Solutions

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1. Complete the details left as exercises in 9.4.1, 9.4.2, 9.6.6, 9.6.9, 9.6.10 and 9.7.10.

9.4.1 Show that $\nvDash_{K_4} p \rightarrow (q \vee \neg q)$ and $\nvDash_{K_4} (p \wedge \neg p) \rightarrow q$

$\nvDash_{K_4} p \rightarrow (q \vee \neg q)$

$$\begin{array}{c} p \rightarrow (q \vee \neg q), -0 \\ p, +1 \\ q \vee \neg q, -1 \\ q, -1 \\ \neg q, -1 \end{array}$$

Counter-model such that:

$$W = \{w_0, w_1\}; p\rho_{w_1}1$$

This can be represented in the following diagram:

$$\begin{array}{cc} w_0 & w_1 \\ & +p \\ & -q \\ & -\neg q \end{array}$$

Let us check that the interpretation works:

p is true at w_1, however both q and $\neg q$ are untrue, therefore $p \rightarrow (q \vee \neg q)$ is not true at w_0.

$\nvDash_{K_4} (p \wedge \neg p) \rightarrow q$

$$(p \wedge \neg p) \rightarrow q, -0$$
$$p \wedge \neg p, +1$$
$$q, -1$$
$$p, +1$$
$$\neg p, +1$$

Counter-model such that:

$$W = \{w_0, w_1\}; p\rho_{w_1}1, p\rho_{w_1}0$$

This can be represented in the following diagram:

w_0	w_1
	$-q$
	$+p$
	$+\neg p$

Let us check that the interpretation works:

Both p and $\neg p$ are true at w_1, making $p \wedge \neg p$ true there. However q is not true at w_1 so $(p \wedge \neg p) \rightarrow q$ is not true at w_0.

9.4.2 Show that in K_4, (if $\vDash \neg A$ then $\vDash A \rightarrow B$) does not hold.

Consider the case where $A = \neg(p \rightarrow p)$ and $B = q$. Then $\vDash \neg A$:

$$\neg\neg(p \rightarrow p), -0$$
$$p \rightarrow p, -0$$
$$p, +1$$
$$p, -1$$
$$\otimes$$

However, $\nvDash A \rightarrow B$:

$$\neg(p \rightarrow p) \rightarrow q, -0$$
$$\neg(p \rightarrow p), +1$$
$$q, -1$$
$$p, +2$$
$$\neg p, +2$$

The counter-model defined by this tablueau can be depicted as follows:

w_0	w_1	w_2
	$-q$	$+p$
		$+\neg p$

Since p is both true and not true at w_2, $p \to p$ is false, and $\neg(p \to p)$ is true, at w_1. However q is not true at w_1, so $\neg(p \to p) \to q$ is not true at w_0.

9.6.6 In K_*, $\models p \to (q \to q)$

$$
\begin{array}{c}
p \to (q \to q), -0 \\
p, +1 \\
q \to q, -1 \\
q, +2 \\
q, -2 \\
\otimes
\end{array}
$$

9.6.9 K_* and N_* validate contraposition: $p \to q \models \neg q \to \neg p$

The below tableau shows this for both K_* and N_*.

$$
\begin{array}{c}
p \to q, +0 \\
\neg q \to \neg p, -0 \\
\neg q, +1 \\
\neg p, -1 \\
q, -1^{\#} \\
p, +1^{\#} \\
\begin{array}{cc}
p, -1^{\#} & q, +1^{\#} \\
\otimes & \otimes
\end{array}
\end{array}
$$

9.6.10 Show that the relation semantics (normal and non-normal) verify $p \wedge \neg q \models \neg(p \to q)$.

The same tableau shows this for relational K_4 and relational N_4

$$
\begin{array}{c}
p \wedge \neg q, +0 \\
\neg(p \to q), -0 \\
p, +0 \\
\neg q, +0 \\
\begin{array}{cc}
p, -0 & \neg q, +0 \\
\otimes & \otimes
\end{array}
\end{array}
$$

9.7.10 Take a $*$ interpretation $< W, N, *, v >$ where $W = \{w_0, w_1, w_2\}$; $N = \{w_0\}, w_0^* = w_0, w_1^* = w_2, w_2^* = w_1$; for every propositional parameter or conditional, D in A, $v_{w_1}(D) = 1$ and $v_{w_2}(D) = 0$; for every propositional parameter or conditional, D, in B, $v_{w_1}(D) = 0$, and $v_{w_2}(D) = 1$. Check that $v_{w_1}(A) = 1$, and $v_{w_1}(B) = 0$, and hence show that $v_{w_0}(A \to B) = 0$.

We must show that, for every formula C made up of parameters and conditionals in A, $v_{w_1}(C) = 1$ and $v_{w_2}(C) = 0$ (and for every formula E in B, $v_{w_1}(E) = 0$ and $v_{w_2}(E) = 1$. This can be shown in an induction on the complexity of A and B similar to that of 9.7.9.

The basis case is that of parameters and conditionals, and is true by stipulation.

The other cases are for A and B of the form $\neg C$, $C \wedge D$, and $C \vee D$.

$\boxed{\neg C}$

For A, by induction hypothesis, $v_{w_2}(C) = 0$. Since w_2 is the star world of w_1, $v_{w_1}(\neg C) = 1$ as required.

For B, by induction hypothesis, $v_{w_2}(C) = 1$. Since w_2 is the star world of w_1, $v_{w_1}(\neg C) = 0$, as required.

$\boxed{C \wedge D}$

For A, by induction hypothesis, $v_{w_1}(C) = 1$, and $v_{w_1}(D) = 1$. So, $v_{w_1}(C \wedge D) = 1$, as required. By induction hypothesis, $v_{w_2}(C) = 0$ and $v_{w_2}(D) = 0$, so $v_{w_2}(C \wedge D) = 0$, as required

For B, by induction hypothesis, $v_{w_1}(C) = 0$, and $v_{w_1}(D) = 0$. So, $v_{w_1}(C \wedge D) = 0$, as required. By induction hypothesis, $v_{w_2}(C) = 1$, and $v_{w_2}(D) = 1$. Thus $v_{w_2}(C \wedge D) = 1$, as required.

$\boxed{C \vee D}$

For A, by induction hypothesis, $v_{w_1}(C) = 1$, and $v_{w_1}(D) = 1$. So, $v_{w_1}(C \vee D) = 1$, as required. By induction hypothesis, $v_{w_2}(C) = 0$ and $v_{w_2}(D) = 0$, so $v_{w_2}(C \vee D) = 0$, as required

For B, by induction hypothesis, $v_{w_1}(C) = 0$, and $v_{w_1}(D) = 0$. So, $v_{w_1}(C \vee D) = 0$, as required. By induction hypothesis, $v_{w_2}(C) = 1$, and $v_{w_2}(D) = 1$. Thus $v_{w_2}(C \vee D) = 1$, as required.

Since all sentences are built from propositional parameters, conditionals, and the above extensional connectives, and we have seen that no matter which of these are used, if A and B are as stipulated, $v_{w_1}(A) = 1$, and $v_{w_1}(B) = 0$, $v_{w_0}(A \rightarrow B) = 0$

2. Show the following in K_4 (where $A \leftrightarrow B$ is $(A \rightarrow B) \wedge (B \rightarrow A)$):

(a) $\vdash A \rightarrow A$

$$\begin{array}{c} A \rightarrow A, -0 \\ A, +1 \\ A, -1 \\ \otimes \end{array}$$

(b) $\vdash A \leftrightarrow \neg\neg A$

$\vdash (A \rightarrow \neg\neg A) \wedge (\neg\neg A \rightarrow A)$

$$\begin{array}{cc} \multicolumn{2}{c}{(A \rightarrow \neg\neg A) \wedge (\neg\neg A \rightarrow A), -0} \\ \swarrow & \searrow \\ A \rightarrow \neg\neg A, -0 & \neg\neg A \rightarrow A, -0 \\ A, +1 & \neg\neg A, +1 \\ \neg\neg A, -1 & A, -1 \\ A, -1 & A, +1 \\ \otimes & \otimes \end{array}$$

(c) $\vdash (A \wedge B) \rightarrow A$

$$\begin{array}{c} (A \wedge B) \rightarrow A, -0 \\ A \wedge B, +1 \\ A, -1 \\ A, +1 \\ B, +1 \\ \otimes \end{array}$$

(d) $\vdash A \rightarrow (A \vee B)$

$$\begin{array}{c} A \rightarrow (A \vee B), -0 \\ A, +1 \\ A \vee B, -1 \\ A, -1 \\ B, -1 \\ \otimes \end{array}$$

(e) $\vdash (A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$

$\vdash ((A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))) \wedge (((A \wedge B) \vee (A \wedge C)) \rightarrow (A \wedge (B \vee C)))$

$$((A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))) \wedge (((A \wedge B) \vee (A \wedge C)) \rightarrow (A \wedge (B \vee C))), -0$$

$(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C)), -0$
$A \wedge (B \vee C), +1$
$(A \wedge B) \vee (A \wedge C), -1$
$A, +1$
$B \vee C, +1$
$A \wedge B, -1$
$A \wedge C, -1$

$B, +1$ $C, +1$

$A, -1$ $\otimes$ $B, -1$ $\otimes$ $A, -1$ $\otimes$ $B, -1$

$A, -1$ $\otimes$ $C, -1$ $\otimes$

$((A \wedge B) \vee (A \wedge C)) \rightarrow (A \wedge (B \vee C)), 0$
$(A \wedge B) \vee (A \wedge C), +1$
$A \wedge (B \vee C), -1$

$A \wedge B, +1$, $A, +1$, $B, +1$ $A \wedge C, +1$, $A, +1$, $C, +1$

$A, -1$ $\otimes$ $B \vee C, -1$, $B, -1$, $C, -1$ $\otimes$ $A, -1$ $\otimes$ $B \vee C, -1$, $B, -1$, $C, -1$ $\otimes$

(f) $A \rightarrow B, A \rightarrow C \vdash A \rightarrow (B \wedge C)$

$A \rightarrow B, +0$
$A \rightarrow C, +0$
$A \rightarrow (B \wedge C), -0$
$A, +1$
$B \wedge C, -1$

$B, -1$ $C, -1$

$A, -1$ $\otimes$ $B, +1$ $\otimes$ $A, -1$ $\otimes$ $B, +1$

$A, -1$ $\otimes$ $C, +1$ $\otimes$

(g) $A \to C, B \to C \vdash (A \vee B) \to C$

$$\begin{array}{c} A \to C, +0 \\ B \to C, +0 \\ (A \vee B) \to C, -0 \\ A \vee B, +1 \\ C, -1 \end{array}$$

- $A, +1$
  - $A, -1$ $\otimes$
  - $C, +1$ $\otimes$
- $B, +1$
  - $A, -1$
    - $B, -1$ $\otimes$
    - $C, +1$ $\otimes$
  - $C, +1$ $\otimes$

(h) $A \to C \vdash (A \wedge B) \to C$

$$\begin{array}{c} A \to C, +0 \\ (A \wedge B) \to C, -0 \\ A \wedge B, +1 \\ C, -1 \\ A, +1 \\ B, +1 \end{array}$$

- $A, -1$ $\otimes$
- $C, +1$ $\otimes$

(i) $\vdash ((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

$$\begin{array}{c} ((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C)), -0 \\ (A \to B) \wedge (A \to C), +1 \\ A \to (B \wedge C), -1 \\ A \to B, +1 \\ A \to C, +1 \\ A, +2 \\ B \wedge C, -2 \end{array}$$

