$\boxed{(+,+,-)}$

C is negative, so $A \to C$ is $A{\cdot}C$, in other words, negative, and the antecedent is equal to $A \to B$. But $B \wedge C$ is equal to B, so the consequent is also $A \to B$, hence the antecedent divides the consequent.

$\boxed{(+,-,+)}$

B is negative, so $A \to B$ is $A \cdot B$, in other words, negative, and the antecedent is equal to $A \to C$. But $B \wedge C$ is equal to C, so the consequent is also $A \to C$, hence the antecedent divides the consequent.

$\boxed{(+,-,-)}$

B and C are negative, so both conjuncts in the antecedent, and the antecedent itself, are negative.

In the remaining cases, A is negative, so $(A \to B) = 0$ and the antecedent is equal to 0.

If the antecedent is negative, and consequent positive, then either one or both conjuncts in the antecedent are negative. If $A \to B$ is negative, then one of A or B is negative. If A is negative, then the consequent is equal to 0. If B is negative, then the consequent is equal to $A \to 0$, which again is equal to 0. If both are negative, the consequent is equal to ∞. If $A \to C$ is negative, then one of A or C is negative. We have already shown that A cannot be negative. If C is negative, then, as for B, the consequent is equal to 0. And again if both are negative, the consequent is equal to ∞. If both conjuncts are negative, then the above reasoning shows the consequent not to be positive.

(A6) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

Suppose this were undesignated. Then it takes the value 0. For $\to$, this happens in one of four cases: (i) $A \neq 0$ and $B = 0$ (ii) $B \neq \infty$ and $A = \infty$ (iii) A and B are both positive, and A does not divide B (iv) A is negative and B is positive.

If $((A \to C) \wedge (B \to C)) \neq 0$ and $((A \vee B) \to C) = 0$, the consequent is a conditional, so there are four further cases:

$((A \vee B) \neq 0$, $C = 0)$, $(C \neq \infty$, $(A \vee B) = \infty)$, $A \vee B$ and C positive; $A \vee B$

does not divide C, $A \vee B$ negative; C positive.

If$(A \vee B) \neq 0$, and $C = 0$, then $(A \to C) = 0$, so the antecedent is equal to 0.

If $C \neq \infty$, and $(A \vee B) = \infty$, then A or B is equal to ∞. If it is A, then $(A \to C) = 0$, and so does the antecedent. If it is B, then $(B \to C) = 0$, and so does the antecedent.

If $A \vee B$ and C are positive, and $A \vee B$ does not divide C, then A or B is positive. If A is positive, then $(A \vee B) = A$, so A does not divide C. But then $(A \to C) = 0$, and so does the antecedent.

If $A \vee B$ is negative and C is positive then, A or B is negative. If A is negative, then $(A \to C) = 0$, and so does the antecedent. If B is negative, then $(A \to B) = 0$, and so does the antecedent.

If $((A \vee B) \to C) \neq \infty$ and $((A \to C) \wedge (B \to C)) = \infty$, then both $A \to C$ and $B \to C$ equal ∞. Hence either C is equal to ∞, or A and B are equal to 0. In the first case, the consequent becomes $(A \vee B) \to \infty$, hence it is equal to ∞. In the second case the consequent becomes $0 \to C$, hence it is equal to ∞.

If antecedent and consequent are positive, and A does not divide B, there are three propositional parameters, and hence eight ways in which the parameters could be assigned positive or negative values:

A	B	C
$+$	$+$	$+$
$+$	$+$	$-$
$+$	$-$	$+$
$+$	$-$	$-$
$-$	$+$	$+$
$-$	$+$	$-$
$-$	$-$	$+$
$-$	$-$	$-$

$\boxed{(+,+,+)}$

Since the antecedent is positive, both conjuncts are positive, so A divides C and B divides C, and since the consequent is positive the LCM of A and B divides C. Which means the antecedent becomes $GCD(\frac{C}{A}, \frac{C}{B})$, while the consequent is $\frac{C}{LCM(A,B)}$. Since $LCM(A, B) = \frac{A \cdot B}{GCD(A,B)}$, $\frac{C}{LCM(A,B)} = \frac{C}{A \cdot B} \cdot GCD(A, B)$. Hence, the consequent equals $GCD(\frac{A \cdot C}{A \cdot B}, \frac{B \cdot C}{A \cdot B})$, which equals the antecedent. Hence the antecedent divides the consequent.

$\boxed{(+,+,-),(+,-,-),(-,+,-),(-,-,-)}$

In all these cases, C is negative, so $A \to C$ and $B \to C$ are negative, meaning the conjunct is negative.

$\boxed{(+,-,+),(-,-,+)}$

In these cases, B is negative, so $(B \to C) = 0$, meaning the antecedent equals 0.

$\boxed{(-,+,+)}$

In this case, A is negative so, as for B, $(A \to C) = 0$, meaning the antecedent equals 0.

The remaining case where all three are negative is clearly contradictory from the above.

If the antecedent is negative, and the consequent is positive, then both conjuncts in the antecedent are negative. So, A is negative, and B and C are positive. But then, $(A \vee B) = A$, so the consequent is $(A \to C) = 0$.

(A7) $\neg\neg A \to A$

Suppose this were undesignated. Then it takes the value 0. For $\to$, this happens in one of four cases: (i) $A \neq 0$ and $B = 0$ (ii) $B \neq \infty$ and $A = \infty$ (iii) A and B are both positive, and A does not divide B (iv) A is negative and B is positive. All the above cases imply that the antecedent and consequent take a different value, but this cannot be.

(A8) $(A \to \neg B) \to (B \to \neg A)$

Suppose this were undesignated. Then it takes the value 0. For $\to$, this happens in one of four cases: (i) $A \neq 0$ and $B = 0$ (ii) $B \neq \infty$ and $A = \infty$ (iii) A and B are both positive, and A does not divide B (iv) A is negative and B is positive.

