Solutions

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1. Using the tableau procedure of 8.3.4 determine whether or not the following are true in FDE. If the inference is invalid, specify a relational counter-model.

(a) $p \wedge q \vdash p$

$$\begin{array}{c} p \wedge q, + \\ p, - \\ p, + \\ q, + \\ \otimes \end{array}$$

(b) $p \vdash p \vee q$

$$\begin{array}{c} p, + \\ p \vee q, - \\ p, - \\ q, - \\ \otimes \end{array}$$

(c) $p \wedge (q \wedge r) \vdash (p \wedge q) \vee (p \wedge r)$

$$\begin{array}{c} p \wedge (q \wedge r), + \\ (p \wedge q) \vee (p \wedge r), - \\ p \wedge q, - \\ p \wedge r, - \\ p, + \\ q \wedge r, + \\ \begin{array}{cc} \begin{array}{c} p, - \\ \otimes \end{array} & \begin{array}{c} q, - \\ \begin{array}{cc} \begin{array}{c} p, - \\ \otimes \end{array} & \begin{array}{c} r, - \\ \begin{array}{cc} \begin{array}{c} q, + \\ \otimes \end{array} & \begin{array}{c} r, + \\ \otimes \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array}$$

(d) $p \wedge (q \vee r) \vdash (p \vee q) \wedge (p \vee r)$

$$\begin{array}{c} p \vee (q \wedge r), + \\ (p \vee q) \wedge (p \vee r) - \\ \begin{array}[t]{cc} \begin{array}[t]{c} p, + \\ \begin{array}[t]{cc} \begin{array}[t]{c} p \vee q, - \\ p, - \\ q, - \\ \otimes \end{array} & \begin{array}[t]{c} p \vee r, - \\ p, - \\ r, - \\ \otimes \end{array} \end{array} \end{array} & \begin{array}[t]{c} q \wedge r, + \\ q, + \\ r, + \\ \begin{array}[t]{cc} \begin{array}[t]{c} p \vee q, - \\ p, - \\ q, - \\ \otimes \end{array} & \begin{array}[t]{c} p \vee r, - \\ p, - \\ r, - \\ \otimes \end{array} \end{array} \end{array} \end{array} \end{array}$$

(e) $p \vdash \neg\neg p$

$$\begin{array}{c} p, + \\ \neg\neg p, - \\ p, - \\ \otimes \end{array}$$

(f) $\neg\neg p \vdash p$

$$\begin{array}{c} \neg\neg p, + \\ p, - \\ p, + \\ \otimes \end{array}$$

(g) $(p \wedge q) \supset r \vdash (p \wedge \neg r) \supset \neg q$

$\neg(p \wedge q) \vee r \vdash \neg(p \wedge \neg r) \vee \neg q$

$$\begin{array}{c} \neg(p \wedge q) \vee r, + \\ \neg(p \wedge \neg r) \vee \neg q, - \\ \neg(p \wedge \neg r), - \\ \neg q, - \\ \neg p \vee \neg\neg r, - \\ \neg p, - \\ \neg\neg r, - \\ r, - \\ \begin{array}[t]{cc} \begin{array}[t]{c} \neg(p \wedge q), + \\ \neg p \vee \neg q, + \\ \begin{array}[t]{cc} \begin{array}[t]{c} \neg p, + \\ \otimes \end{array} & \begin{array}[t]{c} \neg q, + \\ \otimes \end{array} \end{array} \end{array} & \begin{array}[t]{c} r, + \\ \otimes \end{array} \end{array} \end{array}$$

(h) $p \wedge \neg p \vdash p \vee \neg p$

$$\begin{array}{c} p \wedge \neg p, + \\ p \vee \neg p, - \\ p, - \\ \neg p, - \\ p, + \\ \neg p, + \\ \otimes \end{array}$$

(i) $p \wedge \neg p \nvdash q \vee \neg q$

$$\begin{array}{c} p \wedge \neg p, + \\ q \vee \neg q, - \\ q, - \\ \neg q, - \\ p, + \\ \neg p, + \end{array}$$

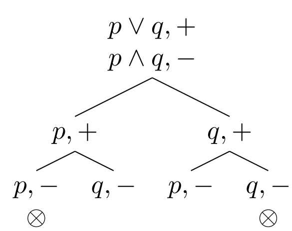
There is a counter-model with the following relations obtaining, and no others:

$$p\rho 1, p\rho 0$$

Let us check that this counter-model works:

$p\rho 1$ and $p\rho 0$, so the antecedent is true, but it is not the case that $q\rho 1$ or $q\rho 0$, so the consequent is not true.

(j) $p \vee q \nvdash p \wedge q$



The counter-model from the left-most open branch has the following relations obtaining, and no others:

$$p\rho 1$$

Let us check that this counter-model works:

$p\rho 1$, so the premise $p \vee q\rho 1$. It is not the case that $q\rho 1$, so it is not the case that $p \wedge q\rho 1$.

(k) $p, \neg(p \wedge \neg q) \nvdash q$

$$
\begin{array}{c}
p, + \\
\neg(p \wedge \neg q), + \\
q, - \\
\neg p \vee \neg\neg q, + \\
\begin{array}{cc}
\neg p, + & \neg\neg q, + \\
 & q, + \\
 & \otimes
\end{array}
\end{array}
$$

There is a counter-model with the following relations obtaining, and no others:

$$p\rho 1, p\rho 0$$

Let us check that this counter-model works:

$p\rho 1$, so the first premise is true, $p\rho 0$, so $p \wedge q\rho 0$, and the second premise $\neg(p \wedge q)\rho 1$. But it is not the case that $q\rho 1$, so the conclusion is not true.

(l) $(p \wedge q) \supset r \vdash p \supset (\neg q \vee r)$

$\neg(p \wedge q) \vee r \vdash \neg p \vee (\neg q \vee r)$

$$
\begin{array}{c}
\neg(p \wedge q) \vee r, + \\
\neg p \vee (\neg q \vee r), - \\
\neg p, - \\
\neg q \vee r, - \\
\neg q, - \\
r, - \\
\begin{array}{cc}
\begin{array}{c}
\neg(p \wedge q), + \\
\neg p \vee \neg q, + \\
\begin{array}{cc}
\neg p, + & \neg q, + \\
\otimes & \otimes
\end{array}
\end{array}
&
\begin{array}{c}
r, + \\
\otimes \\
\\
\\
\end{array}
\end{array}
\end{array}
$$

2. For the inferences of problem 1 that are invalid, determine which ones are valid in K_3 and LP using the appropriate tableaux.

$\boxed{K_3}$

(i) $p \wedge \neg p \vdash_{K_3} q \vee \neg q$

$$\begin{array}{c} p \wedge \neg p, + \\ q \vee \neg q, - \\ q, - \\ \neg q, - \\ p, + \\ \neg p, + \\ \otimes \end{array}$$

(j) $p \vee q \nvdash_{K_3} p \wedge q$

$$\begin{array}{c} p \vee q, + \\ p \wedge q, - \\ \begin{array}{cc} \begin{array}{c} p, + \\ \begin{array}{cc} p, - & q, - \\ \otimes & \end{array} \end{array} & \begin{array}{c} q, + \\ \begin{array}{cc} p, - & q, - \\ & \otimes \end{array} \end{array} \end{array} \end{array}$$

The counter-model from the left-most open branch has the following relations obtaining, and no others:

$$p\rho 1$$

Let us check that this counter-model works:

$p\rho 1$, so the premise $p \vee q\rho 1$. It is not the case that $q\rho 1$, so it is not the case that $p \wedge q\rho 1$.