- $B, -2$
  - $A, -2$ $\otimes$
  - $B, +2$ $\otimes$
- $C, -2$
  - $A, -2$ $\otimes$
  - $B, +2$
    - $A, -2$ $\otimes$
    - $C, +2$ $\otimes$

(j) $\vdash ((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

$$((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C), -0$$
$$(A \to C) \wedge (B \to C), +1$$
$$(A \vee B) \to C, -1$$
$$A \to C, +1$$
$$B \to C, +1$$
$$A \vee B, +2$$
$$C, -2$$

- $A, +2$
  - $A, -2$ $\otimes$
  - $C, +2$ $\otimes$
- $B, +2$
  - $A, -2$
    - $B, -2$ $\otimes$
    - $C, +2$ $\otimes$
  - $C, +2$ $\otimes$

(k) $A \to B \vdash (B \to C) \to (A \to C)$

$$A \to B, +0$$
$$(B \to C) \to (A \to C), -0$$
$$B \to C, +1$$
$$A \to C, -1$$
$$A, +2$$
$$C, -2$$

- $A, -2$ $\otimes$
- $B, +2$
  - $B, -2$ $\otimes$
  - $C, +2$ $\otimes$

(l) $A \to B \vdash (C \to A) \to (C \to B)$

$$A \to B, +0$$
$$(C \to A) \to (C \to B), -0$$
$$C \to A, +1$$
$$C \to B, -1$$
$$C, +2$$
$$B, -2$$

- $A, -2$
  - $C, -2$ $\otimes$
  - $A, +2$ $\otimes$
- $B, +2$ $\otimes$

(m) $A \to B, B \to C \vdash A \to C$

$$A \to B, +0$$
$$B \to C, +0$$
$$A \to C, -0$$
$$A, +1$$
$$C, -1$$

$$\begin{array}{cc} A, -1 & B, +1 \\ \otimes & \begin{array}{cc} B, -1 & C, +1 \\ \otimes & \otimes \end{array} \end{array}$$

3. Show that the following are not true in K_4 and specify a counter-model.

(a) $\nvdash (p \wedge (\neg p \vee q)) \to q$

$$(p \wedge (\neg p \vee q)) \to q, -0$$
$$p \wedge (\neg p \vee q), +1$$
$$q, -1$$
$$p, +1$$
$$\neg p \vee q, +1$$

$$\begin{array}{cc} \neg p, +1 & q, +1 \\ & \otimes \end{array}$$

Counter-model such that:

$$W = \{w_0, w_1\}; p\rho_{w_1}1, p\rho_{w_1}0$$

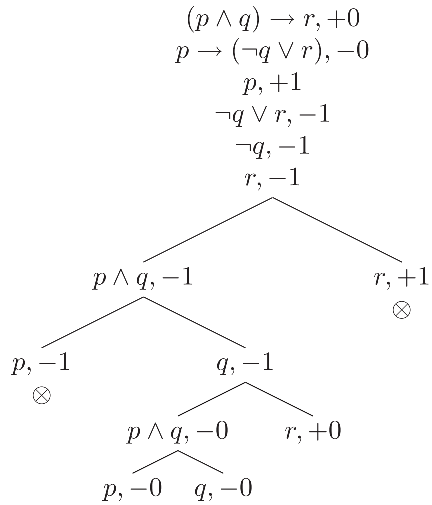
This can be represented in the following diagram:

$$\begin{array}{cc} w_0 & w_1 \\ & +p \\ & -q \\ & +\neg p \end{array}$$

Let us check that the interpretation works:

p is true at w_1, as is $\neg p$, making $p \wedge (\neg p \vee q)$ true at w_1. q is not true at w_1. Therefore the conclusion $(p \wedge (\neg p \vee q)) \to q$ is not true at w_0

(b) $(p \wedge q) \rightarrow r \vdash p \rightarrow (\neg q \vee r)$



Counter-model from open left-most branch such that:

$$W = \{w_0, w_1\}; p\rho_{w_1}1$$

This can be represented in the following diagram:

w_0	w_1
$-p$	$+p$
	$-\neg q$
	$-q$
	$-r$

Let us check that the interpretation works:

$p \wedge q$ is not true at both w_0 and w_1, making the premise $(p \wedge q) \rightarrow r$ true at w_0. p is true, and $\neg q$ and r are both not true at w_1, so the conclusion $p \rightarrow (\neg q \vee r)$ is not true at w_0.

(c) $\vdash p \rightarrow (q \vee \neg q)$

$$p \rightarrow (q \vee \neg q), -0$$
$$p, +1$$
$$q \vee \neg q, -1$$
$$q, -1$$
$$\neg q, -1$$

Counter-model such that:

$$W = \{w_0, w_1\}; p\rho_{w_1}1$$

This can be represented in the following diagram:

$$\begin{array}{cc} w_0 & w_1 \\ & +p \\ & -\neg q \\ & -q \end{array}$$

Let us check that the interpretation works:

p is true at w_1 and both q and $\neg q$ are not true at w_1, making the conclusion $p \to (q \vee \neg q)$ false at w_0.

(d) $\vdash (p \wedge \neg p) \to q$

$$\begin{array}{c} (p \wedge \neg p) \to q, -0 \\ p \wedge \neg p, +1 \\ q, -1 \\ p, +1 \\ \neg p, +1 \end{array}$$

Counter-model such that:

$$W = \{w_0, w_1\}; p\rho_{w_1}1, p\rho_{w_1}0$$

This can be represented in the following diagram:

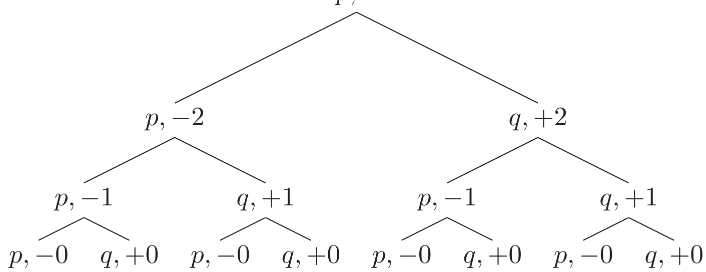
$$\begin{array}{cc} w_0 & w_1 \\ & +p \\ & +\neg p \\ & -q \end{array}$$

Let us check that the interpretation works:

p is true and false at w_1, making $p \wedge \neg p$ true at w_1. q is not true at w_1 making the conclusion $(p \wedge \neg p) \to q$ not true at w_0.

(e) $\vdash (p \to q) \to (\neg q \to \neg p)$

$$(p \to q) \to (\neg q \to \neg p), -0$$
$$p \to q, +1$$
$$\neg q \to \neg p, -1$$
$$\neg q, +2$$
$$\neg p, -2$$



Counter-model from open left-most branch such that:

$$W = \{w_0, w_1, w_2\}; q\rho_{w_2}0$$

This can be represented in the following diagram:

w_0	w_1	w_2
$-p$	$-p$	$-p$
		$-\neg p$
		$+\neg q$

Let us check that the interpretation works:

q is false, and p is neither true nor false, at w_2, meaning that the consequent, $\neg q \to \neg p$, is false at w_1. p is not true at any world, so the antecedent $p \to q$ is true at w_1. Therefore the conclusion $(p \to q) \to (\neg q \to \neg p)$ is not true at w_0.

4. Determine which of the inferences in problem 2 are valid in N_4. Where invalid, specify a counter-model for an instance.

(a) $\vdash_{N_4} A \to A$

$$A \to A, -0$$
$$A, +1$$
$$A, -1$$
$$\otimes$$

(b) $\vdash_{N_4} A \leftrightarrow \neg\neg A$

$\vdash_{N_4} (A \rightarrow \neg\neg A) \wedge (\neg\neg A \rightarrow A)$

$$
\begin{array}{c}
(A \rightarrow \neg\neg A) \wedge (\neg\neg A \rightarrow A), -0 \\
\swarrow \qquad \searrow \\
\begin{array}{cc}
A \rightarrow \neg\neg A, -0 & \neg\neg A \rightarrow A, -0 \\
A, +1 & \neg\neg A, +1 \\
\neg\neg A, -1 & A, -1 \\
A, -1 & A, +1 \\
\otimes & \otimes
\end{array}
\end{array}
$$

(c) $\vdash_{N_4} (A \wedge B) \rightarrow A$

$$
\begin{array}{c}
(A \wedge B) \rightarrow A, -0 \\
A \wedge B, +1 \\
A, -1 \\
A, +1 \\
B, +1 \\
\otimes
\end{array}
$$

(d) $\vdash_{N_4} A \rightarrow (A \vee B)$

$$
\begin{array}{c}
A \rightarrow (A \vee B), -0 \\
A, +1 \\
A \vee B, -1 \\
A, -1 \\
B, -1 \\
\otimes
\end{array}
$$

(e) $\vdash_{N_4} (A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$

$\vdash_{N_4} ((A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))) \wedge (((A \wedge B) \vee (A \wedge C)) \rightarrow (A \wedge (B \vee C)))$

$$((A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))) \wedge (((A \wedge B) \vee (A \wedge C)) \rightarrow (A \wedge (B \vee C))), -0$$

- $(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C)), -0$
  $A \wedge (B \vee C), +1$
  $(A \wedge B) \vee (A \wedge C), -1$
  $A, +1$
  $B \vee C, +1$
  $A \wedge B, -1$
  $A \wedge C, -1$
  - $B, +1$
    - $A, -1$ ⊗
    - $B, -1$ ⊗
  - $C, +1$
    - $A, -1$ ⊗
    - $B, -1$
      - $A, -1$ ⊗
      - $C, -1$ ⊗
- $((A \wedge B) \vee (A \wedge C)) \rightarrow (A \wedge (B \vee C)), 0$
  $(A \wedge B) \vee (A \wedge C), +1$
  $A \wedge (B \vee C), -1$
  - $A \wedge B, +1$
    $A, +1$
    $B, +1$
    - $A, -1$ ⊗
    - $B \vee C, -1$
      $B, -1$
      $C, -1$
      ⊗
  - $A \wedge C, +1$
    $A, +1$
    $C, +1$
    - $A, -1$ ⊗
    - $B \vee C, -1$
      $B, -1$
      $C, -1$
      ⊗

(f) $A \rightarrow B, A \rightarrow C \vdash_{N_4} A \rightarrow (B \wedge C)$

$A \rightarrow B, +0$
$A \rightarrow C, +0$
$A \rightarrow (B \wedge C), -0$
$A, +1$
$B \wedge C, -1$
- $B, -1$
  - $A, -1$ ⊗
  - $B, +1$ ⊗
- $C, -1$
  - $A, -1$ ⊗
  - $B, +1$
    - $A, -1$ ⊗
    - $C, +1$ ⊗

(g) $A \to C, B \to C \vdash_{N_4} (A \vee B) \to C$

$$A \to C, +0$$
$$B \to C, +0$$
$$(A \vee B) \to C, -0$$
$$A \vee B, +1$$
$$C, -1$$

$$A, +1 \qquad\qquad B, +1$$

$$A, -1 \quad C, +1 \qquad\qquad A, -1 \qquad C, +1$$
$$\otimes \qquad \otimes \qquad\qquad\qquad\qquad\qquad \otimes$$

$$B, -1 \quad C, +1$$
$$\otimes \qquad \otimes$$

(h) $A \to C \vdash_{N_4} (A \wedge B) \to C$

$$A \to C, +0$$
$$(A \wedge B) \to C, -0$$
$$A \wedge B, +1$$
$$C, -1$$
$$A, +1$$
$$B, +1$$

$$A, -1 \quad C, +1$$
$$\otimes \qquad \otimes$$

(i) $\nvdash_{N_4} ((p \to q) \wedge (p \to r)) \to (p \to (q \wedge r))$

$$((p \to q) \wedge (p \to r)) \to (p \to (q \wedge r)), -0$$
$$(p \to q) \wedge (p \to r), +1$$
$$p \to (q \wedge r), -1$$
$$p \to q, +1$$
$$p \to r, +1$$

Counter-model such that:

$$W = \{w_0, w_1\}; N = \{w_0\}; p \to q\rho_{w_1}1, p \to r\rho_{w_1}1$$

This can be represented in the following diagram:

$$w_0 \qquad\qquad w_1$$
$$+p \to q$$
$$+p \to r$$
$$-p \to (q \wedge r)$$

Let us check that the interpretation works:

$p \to q$ and $p \to r$ are true at non-normal w_1, making the antecedent $(p \to q) \wedge (p \to r)$ true at w_1. The consequent $p \to (q \wedge r)$ is not true at w_1, meaning that the conclusion $((p \to q) \wedge (p \to r)) \to (p \to (q \wedge r))$ is not true at w_0.