If $(A \to \neg B) \neq 0$ and $(B \to \neg A) = 0$, since the consequent is a conditional, there are four more possibilities:

$B \neq 0$ and $\neg A = 0$, $\neg A \neq \infty$ and $B = \infty$, B and $\neg A$ are positive, and B does not divide $\neg A$, B is negative and $\neg A$ is positive.

If $B \neq 0$ and $\neg A = 0$, $A = \infty$. $\neg B \neq \infty$, so the antecedent is equal to 0.

If $\neg A \neq \infty$ and $B = \infty$, $\neg B = 0$ and $A \neq 0$, so the antecedent is equal to 0.

If B and $\neg A$ are positive, and B does not divide $\neg A$, A and $\neg B$ are negative, so $(A \to \neg B) = (A \to -B) = (- - B \to -A) = (B \to -A) = (B \to \neg A)$. The antecedent and consequent are equal, so the antecedent must divide the consequent.

If B is negative and $\neg A$ is positive, then $\neg B$ is positive, and A is negative, so the antecedent is equal to 0.

If $(B \to \neg A) \neq \infty$ and $(A \to \neg B) = \infty$, then $B = 0$ or $\neg A = \infty$. If $B = 0$, then $\neg B = \infty$, so the antecedent is equal to ∞. If $\neg A = \infty$, then $A = 0$, so the antecedent is equal to ∞.

If antecedent and consequent are positive, and $A \to \neg B$ does not divide $B \to \neg A$, A and B cannot be ∞ or 0, so they are either positive or negative. Further, because one parameter is negated in each conditional, only pairs of A and B with different polarities will come out positive. There are two cases:

$\boxed{+A,-B}$

Then $B \to \neg A$ is $- \to -$, so it is equal to $-(\neg A) \to -B$, which, since A and B are integers, is equal to $A \to \neg B$. Antecedent and consequent are equal, hence antecedent divides consequent.

$\boxed{-A,+B}$

Then $A \to \neg B$ is $- \to -$, so it is equal to $-A \to -(\neg)B$, which, since A and B are integers, is equal to $B \to \neg A$. Antecedent and consequent are equal, hence antecedent divides consequent.

If the antecedent is negative, and the consequent is positive, then $A \to \neg B$ is negative, so A is positive, and $\neg B$ is negative. Hence B is positive, and $\neg$ is negative. But then the consequent is $+ \to -$, and hence negative.

(A9) $(A \to B) \to ((B \to C) \to (A \to C))$

Suppose this were undesignated. Then it takes the value 0. For $\to$, this happens in one of four cases: (i) $A \neq 0$ and $B = 0$ (ii) $B \neq \infty$ and $A = \infty$ (iii) A and B are both positive, and A does not divide B (iv) A is negative and B is positive.

If antecedent is not equal to 0, and consequent is, then the conditional $(B \to C) \to (A \to C)$ is equal to 0, so there are four further cases:

If $(B \to C) \neq 0$, $(A \to C) = 0$, then $A \to C$, so there are four possibilities for A and C:

If $A \neq 0$ and $C = 0$, then, either $B \neq 0$ or $B = 0$. In both cases, the antecedent $A \to B$ is equal to 0.

If $C \neq \infty$ and $A = \infty$, then either $A \neq \infty$ or $A = \infty$. In the first case the antecedent is equal to ∞, in the second case, to 0.

If A and C are positive, and C does not divide A then, either B is positive or negative. If it is negative, then the antecedent is $+ \rightarrow -$, and hence equal to 0. If it is positive, $A \rightarrow B$ is $+ \rightarrow +$, and so equals 0 or $\frac{B}{A}$ (clearly it cannot equal 0). $B \rightarrow C$ is $+ \rightarrow +$ and so, since it cannot equal 0 without making the consequent equal to ∞, it is $\frac{C}{B}$. So, the consequent becomes $\frac{C}{B} \rightarrow \frac{C}{A}$. The stipulation above becomes $\frac{C}{B}$ does not divide $\frac{C}{A}$, but clearly it does. Some basic algebra shows us that:

$$\frac{\frac{C}{A}}{\frac{C}{B}} = \frac{C}{A} \cdot \frac{B}{C} = \frac{B}{A}$$

We already have the fact that $\frac{B}{A}$ is an integer.

If A is negative, and C is positive, then B is either positive or negative. If it is positive, the antecedent is $- \rightarrow +$, and hence 0. If it is negative, $B \rightarrow C$ is $- \rightarrow +$, hence equal to 0. But then the consequent is $0 \rightarrow a$, and hence equal to ∞.

If $(A \rightarrow C) \neq \infty$, $(B \rightarrow C) = \infty$ then, either $B = 0$ or $C = \infty$. In the first case, $A \rightarrow B$ becomes $A \rightarrow 0$. Either $A = 0$ or $A \neq 0$. In the first case the antecedent equals ∞. In the second case, the antecedent equals 0. If $C = \infty$, then the antecedent is ∞.

If $(B \rightarrow C)$ and $(A \rightarrow C)$ are positive, and the former does not divide the latter then, there are 8 cases for distribution of positivity and negativity:

A	B	C
$+$	$+$	$+$
$+$	$+$	$-$
$+$	$-$	$+$
$+$	$-$	$-$
$-$	$+$	$+$
$-$	$+$	$-$
$-$	$-$	$+$
$-$	$-$	$-$

We can rule out the cases where only one of B and C, or only one of A and C are negative, since these will make one of the conditionals negative.

Hence we have only one case to consider: the case where all three are negative:

$\boxed{-,-,-}$

So, $B \to C = -C \to -B = \frac{-B}{-C}$. $A \to C$ is $-C \to -A$ which is $\frac{-A}{-C}$, and the antecedent is $A \to B = -B \to -A = \frac{-A}{-B}$.