(k) $p, \neg(p \wedge \neg q) \vdash_{K_3} q$

$$\begin{array}{c} p, + \\ \neg(p \wedge \neg q), + \\ q, - \\ \neg p \vee \neg\neg q, + \\ \begin{array}{cc} \begin{array}{c} \neg p, + \\ \otimes \\ \end{array} & \begin{array}{c} \neg\neg q, + \\ q, + \\ \otimes \end{array} \end{array} \end{array}$$

$\boxed{LP}$

(i) $p \wedge \neg p \vdash_{LP} q \vee \neg q$

$$\begin{array}{c} p \wedge \neg p, + \\ q \vee \neg q, - \\ q, - \\ \neg q, - \\ \otimes \end{array}$$

(j) $p \vee q \nvdash_{LP} p \wedge q$

$$\begin{array}{c} p \vee q, + \\ p \wedge q, - \\ \begin{array}{cc} \begin{array}{c} p, + \\ \begin{array}{cc} \begin{array}{c} p, - \\ \otimes \end{array} & \begin{array}{c} q, - \\ \end{array} \end{array} \end{array} & \begin{array}{c} q, + \\ \begin{array}{cc} \begin{array}{c} p, - \\ \end{array} & \begin{array}{c} q, - \\ \otimes \end{array} \end{array} \end{array} \end{array} \end{array}$$

Counter-model from the left-most open branch with the following relations obtaining, and no others:

$$p\rho 1$$

Let us check that this counter-model works:

$p\rho 1$, so the premise $p \vee q\rho 1$. It is not the case that $q\rho 1$ so it is not the case that the conclusion $p \wedge q\rho 1$.

(k) $p, \neg(p \wedge \neg q) \nvdash_{LP} q$

$$\begin{array}{c} p, + \\ \neg(p \wedge \neg q), + \\ q, - \\ \neg p \vee \neg\neg q, + \\ \begin{array}{cc} \begin{array}{c} \neg p, + \\ \\ \end{array} & \begin{array}{c} \neg\neg q, + \\ q, + \\ \otimes \end{array} \end{array} \end{array}$$

Counter-model from the left-most open branch with the following relations obtaining, and no others:

$$p\rho 1, p\rho 0$$

Let us check that this counter-model works:

$p\rho 1$, so the first premise is true. $p\rho 0$, so $p \wedge \neg q \rho 0$, and the second premise $\neg(p \wedge \neg q)\rho 1$. It is not the case that $q\rho 1$ so the conclusion is not true.

3. Check all the details omitted in 8.4.2.

$\boxed{f_\neg}$

Suppose that the truth value of A is 1. Then A is true and not false, so $\neg A$ is false and not true, as required.

Suppose that the truth value of A is b. Then A is both true and false. So $\neg A$ is also both true and false, as required.

Suppose that the truth value of A is n. Then A is neither true nor false. So $\neg A$ is also neither true nor false, as required.

Suppose that the truth value of A is 0. Then A is false and not true, so $\neg A$ is true and not false, as required.

$\boxed{f_\wedge}$

For the bottom row and far right column of the truth-table (where one or both of A and B takes 0):

Suppose that A or B is 0. Then A or B is false and not true. Because A or B is false, $A \wedge B$ is false, and because A or B is not true, $A \wedge B$ is not true. So $A \wedge B$ is false and not true: 0.

For the rest of the truth-table:

Suppose that A is 1, and B is 1. Then A and B are both true and not false. So $A \wedge B$ is true and not false: 1.

Suppose that A is 1, and B is b. Then A is true and not false, while B is both true and false. So $A \wedge B$ is true (because A and B are both true), and false (because B is false): b.

Suppose that A is 1 and B is n. Then A is true and not false, while B is neither true nor false. So $A \wedge B$ is not true, because B is not true, and not false, because B is not false: n.

Suppose that A is b, and B is 1. Then A is true and false, and B is true and not false. So $A \wedge B$ is true (because both A and B are true) and false (because A is false): b.

Suppose that A is b, and B is b. Then both A and B are both true and false. So $A \wedge B$ is both true and false: b.

Suppose that A is b, and B is n. Then A is true and false, and B is neither true nor false. So $A \wedge B$ is false, because A is false, and not true, because B is not true: 0.

Suppose that A is n, and B is 1. Then A is neither true nor false, and B is true and not false. Since A is not true $A \wedge B$ is not true. Since neither conjunct is false, $A \wedge B$ is not false either: n.

Suppose that A is n, and B is b. Then A is neither true nor false, and B is true and false. Since A is not true $A \wedge B$ is not true. Since B is false, $A \wedge B$ is false: 0

Suppose that A is n, and B is n. Then A and B are neither true nor false. Thus $A \wedge B$ is neither true nor false: n.

$\boxed{f_\vee}$

For the bottom row and far-left column of the truth-table (where one or both of A and B take 1):

Suppose that A or B is 1. Then A or B is true and not false. Because A or B is true, $A \vee B$ is true, and because A or B is not false, $A \vee B$ is not false. So $A \wedge B$ is true and not false: 1.

For the rest of the truth-table:

Suppose that A is b, and B is b. Then A and B are both true and false. So $A \wedge B$ is both true and false: b.

Suppose that A is b, and B is n. Then A is true and false, while B is neither true nor false. So $A \vee B$ is true (because A is true), and not false (because B is not false): 1.

Suppose that A is b and B is 0. Then A is true and false, while B is false and not true. So $A \vee B$ is true, because A is true, and also false, because both A and B are false: b .

Suppose that A is n, and B is b. Then A is neither true nor false, and B is true and false. Since A is not false $A \vee B$ is not false. Since B is true, $A \wedge B$ is true: 1

Suppose that A is n, and B is n. Then A and B are both neither true nor false. Thus $A \vee B$ is neither true nor false: n.

Suppose that A is n, and B is 0. Then A is neither true nor false, and B is false and not true. Since A is not false $A \vee B$ is not false. Since neither conjunct is true, $A \vee B$ is not true either: n.

Suppose that A is 0, and B is b. Then A is false and not true, and B is both true and false. Since B is true, $A \vee B$ is true. And since both A and B are false, $A \vee B$ is also false: b.

Suppose that A is 0, and B is n. Then A is false and not true, and B is neither true nor false. Since B is not false $A \vee B$ is not false. Since neither A nor B is true, $A \vee B$ is not true: n.

Suppose that A is 0, and B is 0. Then both A and B are false and not true. Since both are false, $A \vee B$ is false. Since neither are true, $A \vee B$ is not true: 0.

4. By checking the truth tables of 8.4.2, note that if A and B have truth value n, then so do $A \vee B$, $A \wedge B$ and $\neg A$. Infer that if A is any formula all of whose propositional parameters take the value n, it, too, takes the value n. Hence infer that there is no formula A such that $\vDash_{FDE} A$.

For the first inference, no work is really needed:

The truth tables provide us with the fact that if A and B have truth value n, then so do $A \vee B$, $A \wedge B$ and $\neg A$. Since the only connectives in FDR are $\vee$ $\wedge$ and $\neg$, (see the definition of $\supset$ in 8.2.1), the the conclusion quickly follows.

For the second inference, Take any formula A. Assign each parameter that occurs in it the value n. It follows that A takes the value n, which is not designated. So A is not a logical truth.

5. Similarly, show that if A is a formula all of whose propositional parameters take the value b, then A takes the value b. Hence, show that if A and B have no propositional parameters in common, $A \nvDash_{FDE} B$. (Hint: Assign all the parameters in A the value b, and all the parameters in B the value n.)

As in 4, the first part follows from looking at the truth table, and the fact that the only connectives in FDE are $\wedge$, $\vee$, and $\neg$.

For the second part, Take any formulas A and B. Assign each parameter in A the value b, and each parameter in B the value n. (This is possible because A and B have no parameters in common.) Then A takes the value b, and B

takes the value n. So $A \nvDash_{FDE} B$.

6. Repeat problem 1 with the * semantics and tableaux of 8.5.

(a) $p \wedge q \vdash p$

$$\begin{array}{c} p \wedge q, +0 \\ p, -0 \\ p, +0 \\ q, +0 \\ \otimes \end{array}$$

(b) $p \vdash p \vee q$

$$\begin{array}{c} p, +0 \\ p \vee q, -0 \\ p, -0 \\ q, -0 \\ \otimes \end{array}$$