(j) $\nvdash_{N_4} ((p \to r) \wedge (q \to r)) \to ((p \vee q) \to r)$

$$
\begin{gathered}
((p \to r) \wedge (q \to r)) \to ((p \vee q) \to r), -0 \\
(p \to r) \wedge (q \to r), +1 \\
(p \vee q) \to r, -1 \\
p \to r, +1 \\
q \to r, +1
\end{gathered}
$$

Counter-model such that:

$$W = \{w_0, w_1\}; N = \{w_0\}; p \to r \rho_{w_1} 1, q \to r \rho_{w_1} 1$$

This can be represented in the following diagram:

$$
\begin{array}{cc}
w_0 & w_1 \\
 & +p \to r \\
 & +q \to r \\
 & -(p \vee q) \to r
\end{array}
$$

Let us check that the interpretation works:

$p \to r$ and $q \to r$ are true at non-normal w_1, making the antecedent $(p \to r) \wedge (q \to r)$ true at w_1. The consequent $(p \vee q) \to r$ is not true at w_1, meaning that the conclusion $((p \to r) \wedge (q \to r)) \to ((p \vee q) \to r)$ is not true at w_0.

(k) $p \to q \nvdash_{N_4} (q \to r) \to (p \to r)$

$$\begin{array}{c} p \to q, +0 \\ (q \to r) \to (p \to r), -0 \\ q \to r, +1 \\ p \to r, -1 \end{array}$$

$$\begin{array}{cc} p, -0 & q, +0 \end{array}$$

$$\begin{array}{cccc} p, -1 & q, +1 & p, -1 & q, +1 \end{array}$$

Counter-model from the open left-hand branch such that:

$$W = \{w_0, w_1\}; N = \{w_0\}; q \to r \rho_{w_1} 1$$

This can be represented in the following diagram:

$$\begin{array}{cc} w_0 & w_1 \\ -p & -q \to r \\ & +p \to r \\ & -p \end{array}$$

Let us check that the interpretation works:

p is not true at all worlds, so the premise $p \to q$ is true at w_0. $q \to r$ is true and $p \to r$ is not true at non-normal w_1, making the conclusion $(q \to r) \to (p \to r)$ untrue at w_0.

(l) $p \to q \nvdash_{N_4} (r \to p) \to (r \to q)$

$$\begin{array}{c} p \to q, +0 \\ (r \to p) \to (r \to q), -0 \\ r \to p, +1 \\ r \to q, -1 \end{array}$$

$$\begin{array}{cc} p, -0 & q, +0 \end{array}$$

$$\begin{array}{cccc} p, -1 & q, +1 & p, -1 & q, +1 \end{array}$$

Counter-model from the open left-hand branch such that:

$$W = \{w_0, w_1\}; N = \{w_0\}; r \to p \rho_{w_1} 1$$

This can be represented in the following diagram:

$$\begin{array}{cc} w_0 & w_1 \\ -p & -r \to q \\ & +r \to p \\ & -p \end{array}$$

Let us check that the interpretation works:

p is not true at all worlds, so the premise $p \to q$ is true at w_0. $r \to p$ is true and $r \to q$ is not true at non-normal w_1, making the conclusion $(r \to p) \to (r \to q)$ untrue at w_0.

(m) $A \to B, B \to C \vdash_{N_4} A \to C$

$$\begin{array}{c} A \to B, +0 \\ B \to C, +0 \\ A \to C, -0 \\ A, +1 \\ C, -1 \end{array}$$

$$\begin{array}{cc} A, -1 & B, +1 \\ \otimes & \begin{array}{cc} B, -1 & C, +1 \\ \otimes & \otimes \end{array} \end{array}$$

5. Repeat problems 2-4 with K_* and N_*

Show the following in K_* (where $A \leftrightarrow B$ is $(A \to B) \wedge (B \to A)$:

(a) $\vdash_{K_*} A \to A$

$$\begin{array}{c} A \to A, -0 \\ A, +1 \\ A, -1 \\ \otimes \end{array}$$

(b) $\vdash_{K_*} A \leftrightarrow \neg\neg A$

$\vdash (A \to \neg\neg A) \wedge (\neg\neg A \to A)$

$$(A \to \neg\neg A) \wedge (\neg\neg A \to A), -0$$

$$\begin{array}{cc}
A \to \neg\neg A, -0 & \neg\neg A \to A, -0 \\
A, +1 & \neg\neg A, +1 \\
\neg\neg A, -1 & A, -1 \\
\neg A, +1^{\#} & \neg A, -1^{\#} \\
A, -1 & A, +1 \\
\otimes & \otimes
\end{array}$$

(c) $\vdash_{K_*} (A \wedge B) \to A$

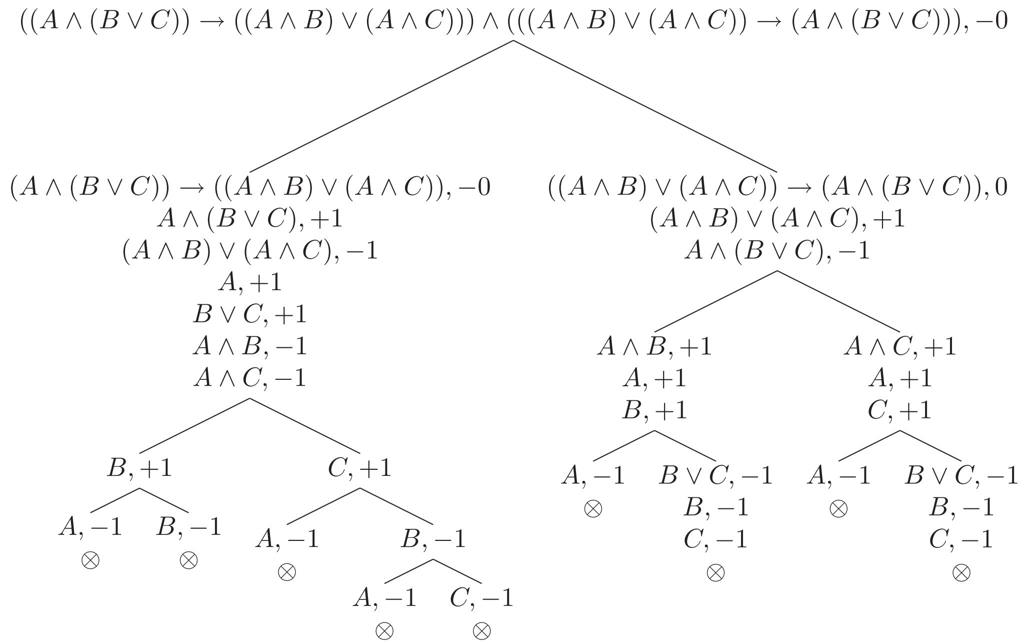
$$\begin{array}{c}
(A \wedge B) \to A, -0 \\
A \wedge B, +1 \\
A, -1 \\
A, +1 \\
B, +1 \\
\otimes
\end{array}$$

(d) $\vdash_{K_*} A \to (A \vee B)$

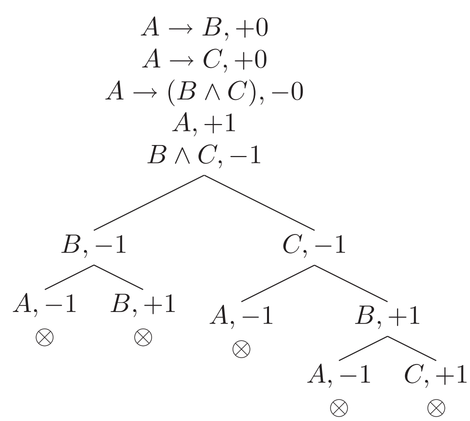
$$\begin{array}{c}
A \to (A \vee B), -0 \\
A, +1 \\
A \vee B, -1 \\
A, -1 \\
B, -1 \\
\otimes
\end{array}$$

(e) $\vdash_{K_*} (A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$

$\vdash ((A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))) \wedge (((A \wedge B) \vee (A \wedge C)) \to (A \wedge (B \vee C)))$



(f) $A \to B, A \to C \vdash_{K_*} A \to (B \wedge C)$



(g) $A \to C, B \to C \vdash_{K_*} (A \lor B) \to C$

$$A \to C, +0$$
$$B \to C, +0$$
$$(A \lor B) \to C, -0$$
$$A \lor B, +1$$
$$C, -1$$

$$A, +1 \qquad\qquad B, +1$$

$$A, -1 \quad C, +1 \qquad A, -1 \qquad C, +1$$
$$\otimes \qquad \otimes \qquad\qquad\qquad\qquad \otimes$$
$$B, -1 \quad C, +1$$
$$\otimes \qquad \otimes$$

(h) $A \to C \vdash_{K_*} (A \land B) \to C$

$$A \to C, +0$$
$$(A \land B) \to C, -0$$
$$A \land B, +1$$
$$C, -1$$
$$A, +1$$
$$B, +1$$

$$A, -1 \quad C, +1$$
$$\otimes \qquad \otimes$$

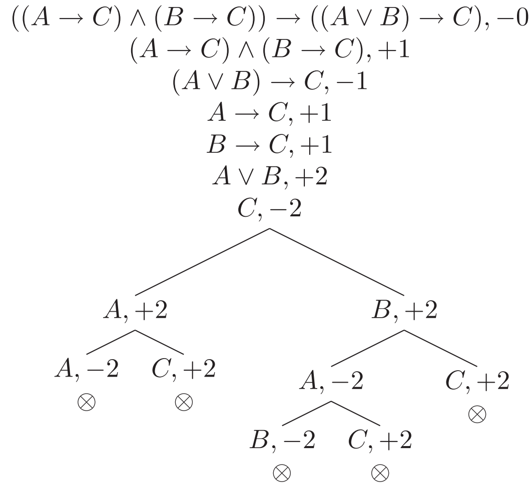
(i) $\vdash_{K_*} ((A \to B) \land (A \to C)) \to (A \to (B \land C))$

$$((A \to B) \land (A \to C)) \to (A \to (B \land C)), -0$$
$$(A \to B) \land (A \to C), +1$$
$$A \to (B \land C), -1$$
$$A \to B, +1$$
$$A \to C, +1$$
$$A, +2$$
$$B \land C, -2$$