Rationalising, the consequent is such that $\frac{B}{C}$ does not divide $\frac{A}{C}$. But clearly it does:

$$\frac{\frac{A}{C}}{\frac{B}{C}} = \frac{A}{C} \cdot \frac{C}{B} = \frac{A}{B}$$

The fact that the antecedent is non-zero gives us that $\frac{A}{B}$ is an integer.

If $(B \to C)$ is negative, and $(A \to C)$ is positive then, either both A and C are negative, or both are positive. In the first case, if B is negative, then $B \to C$ is positive. If B is positive, then $A \to B$ is 0. In the second case, if B is positive, $B \to C$ is positive. If B is negative, then the antecedent is 0.

If the antecedent is equal to ∞ and the consequent is not, then either $A = 0$ or $B = \infty$. In the first case, $(A \to C) = \infty$, and so does the consequent. In the second case, $(B \to C) = \infty$, and so does the consequent.

If the antecedent and consequent are positive, and the antecedent does not divide the consequent then, there are 8 possibilities with regards to distribution of polarity among the parameters, as follows:

A	B	C
$+$	$+$	$+$
$+$	$+$	$-$
$+$	$-$	$+$
$+$	$-$	$-$
$-$	$+$	$+$
$-$	$+$	$-$
$-$	$-$	$+$
$-$	$-$	$-$

Since the antecedent is positive, we can disregard the cases where A and B are not the same polarity, cutting out the middle four. So there are four cases to consider:

A	B	C
$+$	$+$	$+$
$+$	$+$	$-$
$-$	$-$	$+$
$-$	$-$	$-$

$\boxed{+,+,+}$

All parameters are positive, and so is the antecedent, $A \to B$ is equal to $\frac{B}{A}$. The consequent is positive, so either both $B \to C$ and $A \to C$ are positive or negative. If they are negative, then one of the parameters in them is negative. If they are positive, then either their antecedents divide their consequents or not. If not, then they are equal to 0. So they are positive and their antecedents divide their consequents, in other words they are $\frac{C}{B}$ and $\frac{C}{A}$ respectively. Since the consequent is positive, this equals $\frac{\frac{C}{A}}{\frac{C}{B}}$, which equals $\frac{C}{A} \cdot \frac{B}{C}$, which equals $\frac{B}{A}$. The antecedent and consequent are equal, therefore the antecedent divides the consequent.

$\boxed{+,+,-}$

A is positive, and C is negative, so $A \to C$ is $A \cdot C$. B is positive and C is negative, so $B \to C$ is $B \cdot C$. $A \cdot C$ and $B \cdot C$ are negative so $A \cdot C \to B \cdot C$ is $-(B \cdot C) \to -(A \cdot C)$. The consequent is positive, so it is equal to $\frac{-(A \cdot C)}{-(B \cdot C)} = \frac{A}{B}$. The antecedent is also positive, and both A and B are positive, so it equals $\frac{B}{A}$. $\frac{B}{A}$ divides $\frac{A}{B}$:

$\frac{\frac{A}{B}}{\frac{B}{A}} = \frac{A}{B} \cdot \frac{A}{B} = \frac{A^2}{B^2}$, and if $\frac{A}{B}$ is a whole number, $\frac{A^2}{B^2}$ is also.

$\boxed{-,-,+}$

B is positive, and C negative, so $B \to C$ is equal to $C \cdot B$. A is negative and C is positive, so $A \to C$ is equal to 0. $C \cdot B$ is not equal to 0, so the consequent is 0.

$\boxed{-,-,-}$ All parameters are negative, so the antecedent is $-B \to -A = \frac{-A}{-B} = \frac{A}{B}$. The consequent is $(-C \to -B) \to (-C \to -A)$. Since the consequent is positive, neither half can be equal to 0, so this becomes $\frac{-B}{-C} \to \frac{-A}{-C} = \frac{B}{C} \to \frac{A}{C}$. Since the consequent is positive, this is equal to $\frac{\frac{A}{C}}{\frac{B}{C}} = \frac{A}{C} \cdot \frac{C}{B} = \frac{A}{B}$. The antecedent and consequent are equal, therefore the antecedent divides the consequent.

If the antecedent is negative, and the consequent is positive, then $A \to B$ is negative. So, either A or B is negative. The consequent is positive, so neither $B \to C$ nor $A \to C$ is negative. By looking at the possible distributions of polarity, we can see that this case is contradictory:

A	B	C
$+$	$+$	$+$
$+$	$+$	$-$
$+$	$-$	$+$
$+$	$-$	$-$
$-$	$+$	$+$
$-$	$+$	$-$
$-$	$-$	$+$
$-$	$-$	$-$

The cases where both A and B are positive or negative (the first two and last two) can be disposed of, as can the cases where either but not both of B, C (the seventh and third) and A, C (the fourth and fifth) are negative . There are no remaining cases.

(A10) $(A \to B) \to ((C \to A) \to (C \to B))$

Suppose this were undesignated. Then it takes the value 0. For $\to$, this happens in one of four cases: (i) $A \neq 0$ and $B = 0$ (ii) $B \neq \infty$ and $A = \infty$ (iii) A and B are both positive, and A does not divide B (iv) A is negative and B is positive.

If $(A \to B) \neq 0, ((C \to A) \to (C \to B)) = 0$ then there are four further cases:

$\boxed{(C \to A) \neq 0,\ (C \to B) = 0}$

Since $C \to B$ is equal to 0, there are four further cases.

If C is not equal to 0 and B is equal to 0, then A is either equal to 0 or not. If it is equal to 0 then, $C \to A$ is equal to 0, and the consequent is equal to ∞. If it is not equal to 0 then the antecedent is equal to 0.