(c) $p \wedge (q \wedge r) \vdash (p \wedge q) \vee (p \wedge r)$

$$\begin{array}{c} p \wedge (q \wedge r), +0 \\ (p \wedge q) \vee (p \vee r), -0 \\ p \wedge q, -0 \\ p \wedge r, -0 \\ p, +0 \\ q \vee r, +0 \end{array}$$

$$\begin{array}{ccc} p, -0 & & q, -0 \\ \otimes & & \end{array}$$

$$\begin{array}{ccc} p, -0 & & r, -0 \\ \otimes & & \end{array}$$

$$\begin{array}{cc} q, +0 & r, +0 \\ \otimes & \otimes \end{array}$$

(d) $p \wedge (q \vee r) \vdash (p \vee q) \wedge (p \vee r)$

$$
\begin{array}{c}
p \vee (q \wedge r), +0 \\
(p \vee q) \wedge (p \vee r) - 0
\end{array}
$$

$$
\begin{array}{cc}
\begin{array}{c}
p, +0 \\
\begin{array}{cc}
\begin{array}{c} p \vee q, -0 \\ p, -0 \\ q, -0 \\ \otimes \end{array} &
\begin{array}{c} p \vee r, -0 \\ p, -0 \\ r, -0 \\ \otimes \end{array}
\end{array}
\end{array}
&
\begin{array}{c}
q \wedge r, +0 \\
q, +0 \\
r, +0 \\
\begin{array}{cc}
\begin{array}{c} p \vee q, -0 \\ p, -0 \\ q, -0 \\ \otimes \end{array} &
\begin{array}{c} p \vee r, -0 \\ p, -0 \\ r, -0 \\ \otimes \end{array}
\end{array}
\end{array}
\end{array}
$$

(e) $p \vdash \neg\neg p$

$$
\begin{array}{c}
p, +0 \\
\neg\neg p, -0 \\
\neg p, +0^{\#} \\
p, -0 \\
\otimes
\end{array}
$$

(f) $\neg\neg p \vdash p$

$$
\begin{array}{c}
\neg\neg p, +0 \\
p, -0 \\
\neg p, -0^{\#} \\
p, +0 \\
\otimes
\end{array}
$$

(g) $(p \wedge q) \supset r \vdash (p \wedge \neg r) \supset \neg q$

$\neg(p \wedge q) \vee r \vdash \neg(p \wedge \neg r) \vee \neg q$

$$
\begin{array}{c}
\neg(p \wedge q) \vee r, +0 \\
\neg(p \wedge \neg r) \vee \neg q, -0 \\
\neg(p \wedge \neg r), -0 \\
\neg q, -0 \\
p \wedge \neg r, +0^{\#} \\
p, +0^{\#} \\
\neg r, +0^{\#} \\
r, -0 \\
\begin{array}{cc}
\begin{array}{c}
\neg(p \wedge q), +0 \\
p \wedge q, -0^{\#} \\
\begin{array}{cc}
\begin{array}{c} p, -0^{\#} \\ \otimes \end{array} &
\begin{array}{c} q, -0^{\#} \\ q, +0^{\#} \\ \otimes \end{array}
\end{array}
\end{array} &
\begin{array}{c} r, +0 \\ \otimes \end{array}
\end{array}
\end{array}
$$

(h) $p \wedge \neg p \vdash p \vee \neg p$

$$
\begin{array}{c}
p \wedge \neg p, +0 \\
p \vee \neg p, -0 \\
p, -0 \\
\neg p, -0 \\
p, +0 \\
\neg p, +0 \\
\otimes
\end{array}
$$

(i) $p \wedge \neg p \nvdash q \vee \neg q$

$$
\begin{array}{c}
p \wedge \neg p, +0 \\
q \vee \neg q, -0 \\
q, -0 \\
\neg q, -0 \\
p, +0 \\
\neg p, +0 \\
q, +0^{\#} \\
p, -0^{\#}
\end{array}
$$

Counter-model such that:

$$W = \{w_0, w_0^*\}; v_{w_0}(p) = 1, v_{w_0^*}(p) = 0, v_{w_0}(q) = 0, v_{w_0^*}(q) = 1$$

Let us check that this counter-model works:

$v_{w_0}(p) = 1$ and $v_{w_0^*}(p) = 0$ therefore the premise $v_{w_0}(p \wedge \neg p) = 1$. However, $v_{w_0}(q) = 0$ and $v_{w_0^*}(q) = 1$, so the conclusion $v_{w_0}(q \vee \neg q) = 0$.

(j) $p \vee q \nvdash p \wedge q$

$$\begin{array}{cccc} \multicolumn{4}{c}{p \vee q, +0} \\ \multicolumn{4}{c}{p \wedge q, -0} \\ \multicolumn{2}{c}{p, +0} & \multicolumn{2}{c}{q, +0} \\ p, -0 & q, -0 & p, -0 & q, -0 \\ \otimes & & & \otimes \end{array}$$

Counter-model from the left-most open branch such that:

$$W = \{w_0, w_0^*\}; v_{w_0}(p) = 1, v_{w_0}(q) = 0$$

Let us check that this counter-model works:

$v_{w_0}(p) = 1$, so the premise $v_{w_0}(p \vee q) = 1$. $v_{w_0}(q) = 0$ so the conclusion $v_{w_0}(p \wedge q) = 0$.

(k) $p, \neg(p \wedge \neg q) \nvdash q$

$$\begin{array}{cc} \multicolumn{2}{c}{p, +0} \\ \multicolumn{2}{c}{\neg(p \wedge \neg q), +0} \\ \multicolumn{2}{c}{q, -0} \\ \multicolumn{2}{c}{p \wedge \neg q, -0^{\#}} \\ p, -0^{\#} & \neg q, -0^{\#} \\ & q, +0 \\ & \otimes \end{array}$$

Counter-model such that:

$$W = \{w_0, w_0^*\}; v_{w_0}(p) = 1, v_{w_0^*}(p) = 0, v_{w_0}(q) = 0$$

Let us check that this counter-model works:

$v_{w_0}(p) = 1$, so the first premise is designated. $v_{w_0^*}(p) = 0$, so $v_{w_0^*}(p \wedge \neg q) = 0$. Accordingly $v_{w_0} \neg(p \wedge \neg q) = 1$. However $v_{w_0}(q) = 0$; the conclusion is undesignated.

(l) $(p \wedge q) \supset r \vdash p \supset (\neg q \vee r)$

$\neg(p \wedge q) \vee r \vdash \neg p \vee (\neg q \vee r)$

$$
\begin{array}{c}
\neg(p \wedge q) \vee r, +0 \\
\neg p \vee (\neg q \vee r), -0 \\
\neg p, -0 \\
\neg q \vee r, -0 \\
\neg q, -0 \\
r, -0 \\
p, +0^{\#} \\
q, +0^{\#} \\
\begin{array}{cc}
\begin{array}{c}
\neg(p \wedge q), +0 \\
p \wedge q, -0^{\#} \\
\begin{array}{cc}
p, -0^{\#} & q, -0^{\#} \\
\otimes & \otimes
\end{array}
\end{array}
&
\begin{array}{c}
r, +0 \\
\otimes \\
\\
\\
\end{array}
\end{array}
\end{array}
$$

7. Using the * semantics, show that if $A \vDash_{FDE} B$, then $\neg B \vDash_{FDE} \neg A$. (Hint: Assume that there is a counter-model for the consequent.) Why is this not obvious with the many-valued or the relational semantics? (Note that contraposition of this kind does not extend to multiple premise inferences: $p, q \vDash_{FDE} p$, but $p, \neg p \nvDash_{FDE} \neg q$.)

Contrapositive proof:

Suppose that $\neg B \nvDash_{FDE} \neg A$. Then for some interpretation $< W, *, v >$, and some $w \in W$, $v_w(\neg B) = 1$, and $v_w(\neg A) = 0$. Hence, $v_{w^*}(A) = 1$, and $v_{w^*}(B) = 0$. So, $A \nvDash_{FDE} B$.

The proof is not obvious with the relational semantics because there is no easy way of getting from the fact that there is a ρ such that $B\rho 0$ and it is not the case that $A\rho 0$ to the existence of a ρ', such that $A\rho' 1$ and it is not the case that $B\rho' 1$.