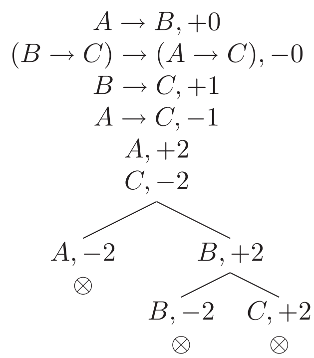
$$B, -2 \qquad\qquad C, -2$$

$$A, -2 \quad B, +2 \qquad A, -2 \qquad B, +2$$
$$\otimes \qquad \otimes \qquad \otimes$$
$$A, -2 \quad C, +2$$
$$\otimes \qquad \otimes$$

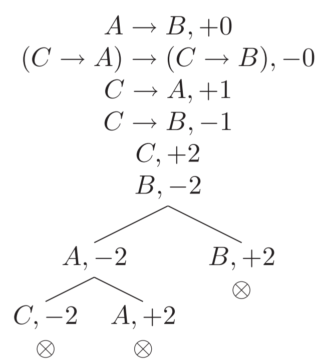
(j) $\vdash_{K_*} ((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$



(k) $A \to B \vdash_{K_*} (B \to C) \to (A \to C)$



(l) $A \to B \vdash_{K_*} (C \to A) \to (C \to B)$



(m) $A \to B, B \to C \vdash_{K_*} A \to C$

$$
\begin{array}{c}
A \to B, +0 \\
B \to C, +0 \\
A \to C, -0 \\
A, +1 \\
C, -1
\end{array}
$$

$$
\begin{array}{cc}
A, -1 & B, +1 \\
\otimes & \begin{array}{cc} B, -1 & C, +1 \\ \otimes & \otimes \end{array}
\end{array}
$$

Show that the following are not true in K_* and specify a counter-model.

(a) $\nvdash_{K_*} (p \wedge (\neg p \vee q)) \to q$

$$
\begin{array}{c}
(p \wedge (\neg p \vee q)) \to q, -0 \\
p \wedge (\neg p \vee q), +1 \\
q, -1 \\
p, +1 \\
\neg p \vee q, +1
\end{array}
$$

$$
\begin{array}{cc}
\neg p, +1 & q, +1 \\
p, -1^{\#} & \otimes
\end{array}
$$

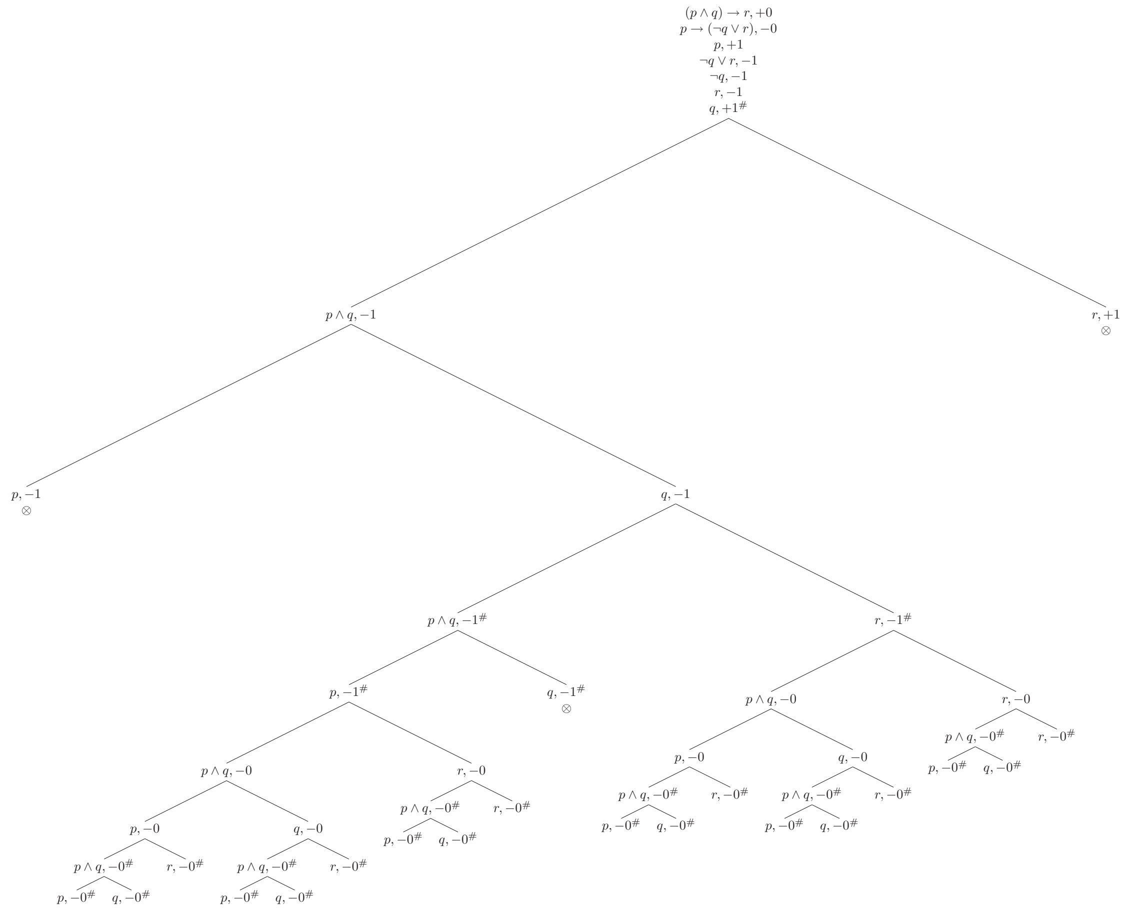
Counter-model such that:

$$W = \{w_0, w_1, w_0^*, w_1^*\}; v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_1^*}(p) = 0$$

Let us check that the interpretation works:

$v_{w_1}(p) = 1$, and $v_{w_1^*}(p) = 0$, making $v_{w_1}(p \wedge (\neg p \vee q)) = 1$. $v_{w_1}(q) = 0$. Therefore the truth-value of the conclusion is $v_{w_0}((p \wedge (\neg p \vee q)) \to q) = 0$

(b) $(p \wedge q) \rightarrow r \nvdash_{K_*} p \rightarrow (\neg q \vee r)$



Counter-model from the open left-most branch such that:

$$W = \{w_0, w_1, w_0^*, w_1^*\}; v_{w_0}(p) = 0, v_{w_0^*}(p) = 0, v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_1^*}(p) = 0, v_{w_1^*}(q) = 1, v_{w_1}(r) = 0$$

Let us check that the interpretation works:

$v_{w_0}(p) = 0$, $v_{w_0^*}(p) = 0$, $v_{w_1^*}(p) = 0$, and $v_{w_1}(q) = 0$, so $p \wedge q$ is false at every world. This means $v_{w_0}((p \wedge q) \rightarrow r) = 1$. However, $v_{w_1}(p) = 1$, while $v_{w_1}(r) = 0$, and $v_{w_1^*}(q) = 1$, so $v_{w_0}(p \rightarrow (\neg q \vee r)) = 0$.

(c) $\nvdash_{K_*} p \rightarrow (q \vee \neg q)$

$$\begin{gathered} p \rightarrow (q \vee \neg q), -0 \\ p, +1 \\ q \vee \neg q, -1 \\ q, -1 \\ \neg q, -1 \\ q, +1^{\#} \end{gathered}$$

Counter-model such that:

$$W = \{w_0, w_1, w_0^*, w_1^*\}; v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_1^*}(q) = 1$$

Let us check that the interpretation works:

p is true at w_1 and both q and $\neg q$ are not true at w_1, making the conclusion $p \rightarrow (q \vee \neg q)$ untrue at w_0.

(d) $\nvdash_{K_*} (p \wedge \neg p) \rightarrow q$

$$\begin{gathered} (p \wedge \neg p) \rightarrow q, -0 \\ p \wedge \neg p, +1 \\ q, -1 \\ p, +1 \\ \neg p, +1 \\ p, -1^{\#} \end{gathered}$$

Counter-model such that:

$$W = \{w_0, w_1, w_0^*, w_1^*\}; v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_1^*}(p) = 0$$

Let us check that the interpretation works:

p, and $\neg p$ are both true at w_1, making $p \wedge \neg p$ true at w_1. q is false at w_1 making the conclusion $(p \wedge \neg p) \rightarrow q$ false at w_0.

(e) $\vdash_{K_*} (A \to B) \to (\neg A \to \neg B)$

(The question asks us to show that this inference is invalid, however it is in fact valid.)

$$\begin{array}{c}
(A \to B) \to (\neg B \to \neg A), -0 \\
A \to B, +1 \\
\neg B \to \neg A, -1 \\
\neg B, +2 \\
\neg A, -2 \\
B, -2^{\#} \\
A, +2^{\#} \\
\begin{array}{cc}
A, -2^{\#} & B, +2^{\#} \\
\otimes & \otimes
\end{array}
\end{array}$$

Determine which of the inferences in problem 2 are valid in N_*. Where invalid, specify a counter-model for an instance.

(a) $\vdash_{N_*} A \to A$

$$\begin{array}{c}
A \to A, -0 \\
A, +1 \\
A, -1 \\
\otimes
\end{array}$$

(b) $\vdash_{N_*} A \leftrightarrow \neg\neg A$

$\vdash_{N_*} (A \to \neg\neg A) \wedge (\neg\neg A \to A)$

$$\begin{array}{c}
(A \to \neg\neg A) \wedge (\neg\neg A \to A), -0 \\
\begin{array}{cc}
A \to \neg\neg A, -0 & \neg\neg A \to A, -0 \\
A, +1 & \neg\neg A, +1 \\
\neg\neg A, -1 & A, -1 \\
\neg A, +1^{\#} & \neg A, -1^{\#} \\
A, -1 & A, +1 \\
\otimes & \otimes
\end{array}
\end{array}$$

(c) $\vdash_{N_*} (A \wedge B) \to A$

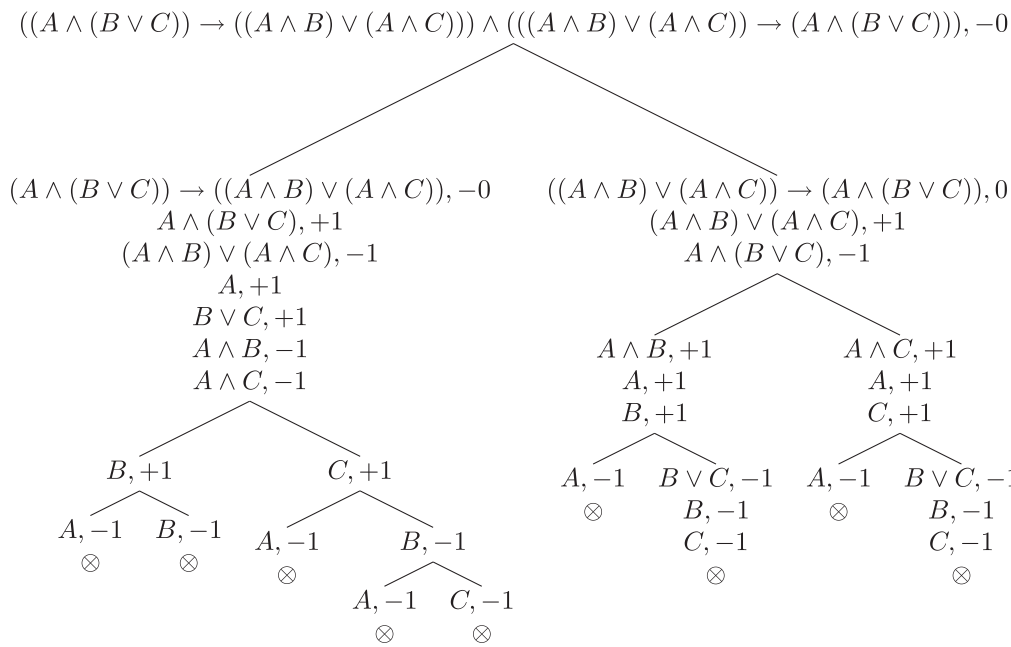
$$\begin{array}{c}
(A \wedge B) \to A, -0 \\
A \wedge B, +1 \\
A, -1 \\
A, +1 \\
B, +1 \\
\otimes
\end{array}$$