If $B \neq \infty$ and $C = \infty$, then either $A = \infty$ or not. If $A = \infty$, then $(A \to B) = 0$. If $A \neq \infty$ then $(C \to A) = 0$.

If C and B are positive, and C does not divide B, then either A is positive or it is not. If it is positive, since $C \to A \neq 0$, $C \to A = \frac{A}{C}$. And since $A \to B$ is not equal to 0, $(A \to B) = \frac{B}{A}$. Since $\frac{A}{C}$ and $\frac{B}{A}$ are whole numbers $\frac{A}{C} \cdot \frac{B}{A}$ is a whole number. But this is clearly $\frac{B}{C}$, so C does divide B. If A is not positive, then it is $0, \infty$ or negative. If it is 0, then $C \to A = 0$. If it is ∞ then $A \to B = 0$. If it is negative, then $C \to A$ is negative.

If C is negative, and B positive then, A is either 0, ∞, negative or positive. In the first case, $C \to A = 0$, in the second $A \to B = 0$. In the third case, $A \to B$ is 0. In the fourth case, $C \to A$ is 0.

$\boxed{(C \to B) \neq \infty,\ (C \to A) = \infty}$

Since $(C \to A) = \infty$, either $C = 0$ or $A = \infty$. In the first case, $C \to B$ is equal to ∞. In the second case, $A \to B$ is equal to 0, (unless B is equal to ∞, but, if that were the case $C \to B$ is equal to ∞.)

$\boxed{C \to B \text{ and } C \to A \text{ are positive and the former does not divide the latter}}$

Since both are positive, if C is positive, A and B are also. And if C is negative, so are A and B. If C is positive, $(C \to B) = \frac{B}{C}$ and $(C \to A) = \frac{A}{C}$. $\frac{B}{C} \cdot \frac{C}{A} = \frac{B}{A}$. Thus $\frac{B}{A}$ is not a whole number; A does not divide B. But then $A \to B$ is equal to 0. If C is negative, $(-B \to -C) = \frac{-C}{-B} = \frac{C}{B}$ and $(-A \to -C) = \frac{-C}{-A} = \frac{C}{A}$. $\frac{C}{B} \cdot \frac{A}{C} = \frac{A}{B}$. Thus $\frac{A}{B}$ is not a whole number; B does not divide A; $-B$ does not divide $-A$. But then $-B \to -A$ is equal to 0.

$\boxed{C \to B \text{ is positive and } C \to A \text{ is negative}}$

Since $C \to A$ is negative the polarity of C and A must be different. So either C is positive and A negative, or vice versa. In the first case, B is either positive or negative (the cases where B is anything else were dealt with above). If B is positive, $A \to B$ is equal to 0. If it is negative, $C \to B$ is equal to $C \cdot B$, and is hence negative. In the case where A is positive, and C negative, again B is either positive or negative. If it is positive, then $C \to B$ is negative. If it is negative, then $C \to A$ is positive.

If $((C \to A) \to (C \to B)) \neq \infty, (A \to B) = \infty$ then, $A = 0$, or $B = \infty$. In the first case, either $C = 0$ or not. If $C = 0$, $(C \to B) = \infty$, and so does the antecedent. If $C \neq 0$ then $(C \to A) = 0$, and the antecedent is equal to ∞. In the second case, $(C \to B) = \infty$ and so does the antecedent.

If $A \to B$ and $(C \to A) \to (C \to B)$ are positive and $A \to B$ does not divide $(C \to A) \to (C \to B)$ then, since antecedent and consequent are conditionals, both both halves must be either positive or negative. (If either or both are 0 or ∞ the whole will not be positive, and if one is positive and one negative, the whole will be negative or 0.)

So we have four possibilities:

A, B	$(C \to A), (C \to B)$
$+$	$+$
$+$	$-$
$-$	$+$
$-$	$-$

$\boxed{+,+}$

A and B are positive, as is $C \to A$, thus C is also positive. So, the antecedent is $\frac{B}{A}$, and the consequent is $\frac{\frac{B}{C}}{\frac{A}{C}} = \frac{B}{C} \cdot \frac{C}{A} = \frac{B}{A}$. Since antecedent and consequent are equal, antecedent divides consequent.

$\boxed{+, -}$

A and B are positive, but $C \to A$ is negative. Looking at the definition of negative $\to$ we can see that this situation is impossible.

$\boxed{-, +}$

A and B are negative, but $C \to A$ and $C \to B$ are positive. Thus C is negative, and $(A \to B) = \frac{A}{B}$, $C \to A = \frac{C}{A}$, $C \to B = \frac{C}{B}$. Thus the consequent is $\frac{C}{B} \cdot \frac{A}{C} = \frac{A}{B}$, and the antecedent is $\frac{A}{B}$. Since antecedent and consequent are equal, antecedent divides consequent.

$\boxed{-, -}$

A and B are negative, and so are $C \to A$ and $C \to B$. Thus C is positive, and $C \to A$ and $C \to B$ are equal to $C \cdot A$ and $C \cdot B$ respectively. The antecedent is equal to $\frac{A}{B}$. And the consequent is equal to $\frac{C \cdot A}{C \cdot B} = \frac{A}{B}$. Antecedent and consequent are equal, so antecedent divides consequent.

If $A \to B$ is negative and $(C \to A) \to (C \to B)$ is positive then, either A or B is negative (but not both), and either both $C \to A$ and $C \to B$ are positive, or they are both negative. So we have four possibilities:

We can see that in all four cases, because the polarities of A and B are different, and because both sides of the conditional involve C, the sides of the conditional will have different polarities. But this is not the case.