8. Test the validity of the inferences in 7.5.2 using the tableaux of this chapter.

$\boxed{K_3}$

(1) $q \vDash_{K_3} p \supset q$

$q \vDash_{K_3} \neg p \vee q$

$$\begin{array}{c} q, + \\ \neg p \vee q, - \\ \neg p, - \\ q, - \\ \otimes \end{array}$$

(2) $\neg p \vDash_{K_3} p \supset q$

$\neg p \vDash_{K_3} \neg p \vee q$

$$\begin{array}{c} \neg p, + \\ \neg p \vee q, - \\ \neg p, - \\ q, - \\ \otimes \end{array}$$

(3) $(p \wedge q) \supset r \vDash_{K_3} (p \supset q) \vee (q \supset r)$

$\neg(p \wedge q) \vee r \vDash_{K_3} (\neg p \vee q) \vee (\neg q \vee r)$

$$\begin{array}{c} \neg(p \wedge q) \vee r, + \\ (\neg p \vee q) \vee (\neg q \vee r), - \\ \neg p \vee q, - \\ \neg q \vee r, - \\ \neg p, - \\ q, - \\ \neg q, - \\ r, - \\ \begin{array}{cc} \neg(p \vee q), + & r, + \\ \neg p \wedge \neg q, + & \otimes \\ \neg p, + & \\ \neg q, + & \\ \otimes & \end{array} \end{array}$$

(4) $(p \supset q) \wedge (r \supset s) \vDash_{K_3} (p \supset s) \vee (r \supset q)$

$(\neg p \vee q) \wedge (\neg r \vee s) \vDash_{K_3} (\neg p \vee s) \vee (\neg r \vee q)$

$$
\begin{array}{c}
(\neg p \vee q) \wedge (\neg r \vee s), + \\
(\neg p \vee s) \vee (\neg r \vee q), - \\
\neg p \vee s, - \\
\neg r \vee q, - \\
\neg p \vee q, + \\
\neg r \vee s, + \\
\neg p, - \\
s, - \\
\neg r, - \\
q, - \\
\begin{array}{cc}
\neg p, + & q, + \\
\otimes & \otimes
\end{array}
\end{array}
$$

(5) $\neg(p \supset q) \vDash_{K_3} p$

$\neg(\neg p \vee q) \vDash_{K_3} p$

$$
\begin{array}{c}
\neg(\neg p \vee q), + \\
p, - \\
\neg\neg p \wedge \neg q, + \\
\neg\neg p, + \\
\neg q, + \\
p, + \\
\otimes
\end{array}
$$

(6) $p \supset r \vDash_{K_3} (p \wedge q) \supset r$

$\neg p \vee r \vDash_{K_3} \neg(p \wedge q) \vee r$

$$
\begin{array}{c}
\neg p \vee r, + \\
\neg(p \wedge q) \vee r, - \\
\neg(p \wedge q), - \\
r, - \\
\neg p \vee \neg q, - \\
\neg p, - \\
\neg q, - \\
\begin{array}{cc}
\neg p, + & r, + \\
\otimes & \otimes
\end{array}
\end{array}
$$

(7) $p \supset q, q \supset r \vDash_{K_3} p \supset r$

$\neg p \vee q, \neg q \vee r \vDash_{K_3} \neg p \vee r$

$$\begin{array}{c} \neg p \vee q, + \\ \neg q \vee r, + \\ \neg p \vee r, - \\ \neg p, - \\ r, - \\ \begin{array}{cc} \neg p, + & q, + \\ \otimes & \begin{array}{cc} \neg q, + & r, + \\ \otimes & \otimes \end{array} \end{array} \end{array}$$

The middle branch closes because of the K_3 closure rule.

(8) $p \supset q \vDash_{K_3} \neg q \supset \neg p$

$\neg p \vee q \vDash_{K_3} \neg\neg q \vee \neg p$

$$\begin{array}{c} \neg p \vee q, + \\ \neg\neg q \vee \neg p, - \\ \neg\neg q, - \\ \neg p, - \\ q, - \\ \begin{array}{cc} \neg p, + & q, + \\ \otimes & \otimes \end{array} \end{array}$$

(9) $\nvDash_{K_3} p \supset (q \vee \neg q)$

$\nvDash_{K_3} \neg p \vee (q \vee \neg q)$

$$\begin{array}{c} \neg p \vee (q \vee \neg q), - \\ \neg p, - \\ q \vee \neg q, - \\ q, - \\ \neg q, - \end{array}$$

Counter-model such that nothing is related to anything.

Let us check that the induced interpretation works:

It is not the case that $p\rho 0$ and not the case that $q\rho 1$ or $q\rho 0$ therefore both of the disjuncts are untrue, and so is the conclusion.

(10)$\not\models_{K_3} (p \wedge \neg p) \supset q$

$\not\models_{K_3} \neg(p \wedge \neg p) \vee q$

$$\begin{array}{c}
\neg(p \wedge \neg p) \vee q, - \\
\neg(p \wedge \neg p), - \\
q, - \\
\neg p, \vee \neg\neg p, - \\
\neg p, - \\
\neg\neg p - \\
p, -
\end{array}$$

Counter-model such that nothing is related to anything.

Let us check that the induced interpretation works:

It is not the case that $p\rho 1$, and it is not the case that $\neg p\rho 0$, so it is not the case that $p \wedge \neg p\rho 0$, so it is not the case that $\neg(p \wedge \neg p)\rho 1$. It is also not the case that $q\rho 1$, therefore it is not the case that $\neg(p \wedge \neg p) \vee q\rho 1$.

$\boxed{Ł_3}$

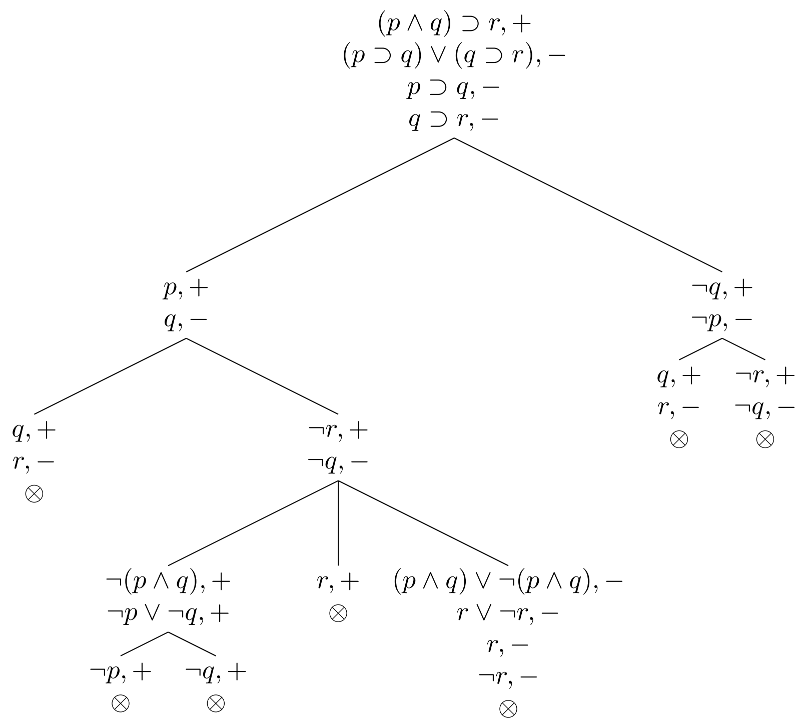
(1) $q \models_{Ł_3} p \supset q$

$$\begin{array}{c}
q, + \\
p \supset q, - \\
\begin{array}{cc}
p, + & \neg q, + \\
q, - & \neg p, - \\
\otimes & \otimes
\end{array}
\end{array}$$