(d) $\vdash_{N_*} A \rightarrow (A \vee B)$

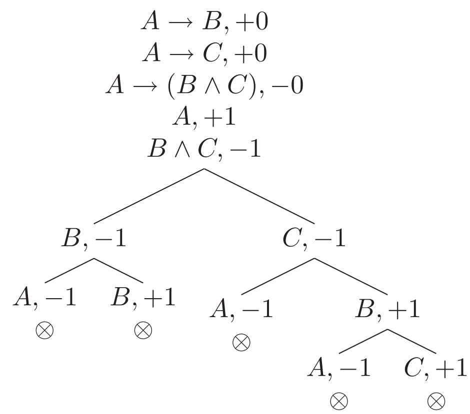
$$A \rightarrow (A \vee B), -0$$
$$A, +1$$
$$A \vee B, -1$$
$$A, -1$$
$$B, -1$$
$$\otimes$$

(e) $\vdash_{N_*} (A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$

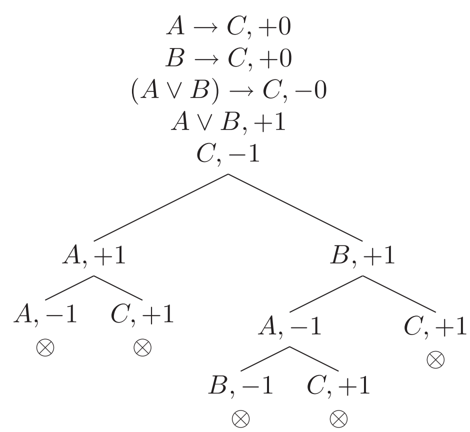
$\vdash_{N_*} ((A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))) \wedge (((A \wedge B) \vee (A \wedge C)) \rightarrow (A \wedge (B \vee C)))$



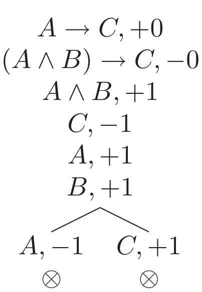
(f) $A \to B, A \to C \vdash_{N_*} A \to (B \wedge C)$



(g) $A \to C, B \to C \vdash_{N_*} (A \vee B) \to C$



(h) $A \to C \vdash_{N_*} (A \wedge B) \to C$



(i) $\nvdash_{N_*} ((p \to q) \wedge (p \to r)) \to (p \to (q \wedge r))$

$$((p \to q) \wedge (p \to r)) \to (p \to (q \wedge r)), -0$$
$$(p \to q) \wedge (p \to r), +1$$
$$p \to (q \wedge r), -1$$
$$p \to q, +1$$
$$p \to r, +1$$

Counter-model such that:

$$W = \{w_0, w_0^*, w_1, w_1^*\}; N = \{w_0\}; v(p \to q)_{w_1} = 1, v(p \to r)_{w_1} = 1, v(p \to (q \wedge r))_{w_1} = 0$$

This can be represented in the following diagram:

w_0	w_1	w_0^*	w_1^*
	$+p \to q$		
	$+p \to r$		
	$-p \to (q \wedge r)$		

Let us check that the interpretation works:

$p \to q$ and $p \to r$ are true at non-normal w_1, making the antecedent $(p \to q) \wedge (p \to r)$ true at w_1. The consequent $p \to (q \wedge r)$ is not true at w_1, meaning that the conclusion $((p \to q) \wedge (p \to r)) \to (p \to (q \wedge r))$ is not true at w_0.

(j) $\nvdash_{N_*} ((p \to r) \wedge (q \to r)) \to ((p \vee q) \to r)$

$$((p \to r) \wedge (q \to r)) \to ((p \vee q) \to r), -0$$
$$(p \to r) \wedge (q \to r), +1$$
$$(p \vee q) \to r, -1$$
$$p \to r, +1$$
$$q \to r, +1$$

Counter-model such that:

$$W = \{w_0, w_0^*, w_1, w_1^*\}; N = \{w_0\}; v(p \to r)_{w_1} = 1, v(q \to r)_{w_1} = 1, v((p \vee q) \to r)_{w_1} = 0$$

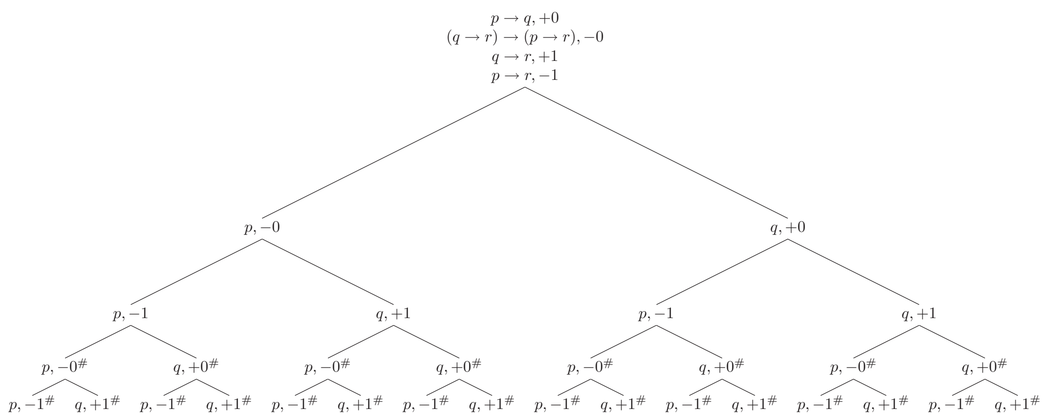
This can be represented in the following diagram:

w_0	w_1	w_0^*	w_1^*
	$+p \to r$		
	$+q \to r$		
	$-(p \vee q) \to r$		

Let us check that the interpretation works:

$p \rightarrow r$ and $q \rightarrow r$ are true at non-normal w_1, making the antecedent $(p \rightarrow r) \wedge (q \rightarrow r)$ true at w_1. The consequent $(p \vee q) \rightarrow r$ is not true at w_1, meaning that the conclusion $((p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow ((p \vee q) \rightarrow r)$ is not true at w_0.

(k) $p \rightarrow q \nvdash_{N_*} (q \rightarrow r) \rightarrow (p \rightarrow r)$



Counter-model from the open left-hand branch such that:

$$W = \{w_0, w_0^*, w_1, w_1^*\}; N = \{w_0\}$$
$$v(p)_{w_0} = 0, v(p)_{w_1} = 0, v(p)_{w_0^*} = 0, v(p)_{w_1^*} = 0, v(q \rightarrow r)_{w_1} = 1, v(p \rightarrow r)_{w_1} = 0$$

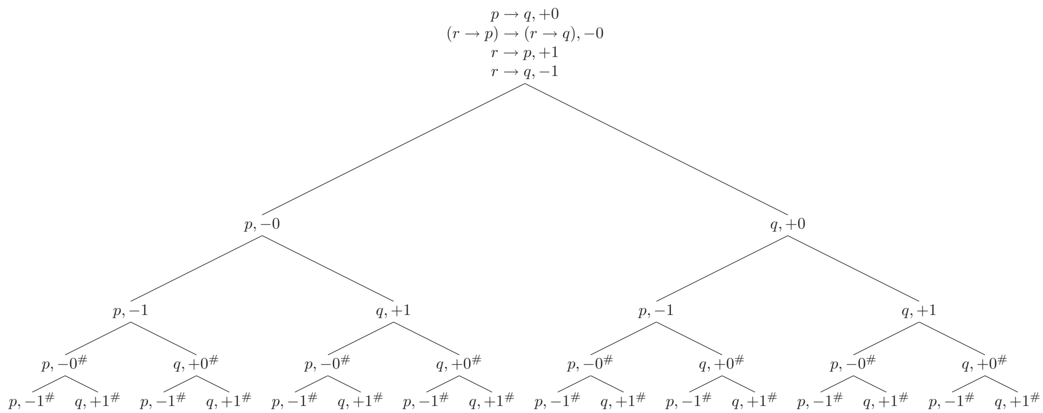
This can be represented in the following diagram:

w_0	w_1	w_0^*	w_1^*
$-p$	$-q \rightarrow r$	$-p$	$-p$
	$+p \rightarrow r$		
	$-p$		

Let us check that the interpretation works:

p is not true at all worlds, so the premise $p \rightarrow q$ is true at w_0. $q \rightarrow r$ is true and $p \rightarrow r$ is not true at non-normal w_1, making the conclusion $(q \rightarrow r) \rightarrow (p \rightarrow r)$ untrue at w_0.

(l) $p \rightarrow q \nvdash_{N_*} (r \rightarrow p) \rightarrow (r \rightarrow q)$



Counter-model from the open left-hand branch such that:

$W = \{w_0, w_1, w_0^*, w_1^*\}; N = \{w_0\}; v(p)_{w_0} = 0, v(p)_{w_1} = 0, v(p)_{w_0^*} = 0, v(p)_{w_1^*} = 0, v(r \rightarrow p)_{w_1} = 1, v(r \rightarrow q)_{w_1} =$

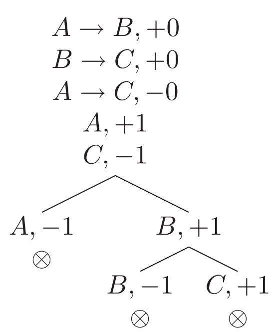
This can be represented in the following diagram:

w_0	w_1	w_0^*	w_1^*
$-p$	$-r \rightarrow q$	$-p$	$-p$
	$+r \rightarrow p$		
	$-p$		

Let us check that the interpretation works:

p is not true at all worlds, so the premise $p \rightarrow q$ is true at w_0. $r \rightarrow p$ is true and $r \rightarrow q$ is not true at non-normal w_1, making the conclusion $(r \rightarrow p) \rightarrow (r \rightarrow q)$ untrue at w_0.

(m) $A \rightarrow B, B \rightarrow C \vdash_{N_*} A \rightarrow C$



6. In the semantics for N_4 and N_*, there may be many normal worlds, but the tableaux show us that it suffices to suppose that there is only one normal world. Why is this?

By the completeness theorem, if an inference is invalid, the tableaux for it does not close and the counter-model this gives has only one normal world. Hence if an inference is invalid, it has a counter-model with only one normal world. On the other hand, if an inference is valid, it is truth preserving at the normal world of all interpretations with only one normal world. Hence, it makes no difference to which inferences are made valid if we define interpretations to have only one normal world.

9. Show by induction that in any interpretation, $< W, R, \rho >$ for I_4, I_3, or W, for any formula A:

If $A\rho_w 1$ and wRw' then $A\rho_{w'} 1$

If $A\rho_w 0$ and wRw' then $A\rho_{w'} 0$

For the basis case, where A is a propositional parameter, the result follows simply from the heredity condition, which all three logics subscribe to.