(A11) $(A \to (A \to B)) \to (A \to B)$

Suppose this were undesignated. Then it takes the value 0. For $\to$, this happens in one of four cases: (i) $A \neq 0$ and $B = 0$ (ii) $B \neq \infty$ and $A = \infty$ (iii) A and B are both positive, and A does not divide B (iv) A is negative and B is positive.

If $(A \to (A \to B)) \neq 0, (A \to B) = 0$ then, there are four further possibilities:

If $A \neq 0, B = 0$, then the antecedent is equal to 0.

If $A = \infty, B \neq \infty$, then the antecedent is equal to 0.

If A, B positive, and A does not divide B, then the antecedent is equal to

$\frac{\frac{B}{A}}{A} = \frac{A^2}{B}$. Since $\frac{B}{A}$ is not a whole number, neither is $\frac{A^2}{B}$, thus the antecedent is equal to 0.

If A is negative, and B is positive, then $A \to B$ is equal to 0, and A is not equal to 0, so the antecedent is equal to 0.

If $(A \to B) \neq \infty, (A \to (A \to B)) = \infty$ then, either A is equal to 0, or $A \to B$ is equal to ∞. In either case, the antecedent is equal to ∞.

If $A \to (A \to B)$ and $A \to B$ are positive, and the former does not divide the latter, then either A and B are positive, in which case the antecedent is equal to $\frac{\frac{B}{A}}{A} = \frac{B}{A^2}$ wheras the consequent is equal to $\frac{B}{A}$, in which case the result when the consequent is divided by the antecedent is $\frac{B}{A} \cdot \frac{A^2}{B} = A$. But A is a whole number by definition so the antecedent does divide the consequent.

Otherwise, A and B are negative, and the antecedent is equal to $-(-B \to -A) \to -A$, wheras the consequent is $-B \to -A$. So, the antecedent is $\frac{-A}{-(\frac{-A}{-B})} = \frac{A}{\frac{A}{B}} = B$, but B is negative, so the antecedent is negative.

If $A \to (A \to B)$ is negative, and $A \to B$ is positive, then the polarities of A and B must be the same, and the polarities of A and $A \to B$ must differ. So, since $A \to B$ is positive, A, and B, are negative. But then the antecedent is equal to 0.

(A12) $A \to ((A \to B) \to B)$

Suppose this were undesignated. Then it takes the value 0. For $\to$, this happens in one of four cases: (i) $A \neq 0$ and $B = 0$ (ii) $B \neq \infty$ and $A = \infty$ (iii) A and B are both positive, and A does not divide B (iv) A is negative and B is positive.

If $A \neq 0, ((A \to B) \to B) = 0$ then, there are four further possibilities:

If $(A \to B) \neq 0, B = 0$, then the only way for $A \to B$ not to equal 0, is for A to equal 0, but then the antecedent is equal to 0.

If $(A \to B) = \infty, B \neq \infty$, then A, the antecedent, is equal to 0.

If $A \to B, B$ positive, and $A \to B$ does not divide B, then A is also positive, so $\frac{B}{A}$ does not divide B. But $\frac{B}{\frac{B}{A}} = \frac{B}{\cdot}\frac{A}{B} = A$, and A is a whole number.

If $A \to B$ is negative, and B is positive, then A is negative, but then $A \to B$ is equal to 0.

If $((A \to B) \to B) \neq \infty, A = \infty$ then, either $B = \infty$ or not. If it does, then the antecedent is equal to ∞. If it does not, then $A \to B$ is equal to 0, and the antecedent is equal to ∞.

If A and $(A \to B) \to B$ are positive, and the former does not divide the latter, then B must be positive so, the consequent is equal to $\frac{B}{\frac{B}{A}} = A$; since antecedent and consequent are equal, antecedent divides consequent.

If A is negative, and $(A \to B) \to B$ is positive, then the polarities of $A \to B$ and B must be the same. But if B is negative, then $A \to B$ is positive, and if B is positive, then $A \to B$ is negative.

(R1) $A, A \to B \vdash B$

Suppose the premises were designated, and conclusion undesignated. Then the value of B is 0. A does not equal 0, so $A \to B$ does equal 0 and one of the premises is undesignated.

(R2) $A, B \vdash A \wedge B$

Suppose the premises were designated, and conclusion undesignated. Then $A \wedge B = 0$. So, either A or B is equal to 0, hence one of the premises is undesignated.

9. Use the results of the previous problem to show that the following do not hold in R:

I will show the following in the many valued logic of question 8 as far as possible. Since this is a sub-logic of that of 10.5.6, the result follows.

(a) $\models p \to (p \to p)$

The simplest countermodel is perhaps that where p is a negative integer. Then $p \to p$ is $-p \to -p = 1$, and so $(p \to (p \to p)) = 0$.

.

(b) $\models p \to (q \to (p \wedge q))$

The simplest countermodel is perhaps that where $p = \infty$ and q is a positive integer. Then $p \wedge q$ equals q, so $q \to (p \wedge q) = q \to q = 1$. $p = \infty$, $(q \to (p \wedge q)) \neq \infty$, so $(p \to (q \to (p \wedge q))) = 0$

(c) $(p \wedge q) \to r \models (p \to r) \vee (q \to r)$

Take a valuation where all parameters are positive, but p does not divide r, and q does not divide r, but the GCD of p and q does divide r. For instance $p = 3$, $q = 4$, $r = 1$. In this case, the conclusion is equal to 0, (since $\frac{1}{3}$ and $\frac{1}{4}$ are not whole numbers), and hence undesignated, but the antecedent is equal to 1 (since the GCD of 3 and 4 is 1, and $\frac{1}{1} = 1$), and so designated.