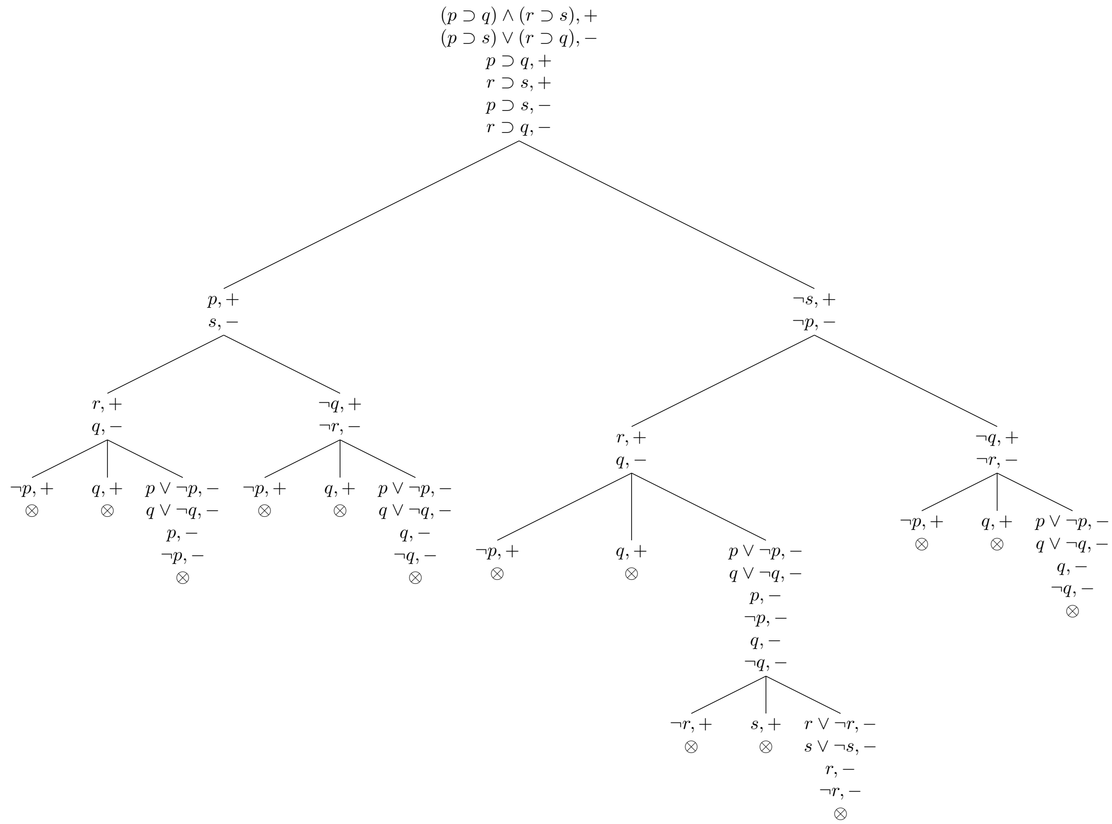
(2) $\neg p \models_{Ł_3} p \supset q$

$$\begin{array}{c}
\neg p, + \\
p \supset q, - \\
\begin{array}{cc}
p, + & \neg q, + \\
q, - & \neg p, - \\
\otimes & \otimes
\end{array}
\end{array}$$

(3) $(p \wedge q) \supset r \vDash_{Ł_3} (p \supset q) \vee (q \supset r)$



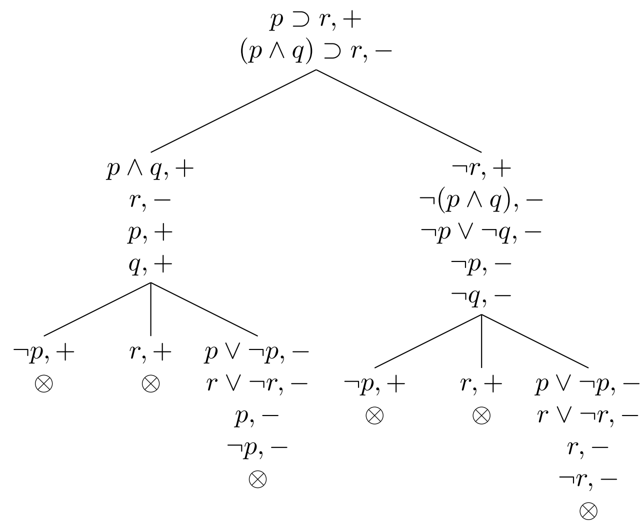
(4) $(p \supset q) \wedge (r \supset s) \vDash_{Ł_3} (p \supset s) \vee (r \supset q)$



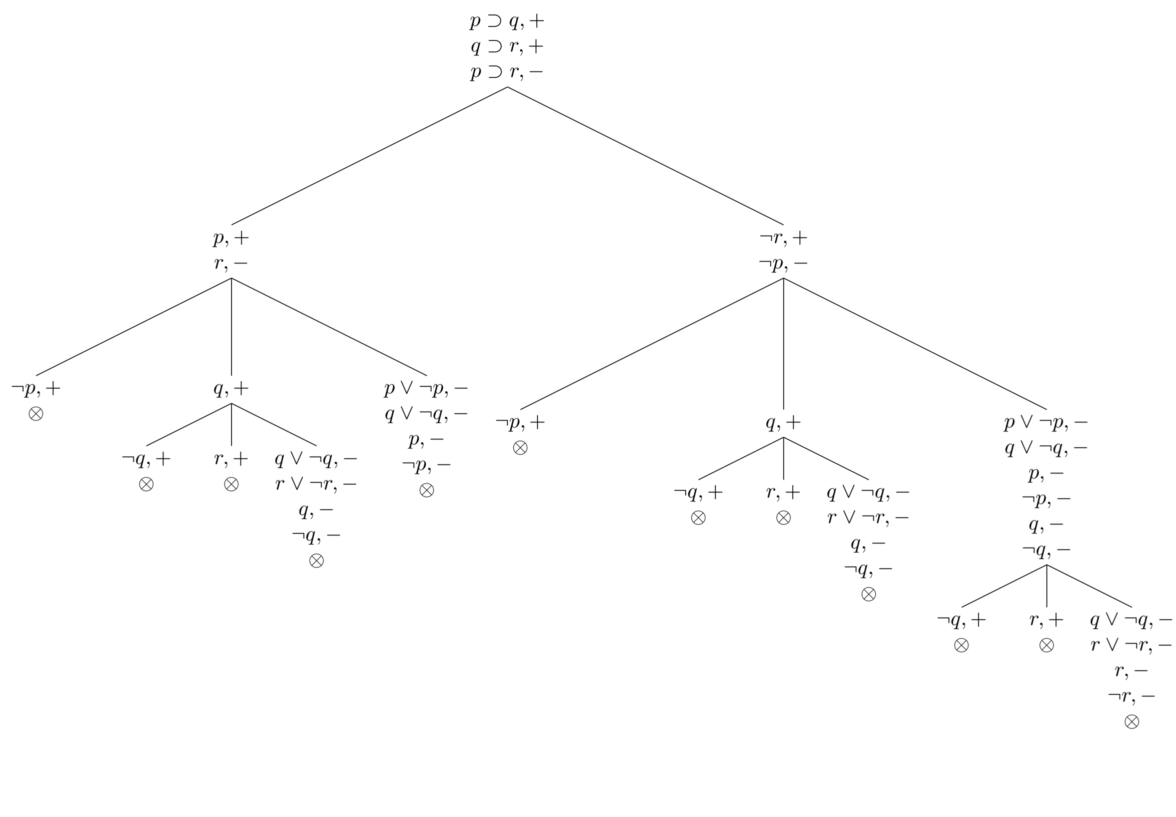
(5) $\neg(p \supset q) \vDash_{Ł_3} p$

$$\begin{array}{c} \neg(p \supset q), + \\ p, - \\ p, + \\ \neg q, + \\ \otimes \end{array}$$

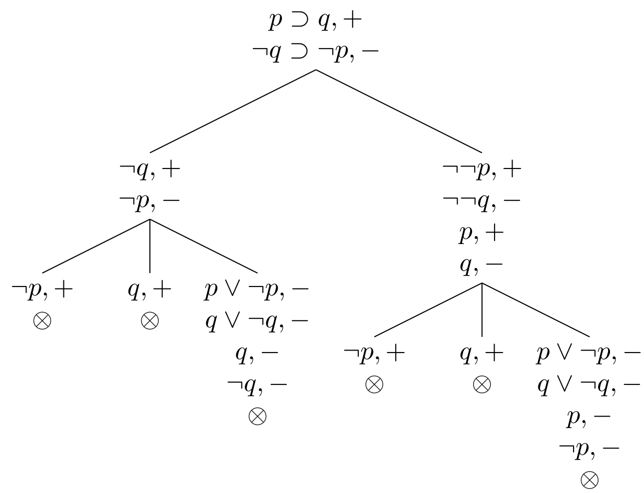
(6) $p \supset r \vDash_{Ł_3} (p \wedge q) \supset r$



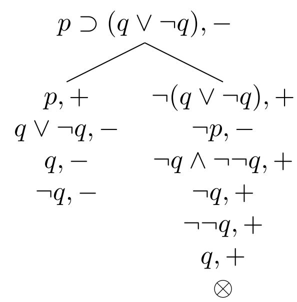
(7) $p \supset q, q \supset r \vDash_{Ł_3} p \supset r$



(8) $p \supset q \vDash_{Ł_3} \neg q \supset \neg p$



(9) $\nvDash_{Ł_3} p \supset (q \vee \neg q)$



There is a counter-model with the following relations obtaining, and no others:

$$p\rho 1$$

Let us check that the induced interpretation works:

It is not the case that $p\rho 0$ or $q \vee \neg q \rho 1$, nor is it the case that none of ($p\rho 1, p\rho 0$, $q \vee \neg q\rho 1$, $q \vee \neg q \rho 0$), because $p\rho 1$. Hence it is not the case that $p \supset (q \vee \neg q)\rho 1$.