We then have four induction cases: for A of the form $\neg B$, $B \wedge C$, $B \vee C$, and $B \sqsupset C$

In the following, let us suppose that wRw'.

$\boxed{\neg B}$

This case will be the same for all three logics, as they all follow the same rules for negation.

Suppose that $\neg B\rho_w 1$, so by the rule for negation $B\rho_w 0$. By induction hypothesis and the heredity condition, $B\rho_{w'} 0$. Hence by the negation rule $\neg B\rho_{w'} 1$, as required.

Suppose that $\neg B\rho_w 0$, so by the rule for negation $B\rho_w 1$. By induction hypothesis and the heredity condition, $B\rho_{w'} 1$. Hence by the negation rule $\neg B\rho_{w'} 0$, as required.

$\boxed{B \wedge C}$

This case will be the same for all three logics, as they all follow the same rules for conjunction.

Suppose that $B \wedge C\rho_w 1$, so by the rule for conjunction $B\rho_w 1$, and $C\rho_w 1$. By induction hypothesis and the heredity condition, $B\rho_{w'}1$ and $C\rho_{w'}1$. Hence by the conjunction rule $B \wedge C\rho_{w'}1$, as required.

Suppose that $B \wedge C\rho_w 0$, so by the rule for conjunction $B\rho_w 0$, or $C\rho_w 0$. By induction hypothesis and the heredity condition, $B\rho_{w'}0$ or $C\rho_{w'}0$. Hence by the conjunction rule $B \wedge C\rho_{w'}0$, as required.

$\boxed{B \vee C}$

This case will be the same for all three logics, as they all follow the same rules for disjunction.

Suppose that $B \vee C\rho_w 1$, so by the rule for disjunction $B\rho_w 1$, or $C\rho_w 1$. By induction hypothesis and the heredity condition, either $B\rho_{w'}1$ or $C\rho_{w'}1$. Hence in either case by the disjunction rule $B \vee C\rho_{w'}1$, as required.

Suppose that $B \vee C\rho_w 0$, so by the rule for disjunction $B\rho_w 0$, and $C\rho_w 0$. By induction hypothesis and the heredity condition, both $B\rho_{w'}0$ and $C\rho_{w'}0$. Hence by the disjunction rule $B \vee C\rho_{w'}0$, as required.

$\boxed{B \sqsupset C}$

Contrapositive proof: Suppose that it is not the case that $B \sqsupset C\rho_{w'}1$. Then for some w'' such that $w'Rw''$, $B\rho_{w''}1$ and it is not the case that $C\rho_{w''}1$. By transitivity wRw'', so it is not the case that $B \sqsupset C\rho_{w'}1$, as required.

The following argument only holds for I_3 and I_4.

Suppose that $B \sqsupset C\rho_w 0$, so by the rule for the negated conditional, $B\rho_w 1$ and $C\rho_w 0$. By the heredity condition and induction hypothesis, $B\rho_{w'}1$ and $C\rho_{w'}0$, hence $B \sqsupset C\rho_{w'}0$, as required

There is a separate case for W:

Contrapositive proof: Suppose it is not the case that that $B \sqsupset C\rho_{w'}0$. Then it is not the case that $B \sqsupset \neg C\rho_{w'}1$. So there is some w'' such that $B\rho_{w''}1$ and it is not the case that $\neg C\rho_{w''}1$. By transitivity, wRw'', and so it is not the case that $B \sqsupset \neg C\rho_w 1$. Hence it is not the case that $B \sqsupset C\rho_w 0$.

10. Determine the truth of the following in I_4, I_3, and the connexive logic W. Where invalid, give a counter-model.

As a general point, it will be helpful to note that tableaux for I_4 and I_3 only differ if they contain nodes of the form $A, +i$ and $A, -i$, and that I_4 only differs from W on tableaux with nodes of the form $\neg(A \sqsupset B)$.

(a) $\vdash \neg(p \wedge q) \sqsupset (\neg p \vee \neg q)$

$\boxed{I_4, I_3, W}$

$$\begin{array}{c} \neg(p \wedge q) \sqsupset (\neg p \vee \neg q), -0 \\ 0r0,\ 0r1,\ 1r1 \\ \neg(p \wedge q), +1 \\ \neg p \vee \neg q, -1 \\ \neg p, -1 \\ \neg q, -1 \\ \neg p \vee \neg q, +1 \\ \begin{array}{cc} \swarrow & \searrow \\ \neg p, +1 & \neg q, +1 \\ \otimes & \otimes \end{array} \end{array}$$

(b) $\vdash \neg(p \vee q) \sqsupset (\neg p \wedge \neg q)$

$\boxed{I_4, I_3, W}$

$$\begin{array}{c} \neg(p \vee q) \sqsupset (\neg p \wedge \neg q), -0 \\ 0r0,\ 0r1,\ 1r1 \\ \neg(p \vee q), +1 \\ \neg p \wedge \neg q, -1 \\ \neg p \wedge \neg q, +1 \\ \otimes \end{array}$$

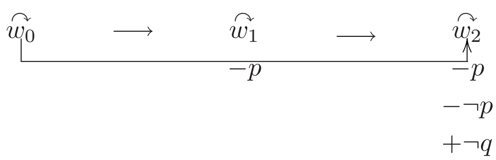
(c) $\nvdash (p \supset q) \supset (\neg q \supset \neg p)$

$\boxed{I_4, W}$

$$(p \supset q) \supset (\neg q \supset \neg p), -0$$
$$0r0,\ 0r1,\ 1r1$$
$$p \supset q, +1$$
$$\neg q \supset \neg p, -1$$
$$1r2,\ 0r2,\ 2r2$$
$$\neg q, +2$$
$$\neg p, -2$$

$p, -2$		$q, +2$	
$p, -1$	$q, +1$	$p, -1$	$q, +1$

Counter-model from open left-most branch such that:

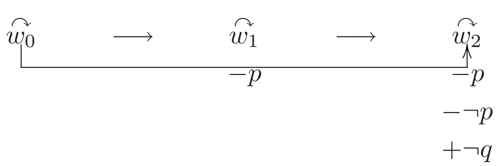


$\boxed{I_3}$

$$(p \supset q) \supset (\neg q \supset \neg p), -0$$
$$0r0,\ 0r1,\ 1r1$$
$$p \supset q, +1$$
$$\neg q \supset \neg p, -1$$
$$1r2,\ 0r2,\ 2r2$$
$$\neg q, +2$$
$$\neg p, -2$$

$p, -2$		$q, +2$
$p, -1$	$q, +1$	$\otimes$

Counter-model from open left-most branch such that:



(d) $\nvdash p \vee \neg p$

$\boxed{I_4, I_3, W}$

$$p \vee \neg p, -0$$
$$0r0$$
$$p, -0$$
$$\neg p, -0$$

Counter-model such that:

$$\overset{\frown}{w_0}$$
$$-p$$
$$-\neg p$$

(e) $\nvdash (\neg p \sqsupset p) \sqsupset p$

$\boxed{I_4, I_3, W}$

$$(\neg p \sqsupset p) \sqsupset p, -0$$
$$0r0,\ 0r1,\ 1r1$$
$$\neg p \sqsupset p, +1$$
$$p, -1$$

$\neg p, -1$	$p, +1$
	$\otimes$

Counter-model such that:

$$\overset{\frown}{w_0} \longrightarrow \overset{\frown}{w_1}$$

w_1: $-p$, $-\neg p$

(f) $\vdash \neg(p \sqsupset q) \sqsupset (p \sqsupset \neg q)$

$\boxed{I_4, I_3}$

$$\neg(p \sqsupset q) \sqsupset (p \sqsupset \neg q), -0$$
0r0, 0r1, 1r1
$$\neg(p \sqsupset q), +1$$
$$p \sqsupset \neg q, -1$$
1r2, 2r2, 0r2
$$p, +2$$
$$\neg q, -2$$
$$p, +1$$
$$\neg q, +1$$
$$\neg q, +2$$
$$\otimes$$

$\boxed{W}$

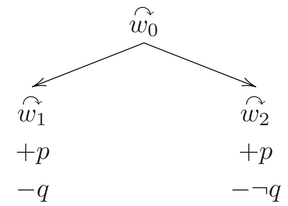
$$\neg(p \sqsupset q) \sqsupset (p \sqsupset \neg q), -0$$
0r0, 0r1, 1r1
$$\neg(p \sqsupset q), +1$$
$$p \sqsupset \neg q, -1$$
$$p \sqsupset \neg q, +1$$
$$\otimes$$

(g) $\nvdash (p \sqsupset q) \vee (p \sqsupset \neg q)$

$\boxed{I_4, I_3, W}$

$$(p \sqsupset q) \vee (p \sqsupset \neg q), -0$$
0r0
$$p \sqsupset q, -0$$
$$p \sqsupset \neg q, -0$$
0r1, 1r1
$$p, +1$$
$$q, -1$$
0r2, 2r2
$$p, +2$$
$$\neg q, -2$$

Counter-model such that:



(h) $\vdash \neg((p \wedge \neg p) \sqsupset (p \vee \neg p))$

$\boxed{I_4, I_3}$

$\neg((p \wedge \neg p) \sqsupset (p \vee \neg p)), -0$

$0r0$

$p \wedge \neg p, -0$ | $\neg(p \vee \neg p), -0$

$p, -0$ | $\neg p, -0$ | $\neg p \wedge \neg\neg p, -0$

$\neg p, -0$ | $\neg\neg p, -0$

$p, -0$

Counter-model from open left-most branch such that:

$\widehat{w_0}$

$-p$

$\boxed{W}$

$\neg((p \wedge \neg p) \sqsupset (p \vee \neg p)), -0$

$0r0$

$(p \wedge \neg p) \sqsupset \neg(p \vee \neg p), -0$

$0r1$, $1r1$

$p \wedge \neg p, +1$

$\neg(p \vee \neg p), -1$

$p, +1$

$\neg p, +1$

$\neg p \wedge \neg\neg p, -1$

$\neg p, -1$ | $\neg\neg p, -1$

$\otimes$ | $p, -1$

$\otimes$

11. Work out the details omitted in 9.7a.8, 9.7a.11, and 9.7a.14.

9.7a.8 Show that for every formula, A, $v_w(p) = 1$ iff $p\rho_w 1$.

Basis case is for propositional parameters, and is true by definition.

$\boxed{B \wedge C}$

$$\begin{aligned} v_w(B \wedge C) = 1 \quad & \text{iff } v_w(B) = 1 \text{ and } v_w(C) = 1 \\ & \text{iff } B\rho_w 1 \text{ and } C\rho_w 1 \\ & \text{iff } B \wedge C\rho_w 1 \end{aligned}$$

$\boxed{B \vee C}$

$$\begin{aligned} v_w(B \wedge C) = 1 \quad & \text{iff } v_w(B) = 1 \text{ or } v_w(C) = 1 \\ & \text{iff } B\rho_w 1 \text{ or } C\rho_w 1 \\ & \text{iff } B \vee C\rho_w 1 \end{aligned}$$

$\boxed{B \sqsupset C}$

$$\begin{aligned} v_w(B \sqsupset C) = 1 \quad & \text{iff} \quad \text{for all } w' \text{ such that } wRw', \text{ either } v_{w'}(B) = 0 \text{ or } v_w(C) = 1 \\ & \text{iff} \quad \text{for all } w' \text{ such that } wRw', \text{ either it is not the case that} \\ & \qquad v_{w'}(B) = 1 \text{ or } v_w(C) = 1 \text{ (since } v \text{ semantics are two valued)} \\ & \text{iff} \quad \text{for all } w' \text{ such that } wRw', \text{ either it is not the case that } B\rho_{w'}0 \text{ or } C\rho_{w'}1 \\ & \text{iff} \quad B \sqsupset C\rho_w 1 \end{aligned}$$

9.7a.11 Check that Aristotle and Boethius are valid in W.