(d) $(p \to q) \wedge (r \to s) \models (p \to s) \vee (r \to q)$

Take a valuation where all parameters are positive, but $\frac{s}{p}$ and $\frac{q}{r}$ are not whole numbers, while the LCM of $\frac{q}{p}$ and $\frac{s}{r}$ is. For instance $p = 3$, $q = 6$, $r = 4, s = 8$. In this case, the conclusion is equal to 0, (since $\frac{8}{6}$ and $\frac{6}{4}$ are not whole numbers), but the antecedent is equal to 2 (since the LCM of $\frac{8}{4} = 2$, and $\frac{6}{3} = 2$, is 2).

(e) $\neg(p \to q) \models p$

There is a mistake in the question - this is in fact valid in the logic of question 8 - here is a proof:

Imagine p were undesignated - i.e. 0. Then $p \to q$ is ∞, and $\neg(p \to q)$ is 0 - i.e. undesignated.

However, the above is invalid in R - I will show this by in the many valued logic of 10.5.6 which is a sublogic of R:

Take the valuation where $p = 1$ and $q = 0$. Then p is undesignated, because all values without primes are undesignated (see p. 10.5.6). However, the value of $p \rightarrow q$ is 0, meaning the value of $\neg(p \rightarrow q)$ is $0'$, which is designated.

10. Show that the following are valid in C_B^+:

(a) $\vdash A > A$

$$\begin{array}{c} A > A, -0 \\ 0 r_A 1 \\ A, +1 \\ A, -1 \\ \otimes \end{array}$$

(b) $\vdash (A > \neg\neg A) \wedge (\neg\neg A > A)$

$$\begin{array}{c} (A > \neg\neg A) \wedge (\neg\neg A > A), -0 \\ \begin{array}{cc} A > \neg\neg A, -0 & \neg\neg A > A, -0 \\ 0 r_A 1 & 0 r_{\neg\neg A} 1 \\ A, +1 & \neg\neg A, +1 \\ \neg\neg A, -1 & A, -1 \\ \neg A, +1^{\#} & \neg A, -1^{\#} \\ A, -1 & A, +1 \\ \otimes & \otimes \end{array} \end{array}$$

(c) $\vdash (A \wedge B) > A$

$$\begin{array}{c} (A \wedge B) > A, -0 \\ 0 r_{A \wedge B} 1 \\ A \wedge B, +1 \\ A, -1 \\ A, +1 \\ B, +1 \\ \otimes \end{array}$$

(d) $A > B, A > C \vdash A > (B \wedge C)$

$$\begin{array}{c} A > B, +0 \\ A > C, +0 \\ A > (B \wedge C), -0 \\ 0 r_A 1 \\ A, +1 \\ B \wedge C, -1 \\ B, +1 \\ C, +1 \\ \begin{array}{cc} B, -1 & C, -1 \\ \otimes & \otimes \end{array} \end{array}$$

(e) $A, A > B \vdash B$

$$\begin{array}{c}
A, +0 \\
A > B, +0 \\
B, -0 \\
\begin{array}{cc}
A, -0 & A, +0 \\
\otimes & 0r_A0 \\
 & B, +0 \\
 & \otimes
\end{array}
\end{array}$$

(f) $A \rightarrow B \vdash A > B$

$$\begin{array}{c}
A \rightarrow B, +0 \\
A > B, -0 \\
0r_A1 \\
A, +1 \\
B, -1 \\
\begin{array}{cc}
A, -1 & B, +1 \\
\otimes & \otimes
\end{array}
\end{array}$$

11. This exercise gives a proof of the relevance of the logic B.

(a) Let $\perp$ and $\perp^*$ be a pair of non-normal worlds such that every propositional parameter is true at $\perp$ and false at $\perp^*$. Suppose that $R\perp\perp\perp$, $R\perp^*\perp\perp^*$, and that each world accesses no other worlds. Show that every formula is true at $\perp$ and false at $\perp^*$

This can be shown by a simple induction:

The atomic case requires no argument.

The cases for all connectives other than $\neg$, and $\rightarrow$ are as in the proof of their classical truth-functionality.

For $\neg A$, suppose $\neg A$ were not true at $\perp$. Then A is not false at $\perp^*$. But, by induction hypothesis, A is false at $\perp^*$.

Suppose $\neg A$ were not false at $\perp^*$. Then A is not true at $\perp$. But by induction hypothesis, A is true at $\perp$.

For $A \rightarrow B$, by induction hypothesis, A and B are true at $\perp$, and false at $\perp^*$. By stipulation $R\perp\perp\perp$, $R\perp^*\perp\perp^*$ and each world accesses no other worlds.

$R\perp\perp\perp$, and A and B are true at $\perp$. Since $\perp$ is not related to any other worlds, $A \rightarrow B$ is true at $\perp$, as required.

$R\perp^*\perp\perp^*$; A is true at $\perp$, and B is false at $\perp^*$. Thus $A \rightarrow B$ is false at $\perp^*$, as required.

(b) Let w and w^* be a pair of non-normal worlds such that $Rw\perp w$ and $Rw^*\perp w^*$. Using part (a), show that: (i) if every parameter in A is true at w and false at w^*, the same is true of A; (ii) if every parameter in B is false at w and true at w^*, the same is true of B.

For (i) and (ii), the argument for w and w^* is the same as that for $\perp$ and $\perp^*$ in 11(a).

(c) Use this to show that if $\vdash_B A \rightarrow C, A$ and C share a parameter.

Suppose A and C do not share any propositional parameter. Then there is an interpretation $I = \langle W, N, R, *, v\rangle$ where $W = \{w_0, w_1, w_0^*, w_1^*\}$, $N = \{w_0, w_0^*\}$, R such that $Rw_1w_1^*w_1$, $Rw_1^*w_1w_1^*$ which makes every propositional parameter in A true at w, and every parameter in C false, hence by (b)(i) this interpretation makes A true at w and C false at w as in (b)(i).Hence it is not the case that $A \rightarrow C$.