(10)$\nvDash_{Ł_3}$ $(p \wedge \neg p) \supset q$

$$
\begin{array}{ccc}
 & (p \wedge \neg p) \supset q, - & \\
 & \swarrow \quad \searrow & \\
p \wedge \neg p, + & & \neg q, + \\
q, - & & \neg(p \wedge \neg p), - \\
p, + & & \neg p \vee \neg\neg p, - \\
\neg p, + & & \neg p, - \\
\otimes & & \neg\neg p, - \\
 & & p, -
\end{array}
$$

There is a counter-model with the following relations obtaining, and no others:

$$q\rho 0$$

Let us check that the counter-model works:

It is not the case that $p \wedge \neg p\rho 0$, and it is not the case that $q\rho 1$. However it is the case that one of $(p \wedge \neg p\rho 1, p \wedge \neg p\rho 0, q\rho 1, q\rho 0)$ is true, because $q\rho 0$.

$\boxed{LP}$

(1) $q \vDash_{LP} p \supset q$

$q \vDash_{LP} \neg p \vee q$

$$
\begin{array}{c}
q, + \\
\neg p \vee q, - \\
\neg p, - \\
q, - \\
\otimes
\end{array}
$$

(2) $\neg p \vDash_{LP} p \supset q$

$\neg p \vDash_{LP} \neg p \vee q$

$$
\begin{array}{c}
\neg p, + \\
\neg p \vee q, - \\
\neg p, - \\
q, - \\
\otimes
\end{array}
$$

(3) $(p \wedge q) \supset r \vDash_{LP} (p \supset q) \vee (q \supset r)$

$\neg(p \wedge q) \vee r \vDash_{LP} (\neg p \vee q) \vee (\neg q \vee r)$

$$\begin{array}{c}
\neg(p \wedge q) \vee r, + \\
(\neg p \vee q) \vee (\neg q \vee r), - \\
\neg p \vee q, - \\
\neg q \vee r, - \\
\neg p, - \\
q, - \\
\neg q, - \\
r, - \\
\begin{array}{cc}
\neg(p \vee q), + & r, + \\
\neg p \wedge \neg q, + & \otimes \\
\neg p, + & \\
\neg q, + & \\
\otimes &
\end{array}
\end{array}$$

(4) $(p \supset q) \wedge (r \supset s) \vDash_{LP} (p \supset s) \vee (r \supset q)$

$(\neg p \vee q) \wedge (\neg r \vee s) \vDash_{LP} (\neg p \vee s) \vee (\neg r \vee q)$

$$\begin{array}{c}
(\neg p \vee q) \wedge (\neg r \vee s), + \\
(\neg p \vee s) \vee (\neg r \vee q), - \\
\neg p \vee s, - \\
\neg r \vee q, - \\
\neg p \vee q, + \\
\neg r \vee s, + \\
\neg p, - \\
s, - \\
\neg r, - \\
q, - \\
\begin{array}{cc}
\neg p, + & q, + \\
\otimes & \otimes
\end{array}
\end{array}$$

(5) $\neg(p \supset q) \vDash_{LP} p$

$\neg(\neg p \vee q) \vDash_{LP} p$

$$\begin{array}{c}
\neg(\neg p \vee q), + \\
p, - \\
\neg\neg p \wedge \neg q, + \\
\neg\neg p, + \\
\neg q, + \\
p, + \\
\otimes
\end{array}$$

(6) $p \supset r \vDash_{LP} (p \wedge q) \supset r$

$\neg p \vee r \vDash_{LP} \neg(p \wedge q) \vee r$

$$
\begin{array}{c}
\neg p \vee r, + \\
\neg(p \wedge q) \vee r, - \\
\neg(p \wedge q), - \\
r, - \\
\neg p \vee \neg q, - \\
\neg p, - \\
\neg q, - \\
\begin{array}{cc}
\neg p, + & r, + \\
\otimes & \otimes
\end{array}
\end{array}
$$

(7) $p \supset q, q \supset r \nvDash_{LP} p \supset r$

$\neg p \vee q, \neg q \vee r \nvDash_{LP} \neg p \vee r$

$$
\begin{array}{c}
\neg p \vee q, + \\
\neg q \vee r, + \\
\neg p \vee r, - \\
\neg p, - \\
r, - \\
\begin{array}{cc}
\neg p, + & q, + \\
\otimes & \begin{array}{cc} \neg q, + & r, + \\ & \otimes \end{array}
\end{array}
\end{array}
$$

There is a counter-model with the following relations obtaining, and no others:

$$p\rho 1, q\rho 1, q\rho 0, r\rho 0$$

Let us check that this counter-model works:

$q\rho 1$ so $\neg p \vee q\rho 1$. $q\rho 0$ so $\neg q\rho 1$, and $\neg q \vee r\rho 1$. Neither $\neg p$ nor r are related to 1, so it is not the case that $\neg p \vee r\rho 1$.

(8) $p \supset q \vDash_{LP} \neg q \supset \neg p$

$\neg p \vee q \vDash_{LP} \neg\neg q \vee \neg p$

$$\begin{array}{c}
\neg p \vee q, + \\
\neg\neg q \vee \neg p, - \\
\neg\neg q, - \\
\neg p, - \\
q, - \\
\begin{array}{cc}
\neg p, + & q, + \\
\otimes & \otimes
\end{array}
\end{array}$$

(9) $\vDash_{LP} p \supset (q \vee \neg q)$

$\vDash_{LP} \neg p \vee (q \vee \neg q)$

$$\begin{array}{c}
\neg p \vee (q \vee \neg q), - \\
\neg p, - \\
q \vee \neg q, - \\
q, - \\
\neg q, - \\
\otimes
\end{array}$$

Closes due to the LP closure rule.

(10)$\vDash_{LP} (p \wedge \neg p) \supset q$

$\vDash_{LP} \neg(p \wedge \neg p) \vee q$

$$\begin{array}{c}
\neg(p \wedge \neg p) \vee q, - \\
\neg(p \wedge \neg p), - \\
q, - \\
\neg p, \vee \neg\neg p, - \\
\neg p, - \\
\neg\neg p - \\
p, - \\
\otimes
\end{array}$$

Closes due to the LP closure rule.

$\boxed{RM_3}$

(1) $q \vDash_{RM_3} p \supset q$

$$\begin{array}{c} q, + \\ p \supset q, - \\ \begin{array}{cc} p, + & \neg q, + \\ q, - & \neg p, - \\ \otimes & \end{array} \end{array}$$

There is a counter-model with the following relations obtaining, and no others:

$$p\rho 1, q\rho 1, q\rho 0$$

Let us check that this counter-model works:

The premise, q is true, but it is also false, while p is true rendering the conclusion $p \supset q$, undesignated.

(2) $\neg p \nvDash_{RM_3} p \supset q$

$$\begin{array}{c} \neg p, + \\ p \supset q, - \\ \begin{array}{cc} p, + & \neg q, + \\ q, - & \neg p, - \\ & \otimes \end{array} \end{array}$$

There is a counter-model with the following relations obtaining, and no others:

$$p\rho 1, p\rho 0, q\rho 0$$

Let us check that this counter-model works:

$p\rho 0$ so $\neg p\rho 1$. It is not the case that either, (it is not the case that $p\rho 1$) or (it is not the case that $q\rho 0$) or all of $(p\rho 1, p\rho 0, q\rho 1, q\rho 0)$, because it is not the case that $q\rho 1$. So it is not the case that p $\supset q\rho 1$.