$\vdash \neg(A \sqsupset \neg A)$

$$\begin{array}{c} \neg(A \sqsupset \neg A), -0 \\ 0r0 \\ A \sqsupset \neg\neg A, -0 \\ 0r1,\ 1r1 \\ A, +1 \\ \neg\neg A, -1 \\ A, -1 \\ \otimes \end{array}$$

$\vdash (A \sqsupset B) \sqsupset \neg(A \sqsupset \neg B)$

$$
\begin{array}{c}
(A \sqsupset B) \sqsupset \neg(A \sqsupset \neg B), -0 \\
0r0,\ 0r1,\ 1r1 \\
A \sqsupset B, +1 \\
\neg(A \sqsupset \neg B), -1 \\
A \sqsupset \neg\neg B, -1 \\
1r2,\ 2r2,\ 0r2 \\
A, +2 \\
\neg\neg B, -2 \\
B, -2 \\
\swarrow \qquad \searrow \\
A, -2 \qquad B, +2 \\
\otimes \qquad\quad \otimes
\end{array}
$$

9.7a.14 Show that the class of logical truths in W is inconsistent by showing that $(p \wedge \neg p) \sqsupset \neg(p \wedge \neg p)$ is valid.

$\vdash_W (p \wedge \neg p) \sqsupset \neg(p \wedge \neg p)$

$$
\begin{array}{c}
(p \wedge \neg p) \sqsupset \neg(p \wedge \neg p), -0 \\
0r0,\ 0r1,\ 1r1 \\
p \wedge \neg p, +1 \\
\neg(p \wedge \neg p), -1 \\
\neg p \vee \neg\neg p, -1 \\
p, +1 \\
\neg p, +1 \\
\neg p, -1 \\
\neg\neg p, -1 \\
p, -1 \\
\otimes
\end{array}
$$

12. Show that in I_3 and I_4:

(a) if $\vDash A \vee B$ then $\vDash A$ or $\vDash B$. (Hint: see 6.10, problem 5.)

(b) if $\vDash \neg(A \wedge B)$ then $\vDash \neg A$ or $\vDash \neg B$.

(a) Show that if $\vDash A \vee B$ then $\vDash A$ or $\vDash B$.

Contrapositive proof. Suppose that $\nvDash A$ and $\nvDash B$. Then there is an interpretation, I_A, and a world, w_A, in the interpretation, where A is not true. Similarly for I_B and w_B. We can assume that the worlds of the two interpretations are distinct.

Now let us construct the interpretation, I, whose worlds are those of I_A and I_B, which relate to each other exactly as they do in those interpretations, and where the truth values at each world are the same as those in those interpretations. In addition, there is one new world, w, such that w relates to itself, w_A, w_B, and all the worlds that w_A and w_B relate to. Further all propositional parameters relate to neither 0 nor 1 at w. (Thus for any propositional parameter p, if relates to 0 or 1 in w, it relates to 0 or 1 in w_A and w_B, and heredity is vacuously satisfied.)

This is an interpretation for both I_3 and I_4. Every formula has the same truth value at w_A in I and I_A, since we have not done anything to change these. Similarly, every formula has the same truth value at w_B in I and I_B. Now suppose that $\models A \vee B$. Then $A \vee B$ is true at w, so either A or B is true at w. If it is A, then by heredity, A is true at w_A, which it is not; similarly for B. Hence, $\not\models A \vee B$.

(b) Show that if $\models \neg(A \wedge B)$ then $\models \neg A$ or $\models \neg B$.

Suppose that $\models \neg(A \wedge B)$. Then $\models \neg A \vee \neg B$. So, by part (a), $\models \neg A$ or $\models \neg B$.

13. Find an inference that is valid in I_4, but not in intuitionist logic. Find an inference that is valid in intuitionist logic, but not in I_4. (Hint, see 9.6.9.)

$\models_{I_4} \neg\neg A \supset A$

(Shown in 9.7a.5)

$\not\models_I \rightharpoondown\rightharpoondown p \supset p$

(Shown in 6.4.11)

$\vdash_I (p \wedge \rightharpoondown p) \supset q$

$$(p \wedge \rightharpoondown p) \supset q, -0$$
$$0r0,\ 0r1,\ 1r1$$
$$p \wedge \rightharpoondown p, +1$$
$$q, -1$$
$$p, +1$$
$$\rightharpoondown p, +1$$
$$p, -1$$
$$\otimes$$

$\not\models_{I_4} (p \wedge \neg p) \supset q$

(Shown in 9.7a.5)

15.* Fill in the details omitted in 9.8.

9.8.3: Cases for $\neg(A \to B), +i$ and $\neg(A \to B), -i$ in the Soundness Lemma for K_4.

Suppose that we apply the rule to $\neg(A \to B), +i$. Then by assumption, $\neg(A \to B)$ is true, and $A \to B$ false at $f(i)$. Hence there is some w such that A is true at w and B is false. Let f' be the same as f, except that $f'(j) = w$. Then f' shows I to be faithful to the extended branch.

Suppose that we apply the rule to $\neg(A \to B), -i$. Then by assumption, $\neg(A \to B)$ is not true, and $A \to B$ not false at $f(i)$. Hence for any j on the branch, either A is not true or B is not false at$f(j)$. In the first case f shows I to be faithful to the left-hand branch, in the second it shows I to be faithful to the right-hand branch.

9.8.6: Cases for negated $\to$ in the Completeness Lemma for K_4.

Suppose that $\neg(B \to C), +i$ is on b. Then there is a j such that $B, +j$ and $\neg C, +j$ are on b. By induction hypothesis, B is true at w_j and C is false at w_j. Thus $B \to C$ is false, and $\neg(B \to C)$ true at w_i, as required.

Suppose that $\neg(B \to C), -i$ is on b. Then for all j, either $B, -j$, or $\neg C, -j$ is on b. By induction hypothesis, either B is not true at w_j, or $\neg C$ is not true (and hence C is not false) at w_j. In either case, $B \to C$ is not false at w_i, so $\neg(B \to C)$ is not true at w_i, as required.

9.8.13: Soundness and Completeness for N_*

The proof modifies the proof for K_*

Soundness Theorem: The tableau system for N_* is sound with respect to its semantics.

Proof:
The proof is exactly the same as for K_*, except that in the definition of faithfulness, we add the clause: $f(0) \epsilon N$. In the Soundness Lemma, the rules for $\rightarrow$ are applied only to 0, and $f(0)$ is normal.

Completeness Theorem: The tableau system for N_* is complete with respect to its semantics.

Proof:
This proof modifies that for K_* as follows. $W = \{w_x : x \text{ or } \overline{x} \text{ occurs on } b\}$, $N = \{w_0\}$, $w_i^* = w_{i\#}$ and $w_{i\#}^* = w_i$; If irj is on b then $f(i)Rf(j)$. Further, for every parameter, p:

$v_{w_i}(p) = 1$ if $p, +i$ occurs on b

$v_{w_i}(p) = 0$ if $p, -i$ occurs on b

And for every formula $A \rightarrow B$ and $i > 0$:

$v_{w_i}(A \rightarrow B) = 1$ if $A \rightarrow B, +i$ occurs on b

$v_{w_i}(A \rightarrow B) = 0$ if $A \rightarrow B, -i$ occurs on b

The rest of the proof is then as for K_*. Only the induction cases for $\rightarrow$ in the Completeness Lemma are different. In these, if w_i is normal, the arguments are exactly the same. If w_i is non-normal, the result holds simply by definition.

9.8.16: Case for $-$ in the Soundness and Completeness Lemmas for W.

Soundness:

Suppose that we apply the rule to $\neg(A \sqsupset B), +i$. By assumption, $A \sqsupset B$ is false at $f(i)$. So $A \sqsupset \neg B$ is true at $f(i)$, as required.

Suppose that we apply the rule to $\neg(A \sqsupset B), -i$. By assumption, $A \sqsupset B$ is not true at $f(i)$. So $A \sqsupset \neg B$ is not true at $f(i)$, as required.

Completeness:

Suppose that $\neg(A \sqsupset B), +i$ is on the branch. Then so is $A \sqsupset \neg B, +i$, and for every j such that irj is on the branch, either $A, -j$ or $\neg B, +j$ is on the branch. By construction and induction hypothesis, for all w_j such that $w_i R w_j$, either A is not true at w_j or B is false there. Hence, $A \sqsupset B$ is false at w_i.

Suppose that $\neg(A \sqsupset B), -i$ is on the branch. Then $A \sqsupset \neg B, -i$ is on the branch. Hence, for all j such that irj are on the branch, either $A, -j$ or $\neg B, -j$ are not on the branch. By induction hypothesis, for all w_j such that $w_i R w_j$, either A is not true at w_j, or B is not false there. Hence $A \sqsupset B$ is not false at w_i.

16.* Design tableaux for the systems of 9.7.14, and prove them sound and complete.

There are three restrictions discussed in 9.7.14: Logics with no gluts, no gaps, and neither gluts nor gaps. I will deal with them in turn.

1. No gluts

A simple way to rule out gluts, is to make $\rho_@$ satisfy the *exclusion* condition. In other words, set up the logic such that it is not the case that for any propositional parameter p, $p\rho_@ 1$ and $p\rho_@ 0$.

Further, to rule out truth value gluts while accommodating formulas containing $\rightarrow$, we need to modify its falsity conditions as follows:

$A \rightarrow B\rho_@ 0$ iff (for some w', $A\rho_{w'}1$ and $B\rho_{w'}0$) and (it is not the case that $A \rightarrow B\rho_@ 1$

Now we need new tableaux rules to reflect the semantic definitions. For simplicity this logic will be based on K_4. In K_4 all worlds are related to one another, so there will be no mention of the accessibility relation.

For the exclusion restriction:

Closure rule: Branches close if they contain lines of the form $A, +0$ and $\neg A, +0$.

And for the new falsity conditions for $\rightarrow$:

Tableaux rule:

$$\begin{array}{c} \neg(A \rightarrow B), +0 \\ \downarrow \\ A \rightarrow B, -0 \\ A, +i \\ \neg B, +i \end{array}$$

Tableaux rule:

$$\begin{array}{ccc} & \neg(A \rightarrow B), -0 & \\ \swarrow & \downarrow & \searrow \\ A \rightarrow B, +0 & A, -i & \neg B, -i \end{array}$$

The first rule is applied for an i new to the branch; the second rule is applied for all i on the branch.

We must now show the new rules to be sound and complete. The Soundness and Completeness proofs for this logic modify those for K_4 (9.8.2 - 9.8.7).

Soundness:

We must first modify the definition of faithfulness, and the statement of the Soundness Lemma, and then produce new arguments for the new rules in the Soundness Lemma.

- *Definition:*

Let $I = < W, \rho, @ >$ be any relational interpretation and b be any branch of a tableau. Then I is faithful to b iff there is a map, f, from the natural numbers to W such that:

for every node $A, +i$ on b, $A\rho_{f(i)}1$ in I.

for every node $A, -i$ on b, it is not the case that $A\rho_{f(i)}1$ in I.

Further, $f(0) = @$.