12. By defining suitable accessibility relations for $>$, modify the proof of the previous question to show the same for $>$ in C_B^+ (Hint: For every non-normal world, w, set $f_A(w) = \{w\}$.)

(a) Let $\perp$ and $\perp^*$ be a pair of non-normal worlds such that every propositional parameter is true at $\perp$ and false at $\perp^*$. Suppose that for all non-normal worlds w, for all formulas A $f_A(w) = \{w\}$, and that no other restrictions on $>$ obtain.Show that every formula is true at $\perp$ and false at $\perp^*$

This can be shown by a simple induction:

The atomic case requires no argument.

The cases for all connectives other than $\neg$, and $\rightarrow$ are as in the proof for C.

For $\neg A$, suppose $\neg A$ were not true at $\perp$. Then A is not false at $\perp^*$. But, by induction hypothesis, A is false at $\perp^*$.

Suppose $\neg A$ were not false at $\perp^*$. Then A is not true at $\perp$. But by induction hypothesis, A is true at $\perp$.

Suppose $A > B$ were not true at $\perp$. Then there is a world w' such that $w' \in f_A(\perp)$, and B is false at w'. But the only such w' is $\perp$ itself, and by induction hypothesis B is true at $\perp$.

Suppose $A > B$ were not false at $\perp^*$. Then there is no world w' such that $w' \in f_A(\perp)$, and B is false at w'. But $\perp^*$ itself is such a w', and B is false at $\perp^*$.

(b) Let w and w^* be a pair of non-normal worlds such that $w \in f_A(\perp)$ and $w^* \in f_A(\perp^*)$. Using part (a), show that: (i) if every parameter in A is true at w and false at w^*, the same is true of A; (ii) if every parameter in B is false at w and true at w^*, the same is true of B.

For (i) and (ii) the argument is the same as that in 12(a).

(c) Use this to show that if $\vdash_B A \rightarrow C, A$ and C share a propositional parameter.

Suppose A and C do not share any propositional parameter. Then there is an interpretation $I = \langle W, N, *, v, f \rangle$ such that $W = \{w_0, w_1, w_0^*, w_1^*\}$, $N = \{w_0, w_0^*\}$, $w_1 \in f_A(w_1) w_1 \in f_A(w_1^*)$ which makes all parameters in A true, and all parameters in C false. Likewise w^* makes all parameters in A false, and those in C true. By (b)(i), w makes A true, and C false. Hence it is not the case that $A \rightarrow C$.

13. Let $D(n)$ be the disjunction of all formulas of the form $p_i \leftrightarrow p_j$ for all i and j such that $0 \leq i < j \leq n$. Using the interpretation of problem 8, show that for all n, $D(n)$ is not logically valid in R. Hence show that neither R nor any weaker relevant logic is finitely-many valued. (Hint: See the similar proofs for modal and intuitionist logics, 7.11.1-7.11.4.)

Definition: Let $A \leftrightarrow B$ be $(A \rightarrow B) \wedge (B \rightarrow A)$.
Lemma: For no n is D_{n+1} a logical truth of any relavent logic weaker than R.

I will simply show the result in the many valued logic of question 8, that R has been shown to be a sub-logic of. Clearly if the result holds in that logic, it will hold in the weaker R.

(Is this true - it would seem to be disproven by the example we found above which was invalid in R, and valid in the many valued logic of question 8!)

D_{n+1} is the disjunction of all sentences of the form $(p_i \rightarrow p_j) \wedge (p_j \rightarrow p_i)$, where $1 \leq i < j \leq n+1$. There is an interpretation where p_i takes the number i, (and p_j the number j). This interpretation makes all disjunctions in D_{n+1} undesignated, and hence D_{n+1} itself undesignated also.

To see this, simply note that if a divides b, $a \rightarrow b = b/a$; otherwise $a \rightarrow b = 0$. Clearly since p_i and p_j are distinct, one of them is a lower number than the other. One of the conjunctions will have the lower number on its LHS (e.g. $2 \rightarrow 3$), and this will always make one conjunction equal to 0. If one of the conjunctions is equal to 0, the whole conjunction is also. since thi sis true for any i and j, for any n the interpretation where $p_i = i$ and $p_j = j$ shows D_n to be invalid.

Theorem: No relevant logic weaker than R is a finitely many-valued logic.

Suppose that there were such, and that it had n truth-values. Since we have A2 in R, $\models_R A \rightarrow (A \vee B)$, $A \models_R (A \vee B)$ and,

(i) whenever $A \in D, A \vee B \in D$

Since we have A3 in R, $\models_R (A \wedge B) \rightarrow A$, $A \wedge B \models_R A$ and,

(ii) Whenever $A \wedge B \in D, A \in D$.

(and the same for B in both cases). Moreover, since $\models_R p \rightarrow p$, $\models_R p \leftrightarrow p$:
(iii) for any $x \in V$, $f_{\rightarrow}(x, x) \in D$.

Now consider any interpretation v. Since there are only n truth values, for some j and k such that $1 \leq j < k \leq n+1$, $v(p_j) = v(p_k)$. Hence by (iii) $v(p_j \rightarrow p_k) \in D$, and $v(p_k \rightarrow p_j) \in D$. By (ii), $v(p_k \leftrightarrow p_j) \in D$, and by (i),

$v(D_{n+1}) \in D$. Thus D_{n+1} is logically valid in R, which it is not by the preceding lemma.

15. *Check the details omitted in 10.8.

10.8.2 Check the cases for T8 - T11 in the Soundness Lemma for B+C8-C11.

T8: Suppose that $rxyz$ appears on b, and that we apply the rule to get $rx\bar{z}\bar{y}$. By assumption $Rf(x)f(y)f(z)$, so by C8 $Rf(x)f(z^*)f(y^*)$, as required.