(3) $(p \wedge q) \supset r \vDash_{RM_3} (p \supset q) \vee (q \supset r)$

$$(p \wedge q) \supset r, +$$
$$(p \supset q) \vee (q \supset r), -$$
$$p \supset q, -$$
$$q \supset r, -$$

- $p, +$ / $q, -$
  - $q, +$ / $r, -$ / $\otimes$
  - $\neg r, +$ / $\neg q, -$ / $\otimes$
- $\neg q, +$ / $\neg p, -$
  - $q, +$ / $r, -$
    - $p \wedge q, -$
      - $p, -$ / $\otimes$
      - $q, -$ / $\otimes$
    - $\neg r, -$ / $\otimes$
    - $(p \wedge q) \wedge \neg(p \wedge q), +$ / $r \wedge \neg r, +$ / $r, +$ / $\neg r, +$ / $\otimes$
  - $\neg r, +$ / $\neg q, -$ / $\otimes$

(4) $(p \supset q) \wedge (r \supset s) \vDash_{RM_3} (p \supset s) \vee (r \supset q)$

- $(p \supset q) \wedge (r \supset s), +$; $(p \supset s) \vee (r \supset q), -$; $p \supset q, +$; $r \supset s, +$; $p \supset s, -$; $r \supset q, -$
  - $p, +$; $s, -$
    - $r, +$; $q, -$
      - $p, -$; $\otimes$
      - $\neg q, -$; $\otimes$
      - $p \wedge \neg p, +$; $q \wedge \neg q, +$; $q, +$; $\neg q, +$; $\otimes$
    - $\neg q, +$; $\neg r, -$
      - $p, -$; $\otimes$
      - $\neg q, -$; $\otimes$
      - $p \wedge \neg p, +$; $q \wedge \neg q, +$; $p, +$; $\neg p, +$; $q, +$; $\neg q, +$
        - $r, -$; $\otimes$
        - $\neg s, -$; $\otimes$
        - $r \wedge \neg r, +$; $s \wedge \neg s, +$; $s, -$; $\neg s, -$; $\otimes$
  - $\neg s, +$; $\neg p, -$
    - $r, +$; $q, -$
      - $p, -$; $\otimes$
      - $\neg q, -$; $\otimes$
      - $p \wedge \neg p, +$; $q \wedge \neg q, +$; $q, +$; $\neg q, +$; $\otimes$
    - $\neg q, +$; $\neg r, -$
      - $p, -$; $\otimes$
      - $\neg q, -$; $\otimes$
      - $p \wedge \neg p, +$; $q \wedge \neg q, +$; $p, +$; $\neg p, +$; $\otimes$

(5) $\neg(p \supset q) \vDash_{RM_3} p$

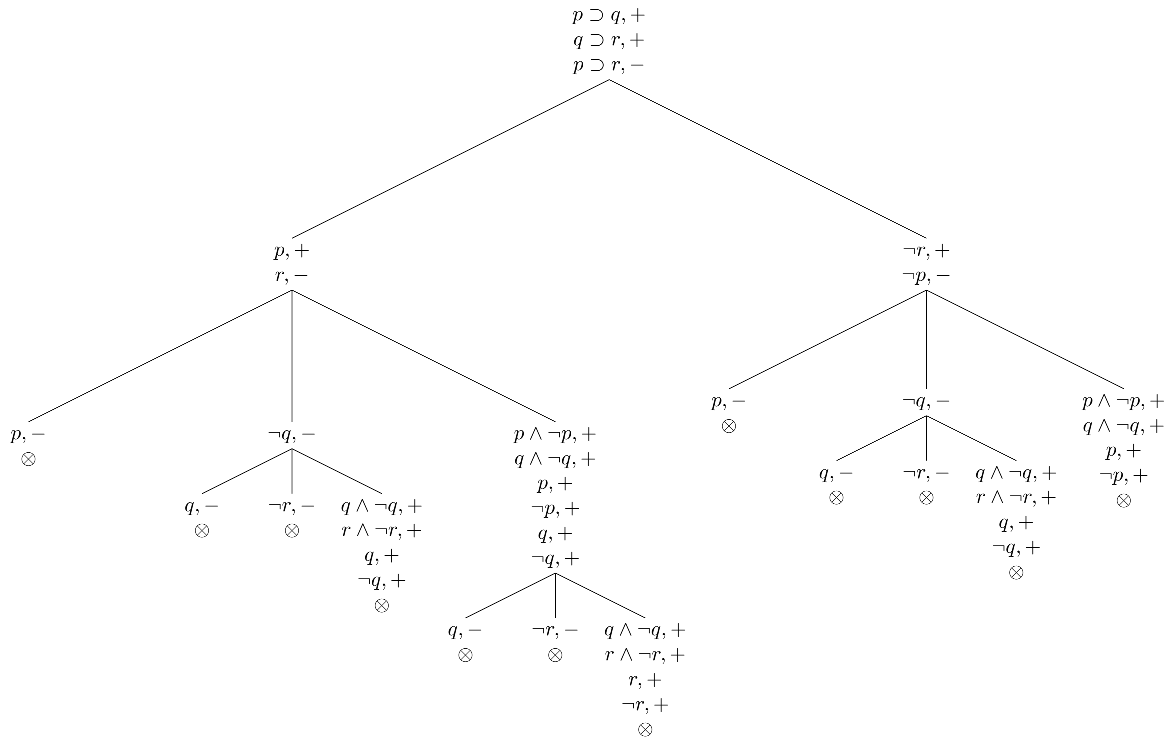
$$\neg(p \supset q), +$$
$$p, -$$
$$p, +$$
$$\neg q, +$$
$$\otimes$$

(6) $p \supset r \vDash_{RM_3} (p \wedge q) \supset r$

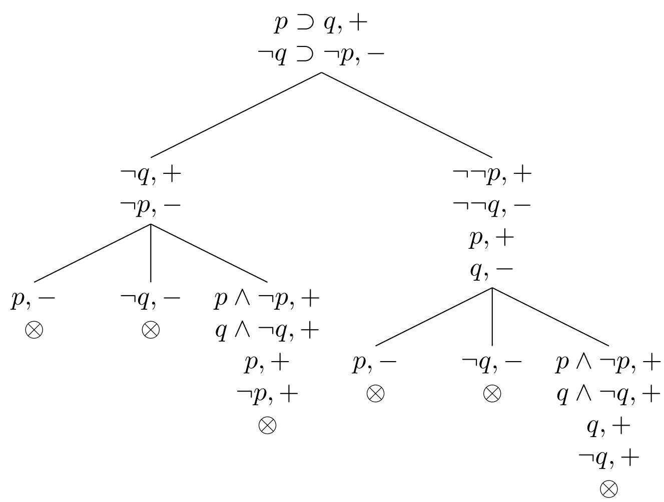
$p \supset r, +$
$(p \wedge q) \supset r, -$

- $p \wedge q, +$
  $r, -$
  $p, +$
  $q, +$
  - $p, -$
    $\otimes$
  - $\neg r, -$
    $\otimes$
  - $p \wedge \neg p, +$
    $r \wedge \neg r, +$
    $r, +$
    $\neg r, +$
    $\otimes$
- $\neg r, +$
  $\neg(p \wedge q), -$
  $\neg p \vee \neg q, -$
  $\neg p, -$
  $\neg q, -$
  - $p, -$
    $\otimes$
  - $\neg r, -$
    $\otimes$
  - $p \wedge \neg p, +$
    $r \wedge \neg r, +$
    $p, +$
    $\neg p, +$
    $\otimes$

(7) $p \supset q, q \supset r \models_{RM_3} p \supset r$



(8) $p \supset q \models_{RM_3} \neg q \supset \neg p$



(9) $\not\models_{RM_3} p \supset (q \vee \neg q)$

$$
\begin{array}{c}
p \supset (q \vee \neg q), - \\
\begin{array}{cc}
p, + & \neg(q \vee \neg q), + \\
q \vee \neg q, - & \neg p, - \\
q, - & \neg q \wedge \neg\neg q, + \\
\neg q, - & \neg q, + \\
\otimes & \neg\neg q, + \\
 & q, +
\end{array}
\end{array}
$$

There is a counter-model with the following relations obtaining, and no others:

$$p\rho 1, q\rho 1, q\rho 0$$

Let us check that the counter-model works:

$p\rho 1$, $q \vee \neg q\rho 0$, and it is not the case that $p\rho 0$. Therefore it is not the case that $p \supset (q \vee \neg q)\rho 1$.

(10)$\not\models_{RM_3} (p \wedge \neg p) \supset q$

$$
\begin{array}{c}
(p \wedge \neg p) \supset q, - \\
\begin{array}{cc}
p \wedge \neg p, + & \neg q, + \\
q, - & \neg(p \wedge \neg p), - \\
p, + & \neg p \vee \neg\neg p, - \\
\neg p, + & \neg p, - \\
 & \neg\neg p, - \\
 & p, - \\
 & \otimes
\end{array}
\end{array}
$$

There is a counter-model with the following relations obtaining, and no others:

$$p\rho 1, p\rho 0, q\rho 0$$

Let us check that the counter-model works:

Since $p\rho 1$ and $p\rho 0$, $p \wedge \neg p\rho 1$. Also $q\rho 0$ and it is not the case that $q\rho 1$. Therefore it is not the case that$(p \wedge \neg p) \supset q\rho 1$.