- *Soundness Lemma:*

Let b be any branch of a tableau, and $I = < W, \rho, @ >$ be any interpretation for this logic. If I is faithful to b, and a tableau rule is applied to it, then it produces at least one extension, b', such that I is faithful to b'.

- *Modifications to the proof:*

Let f be a function which shows I to be faithful to b.

Suppose that we apply the rule to $\neg(A \to B), +0$. Then by assumption, $A \to B$ is false at $f(0)$, that is, @. By the falsity conditions for $\to$ at @, (for some w, A is true at w, and B is false at w) and (it is not the case that $A \to B$ is true at @). Let f' be the same as f, except $f'(i) = w$. Then f' shows I to be faithful to the extended branch.

Suppose that we apply the rule to $\neg(A \to B), -0$. Then by assumption, $A \to B$ is not false at $f(0)$, that is, @. By the falsity conditions for $\to$ at @, either (for every w, A is not true at w or B is not false at w) or ($A \to B$ is true at @). Whichever is the case, f shows I to be faithful to one of the extended branches.

Completeness:

We must now modify the definition of the induced interpretation, and the statement of the Completeness Lemma, and then check that the induced interpretation is how we want it to be for the new cases:

- *Definition:*

Let b be an open branch of a tableau. The interpretation $I = < W, \rho, @ >$, induced by b is defined as in 9.8.5. $W = \{w_i : i < 0 \text{ occurs on } b\}$. $w_0 = @$. For every parameter, p:

$p\rho_{w_i}1$ iff $p, +i$ occurs on b

$p\rho_{w_i}0$ iff $\neg p, +i$ occurs on b

The new closure rule ensures that the interpretation meets the exclusion restraint.

- *Completeness Lemma:*

Let b be any open completed branch of a tableau. Let $I = < W, \rho, @ >$ be the interpretation induced by b. Then:

if $A, +i$ is on b, then A is true at w_i
if $A, -i$ is on b, then it is not the case that A is true at w_i.
if $\neg A, +i$ is on b, then A is false at w_i
if $\neg A, -i$ is on b, then it is not the case that A is false at w_i

• *Modifications to the proof:*

Suppose that $\neg(A \to B), +0$ is on b, then the following are on b: $A \to B, -0$, $A, +i$, and $\neg B, +i$, for some i. By construction and induction hypothesis, $A \to B$ is not true at @, and for some w_i, A is true at w_i, and B is false at w_i. Hence $A \to B$ is false at @, i.e., $\neg(A \to B)$ is true.

Suppose that $\neg(A \to B), -0$ is on b, then either $A \to B, +0$ is on b, or for every i on b, either $A, -i$ is on b, or $\neg B, -i$ is on b. By construction and induction hypothesis, either $A \to B$ is true at @, or for every w_i, either A is not true at w_i, or B is not false there. Hence $A \to B$ is not false at @.

■

2. No gaps

A simple way to rule out gaps, is to make $\rho_@$ satisfy the *exhaustion* condition. In other words, set up the logic such that for all p either $p\rho 1$ or $p\rho 0$.

Further, to rule out truth value gaps while accommodating formulas containing $\to$, we need to modify its falsity conditions as follows:

$A \to B\rho_@ 0$ iff (for some w', $A\rho_{w'}1$ and $B\rho_{w'}0$) or (it is not the case that $A \to B\rho_@ 1$

Now we need new tableaux rules to reflect the semantic definitions. For simplicity this logic will also be based on K_4, so as all worlds are related to one another, there will be no mention of the accessibility relation.

For the exhaustion condition:

Closure rule: Branches close if they contain lines of the form $A, -0$ and $\neg A, -0$.

And for the new falsity conditions for $\rightarrow$:

Tableaux rule:

$$\begin{array}{ccc} & \neg(A \rightarrow B), +0 & \\ \swarrow & & \searrow \\ A, +i & & A \rightarrow B, -0 \\ \neg B, +i & & \end{array}$$

Tableaux rule:

$$\begin{array}{ccc} & \neg(A \rightarrow B), -0 & \\ & A \rightarrow B, +0 & \\ \swarrow & & \searrow \\ A, -i & & \neg B, -i \end{array}$$

The first rule is applied for an i new to the branch; the second rule is applied for all i on the branch.

We must now show the new rules to be sound and complete. The Soundness and Completeness proofs for this logic modify those for LP (8.7.9).

Soundness:

We must first modify the definition of faithfulness, and the statement of the Soundness Lemma, and then produce new arguments for the new rules in the Soundness Lemma.

- *Definition:*

Let $I = < W, \rho, @ >$ be any relational interpretation and b be any branch of a tableau. Then I is faithful to b iff there is a map, f, from the natural numbers to W such that:

for every node $A, +i$ on b, $A\rho_{f(i)}1$ in I.

for every node $A, -i$ on b, it is not the case that $A\rho_{f(i)}1$ in I.

Further, $f(0) = @$

- *Soundness Lemma:*

Let b be any branch of a tableau, and $I = < W, \rho, @ >$ be any interpretation for this logic. If I is faithful to b, and a tableau rule is applied to it, then it produces at least one extension, b', such that I is faithful to b'.

- *Modifications to the proof:*

Let f be a function which shows I to be faithful to b.

Suppose that we apply the rule to $\neg(A \rightarrow B), +0$. Then, by assumption, $A \rightarrow B$ is false at $f(0)$, that is, @. By the falsity conditions for $\rightarrow$ at @, (for some w, A is true at w and B is false at w), or (it is not the case that $A \rightarrow B$ is true at @). Let f' be the same as f, except that $f'(i) = w$. Then f' shows I to be faithful to one of the extended branches.

Suppose that we apply the rule to $\neg(A \rightarrow B), -0$. Then, by assumption, $A \rightarrow B$ is not false at $f(0)$, that is, @. By the falsity conditions for $\rightarrow$ at @, (for every w, either A is not true at w, or B is not false at w) and ($A \rightarrow B$ is true at @). Whichever is the case, f shows I to be faithful to one of the extended branches.

Completeness:

We must now modify the definition of the induced interpretation, and the statement of the Completeness Lemma, and then check that the induced interpretation is how we want it to be for the new cases:

- *Definition:*

Let b be an open branch of a tableau. The interpretation $I = < W, \rho, @ >$ induced by b is defined as in 8.7.9. $W = \{w_i : i < 0 \text{ occurs on } b\}$. $w_0 = @$. For every parameter, p:

$p\rho_{w_i}1$ iff $p, -i$ is not on b

$p\rho_{w_i}0$ iff $\neg p, -i$ is not on b

The new closure rule implies that one or the other of these must hold, so the exhaustion condition holds for this interpretations of this logic.

- *Completeness Lemma:*

Let b be any open completed branch of a tableau. Let $I = < W, \rho, @ >$ be the interpretation induced by b. Then:

if $A, +i$ is on b, then A is true at w_i
if $A, -i$ is on b, then it is not the case that A is true at w_i.
if $\neg A, +i$ is on b, then A is false at w_i
if $\neg A, -i$ is on b, then it is not the case that A is false at w_i

- *Modifications to the proof:*

Suppose that $\neg(A \to B), +0$ is on b, then either $A \to B, -0$ is on b or, for some i, $A, +i$ and $\neg B, +i$ are on b. By construction and induction hypothesis, either $A \to B$ is not true at @, or for some w_i, A is true at w_i and B is false there. Hence $A \to B$ is false at @.

Suppose that $\neg(A \to B), -0$ is on b, then $A \to B, +0$ is on b, and for every i on b, either $A, -i$ is on b or $\neg B, -i$ is on b. By construction and induction hypothesis, $A \to B$ is true at @, and for all w_i, either A is not true at w_i, or B is not false there. Hence $A \to B$ is not false at @.

■

3. No gluts or gaps

A simple way to rule out gluts and gaps in @, is to take a star logic and specify that $@ = @^*$.

We then need tableaux rules to reflect this semantic constraint:

Tableaux rule:

$$\begin{array}{c} A, +0 \\ \downarrow \\ A, +0^{\#} \end{array}$$

Tableaux rule:

$$\begin{array}{c} A, +0^{\#} \\ \downarrow \\ A, +0 \end{array}$$

No new closure rules are necessary.

We must now show the new rules to be sound and complete. The Soundness and Completeness proofs for this logic modify those for K_* (9.8.11 - 8.8.12)

Soundness:

We must first modify the definition of faithfulness, and the statement of the Soundness Lemma, and then produce new arguments for the new rules in the Soundness Lemma.

- *Definition:*

Let $I =< W, *, v, @ >$ be any Routley interpretation, and b be any branch of a tableau. Then I is faithful to b iff there is a map, f, from the natural numbers to W, such that:

$f(0) = @$

for every node $A, +x$ on b, A is true at $f(x)$ in I

for every node $A, -x$ on b, A is false at $f(x)$ in I

where $f(i^{\#})$ is, by definition $f(i)^*$.

- *Soundness Lemma:*

Let b be any branch of a tableau, and $I =< W, *, v, @ >$ be any Routley interpretation. If I is faithful to b, and a tableau rule is applied, then it produces at least one extension, b', such that I is faithful to b'.

- *Modifications to the proof:*

Let f be a function which shows I to be faithful to b.

Suppose that we apply the rule to $A, +0$. By induction hypothesis, $v_{f(0)}(A) = 1$. So $v_@(A) = 1$, and since $@ = @^*$, $v_{@^*}(A) = 1$. That is, $v_{f(0)^*}(A) = 1$ and $v_{f(0^{\#})}(A) = 1$, as required.

Suppose that we apply the rule to $A, +0^{\#}$. By induction hypothesis, $v_{f(0^{\#})}(A) = 1$. So $v_{f(0)^*}(A) = 1$, and $v_{@^*}(A) = 1$. Since $@ = @^*$, $v_@(A) = 1$, and $v_{f(0)}(A) = 1$, as required.

Completeness:

We must now modify the definition of the induced interpretation, and the statement of the Completeness Lemma, and then check that the induced interpretation is how we want it to be for the new cases:

- *Definition:*

Let b be an open branch of a tableau. The interpretation, $I =< W, *, v, @ >$ induced by b is defined as in 9.8.12, except in the case where $x = 0$: $W = \{w_0\} \cup \{w_x : x > 0$ and either x or $\bar{x}$ occurs on $b\}$. Additionally $w_0 = @$ and $@ = @^*$.

v is such that:

$v_{w_x}(p) = 1$ if $p, +x$ is on b

$v_{w_x}(p) = 0$ if $p, -x$ is on b

Note also that by the definition of $*$, $w_x^{**} = w_x$ and by the definition of @, $@^{**} = @$ i.e. the induced interpretation is a Routley interpretation.

- *Completeness Lemma:*

Let b be any open completed branch of a tableau. Let $I =< W, *, v, @ >$ be the interpretation induced by b. Then:

if $A, +x$ is on b, A is true at w_x

if $A, -x$ is on b, A is false at w_x

- *Modifications to the proof:*

Since $@^*$ has been redefined, we need new arguments for negation when $x = 0$:

If $\neg B, +0$ is on b, then $B, -0^{\#}$ is on b. By the new rule, $B, -0$ is on b. By induction hypothesis, B is false at $w_0 = @$. Hence B is false at $@^*$, and $\neg B$ is true at @, as required.

If $\neg B, -0$ is on b, then $B, +0^{\#}$ is on b. By the new rule, $B, +0$ is on b. By induction hypothesis, B is true at $w_0 = @$. Hence B is true at $@^*$, and $\neg B$ is false at @ as required.

■