T9: Suppose that $rxyz$ and $rzuv$ appear on b and that we apply the rule to get $rxuj, ryjv$. By assumption, $Rf(x)f(y)f(z)$, and $Rf(z)f(u)f(v)$, so by C9, $Rf(x)f(u)f(j)$ and $Rf(y)f(j)f(v)$, as required.

T10: Suppose that $rxyz$ and $rzuv$ appear on b, and that we apply the rule to get $ryuj, rxjv$. By assumption, $Rf(x)f(y)f(z)$, and $Rf(z)f(u)f(y)$, so by C10, $Rf(x)f(u)f(j)$ and $Rf(x)f(j)f(v)$, as required.

T11: Suppose that $rxyz$ appears on b, and that we apply the rule to get $rxyj, rjvz$. By assumption, $Rf(x)f(y)f(z)$, so by C11, $Rf(x)f(y)f(j)$ and $Rf(j)f(y)f(z)$, as required.

Check that each of the constraints C8-C11 is satisfied, given that the appropriate rule is in force.

C8: Suppose that T8 has been applied. Then $rxyz$ appears on the branch, and so does $rx\bar{z}\bar{y}$. By induction hypothesis, $Rw_xw_yw_z$ and $Rw_xw_z^*w_y^*$, as required.

C9: Suppose that T9 has been applied. Then $rxyz$ appears on the branch, and so does $rxuj, ryjv$. By induction hypothesis, $Rw_xw_yw_z$ and $Rw_xw_uw_j$, and $Rw_yw_jw_v$, as required.

C10: Suppose that T10 has been applied. Then $rxyz$ appears on the branch, and so does $ryuj, rxjv$. By induction hypothesis, $Rw_xw_yw_z$ and $Rw_yw_uw_j$, and $Rw_xw_jw_v$, as required.

C11: Suppose that T11 has been applied. Then $rxyz$ appears on the branch, and so does $rxyj, rjvz$. By induction hypothesis, $Rw_xw_yw_z$ and $Rw_yw_uw_j$, and $Rw_xw_jw_v$, as required.

10.8.2b Check the new rules of 10.4a.3 in the Soundness Lemma for the tableaux for content-inclusion.

Suppose that the rule to add $x \preceq x$ to the branch is applied to b. If $x \preceq x$ is added to the branch, $f(x) \sqsubseteq f(x)$ since $\sqsubseteq$ is reflexive, as required.

Suppose that $x \preceq y$ and $y \preceq z$ appear on b, and that the rule to add $x \preceq z$ is applied. By induction hypothesis, $w_x \sqsubseteq w_y$ and $w_y \sqsubseteq w_z$, so if $v_{w_x}(p) = 1$, $v_{w_y} = 1$, and if $v_{w_y} = 1$ then $v_{w_z} = 1$. Thus if $v_{w_x}(p) = 1$, $v_{w_z} = 1$, as required.

Suppose that $x \preceq y$ and $p, +x$ appear on b, and that the rule to add $p, +y$ is applied. By induction hypothesis, $w_x \sqsubseteq w_y$, so if $v_{w_x}(p) = 1$, $v_{w_y} = 1$, and $v_{w_x}(p) = 1$. Thus $v_{w_y}(p) = 1$, as required.

Suppose that $x \preceq y$ appears on b, and that the rule to add $\bar{y} \preceq \bar{z}$ is applied. By induction hypothesis, $w_x \sqsubseteq w_y$, so if $v_{w_x}(p) = 1$, $v_{w_y} = 1$. Thus if $v_{w_y^*} = 1$, $v_{w_x^*} = 1$, as required.

Suppose that $x \preceq y$ and $ryzw$ and that the rule to split the branch with x and $z \preceq w$, or $\overline{x}$, and $rxzw$, is applied. By induction hypothesis, $w_x \sqsubseteq w_y$ and $Rw_yw_zw_w$, so if $v_{w_x}(p) = 1$, $v_{w_y} = 1$. By clause 3, w_x is either normal or not. If it is normal, then $w_z \sqsubseteq w_w$, and I is faithful to the left branch. If it is not normal, then $Rw_xw_zw_w$, and I is faithful to the right branch, as required.

Suppose that the rule to add *\$0* to the branch is applied to b. w_0 is normal, as required.

Suppose that *\$x* appears on b, and that the rule to add $rxyy$ is applied. By induction hypothesis, w_x is normal. Thus for any y, $Rw_xw_yw_y$, as required.

Suppose that *\$x* and $rxyz$ appear on b, and that the rule to add $y = z$ is applied. By induction hypothesis, w_x is normal, and $Rw_xw_yw_z$, so $w_y = w_z$, as required.

10.8.2d: Check that the induced interpretation has the right property in each case for each content-inclusion rule T12-T16.

Suppose that T12 has been applied on b. By induction hypothesis, $Rw_xw_yw_z$, so by C12, for some w_x such that $w_y \sqsubseteq w_j$, $Rw_bw_xw_z$, as required by C12.

Suppose that T13 has been applied on b. By induction hypothesis, w_x is normal, and $w_x^* \sqsubseteq w_x$, as required by C13.

Suppose that T14 has been applied on b. By induction hypothesis, either w_x is normal, in which case $w_x \sqsubseteq w_x$, or w_x is non-normal, in which case $Rw_xw_x^*w_x$, as required by C14.

Suppose that T15 has been applied on b. By induction hypothesis, $Rw_xw_yw_z$ and $w_x \sqsubseteq w_z$, as required by C15.

Suppose that T16 has been applied on b. By induction hypothesis, $Rw_xw_yw_z$, and either $w_x \sqsubseteq w_z$, or $w_y \sqsubseteq w_z$, as required by C16.