10.* Check the details omitted in 8.7.3.

$\boxed{A \vee B}$

Suppose that we apply the rule for $A \vee B, +$. Since ρ is faithful to the branch, $A \vee B\rho 1$. So either $A\rho 1$, or $B\rho 1$; in either case, ρ is faithful one of the branches.

Suppose that we apply the rule for $A \vee B, -$. Since ρ is faithful to the branch, it is not the case that $A \vee B\rho 1$. So it is not the case that $A\rho 1$ and it is not the case that $B\rho 1$, as required.

$\boxed{\neg(A \wedge B)}$

$\neg(A \wedge B)\rho 1$ iff $A \wedge B\rho 0$ iff $\neg A\rho 1$ or $\neg B\rho 1$, iff $\neg A \vee \neg B\rho 1$.

So, if ρ is faithful to a branch and we apply a rule to $\neg(A \wedge B), +$ then $\neg(A \wedge B)\rho 1$. Hence $\neg A \vee \neg B\rho 1$ as required. And if ρ is faithful to a branch and we apply a rule to $\neg(A \wedge B), -$ then it is not the case that $\neg(A \wedge B)\rho 1$, hence it is not the case that $\neg A \vee \neg B\rho 1$ as required.

$\boxed{\neg(A \vee B)}$

$\neg(A \vee B)\rho 1$ iff $A \vee B\rho 0$ iff $\neg A\rho 1$ and $\neg B\rho 1$ iff $\neg A \wedge \neg B\rho 1$.

So if ρ is faithful to a branch and we apply a rule to $\neg(A \vee B), +$, then $\neg(A \wedge B)\rho 1$. Hence $\neg A \wedge \neg B\rho 1$ as required.And if ρ is faithful to a branch and we apply a rule to $\neg(A \vee B), -$ then it is not the case that $\neg(A \vee B)\rho 1$, hence it is not the case that $\neg A \wedge \neg B\rho 1$ as required.

$\boxed{\neg\neg A}$

$\neg\neg A\rho 1$ iff $\neg A\rho 0$ iff $A\rho 1$.

So if ρ is faithful to a branch and we apply a rule to $\neg\neg A, +$, then $\neg\neg A\rho 1$. Hence $A\rho 1$ as required. And if ρ is faithful to a branch and we apply a rule to $\neg\neg A, -$, then it is not the case that $\neg\neg A\rho 1$, hence it is not the case that $A\rho 1$ as required.

11. *Show that the tableaux of 8.4a.4 and 8.4a.5 are sound and complete with respect to the semantics of Ł$_3$ and RM_3. (Hint: consult 8.7.8 and 8.7.9.)

The proof for each will be a modification of the proofs for K_3 and LP respectively.

$\boxed{Ł_3}$

This proof will be a modification of the proof for K_3. The only points at which the two systems differ is the conditional. Thus there will be new cases in the Soundness Lemma for the new conditional tableau rules, and corresponding new cases in the Completeness Lemma.

Soundness:

We have four new rules for $\supset$.

$A \supset B, +$

Suppose that we apply the rule for $A \supset B, +$. Since ρ is faithful to the branch, $A \supset B\rho 1$. Hence $A\rho 0$ or $B\rho 1$ or (none of $A\rho 1, A\rho 0, B\rho 1, B\rho 0$).

In the first case ρ is faithful to the left branch. In the second case, to the middle branch. And in the third case, ρ is faithful to the right branch.

$A \supset B, -$

Suppose that we apply the rule for $A \supset B, -$. Since ρ is faithful to the branch, it is not the case that $A \supset B\rho 1$. Hence (it is not the case that $A\rho 0$) and (it is not the case that $B\rho 1$) and (at least one of: $A\rho 1$, $A\rho 0$, $B\rho 1$, $B\rho 0$). It follows that one of $A\rho 1$, $B\rho 0$. In the first case ρ is faithful to the left branch. In the second case, it is faithful to the right branch.

$\neg(A \supset B), +$

Suppose that we apply the rule for $\neg(A \supset B), +$. Since ρ is faithful to the branch, $\neg(A \supset B)\rho 1$. So, by the rule for $\neg$, $A \supset B\rho 0$. Thus $A\rho 1$ and $B\rho 0$, as required.

$\neg(A \supset B), -$

Suppose that we apply the rule for $\neg(A \supset B), -$. Since ρ is faithful to the branch, it is not the case that $A \supset B\rho 0$. Hence it is not the case that $A\rho 1$ or it is not the case that $B\rho 0$. In the first case ρ is faithful to the left branch, in the second case, it is faithful to the right branch.

Completeness:

The K_3 closure rule guarantees that the induced interpretation is an L_3 interpretation, however the four new rules also need corresponding cases in the Completeness Lemma.

$A \supset B, +$

If $A \supset B, +$ occurs on b, then $\neg A, +$ or $B, +$ or $A \vee \neg A, -$ and $B \vee \neg B, -$ occur on b. By induction hypothesis, either $A\rho 0$, (in which case $A \supset B\rho 1$) or $B\rho 1$ (in which case $A \supset B\rho 1$), or it is not the case that $A \vee \neg A\rho 1$ and $B \vee \neg B\rho 1$, (in which case $A \supset B\rho 1$).

$A \supset B, -$

If $A \supset B, -$ occurs on b then either $A, +$ and $B, -$ occur on b, or $\neg B, +$ and $\neg A, -$ occur in b. Consider the first case. By induction hypothesis, $A\rho 1$, and it is not the case that $B\rho 1$. Since this is an Ł$_3$ interpretation, it is not the case that $A\rho 0$. Hence (it is not the case that $A\rho 0$) and (it is not the case that $B\rho 1$) and (at least one of: $A\rho 1$, $A\rho 0$, $B\rho 1$, $B\rho 0$). Hence it is not the case that $A \supset B\rho 1$. In the second case, by induction hypothesis, $B\rho 0$ and it is not the case that $A\rho 0$. Since this is an Ł$_3$ interpretation, it is not the case that $B\rho 1$. Hence (it is not the case that $A\rho 0$) and (it is not the case that $B\rho 1$) and (at least one of: $A\rho 1$, $A\rho 0$, $B\rho 1$, $B\rho 0$). Hence it is not the case that $A \supset B\rho 1$.

$\neg(A \supset B), +$

If $\neg(A \supset B), +$ occurs on b, then $A, +$ and $\neg B, +$ occur on b. By induction hypothesis $A\rho 1$ and $B\rho 0$. Hence $A \supset B\rho 0$, as required.

$\neg(A \supset B), -$

If $\neg(A \supset B), -$ appears on b, then either $A, -$ or $\neg B, -$ appear on b. By induction hypothesis, either it is not the case that $A\rho 1$ or it is not the case that $B\rho 0$ In both cases it is not the case that $A \supset B\rho 0$, as required.

$\boxed{RM_3}$

This proof will be a modification of the proof for LP. The only points at which the two systems differ is the conditional. Thus there will be new cases in the Soundness Lemma for the new conditional tableau rules, and corresponding new cases in the Completeness Lemma.

Soundness:

$A \supset B, +$

Suppose that we apply the rule for $A \supset B, +$. Since ρ is faithful to the branch, $A \supset B\rho 1$. Hence it is not the case that $A\rho 1$ or it is not the case that $B\rho 0$ or ($A\rho 1$ and $A\rho 0$ and $B\rho 1$ and $B\rho 0$).

In the first case ρ is faithful to the left branch. In the second case, to the middle branch. And in the third case, ρ is faithful to the right branch.

$A \supset B, -$

Suppose that we apply the rule for $A \supset B, -$. Since ρ is faithful to the branch it is not the case that $A \supset B\rho 1$. Hence, $A\rho 1$, $B\rho 0$, and not all of ($A\rho 1$, $A\rho 0$, $B\rho 1$, $B\rho 0$). Hence either it is not the case that $A\rho 0$ or it is not the case that $B\rho 1$. In the first case, ρ is faithful to the right hand branch. In the second case ρ is faithful to the left hand branch.

$\neg(A \supset B), +$

Suppose that we apply the rule for $\neg(A \supset B), +$. Since ρ is faithful to the branch, $\neg(A \supset B)\rho 1$. So, by the rule for $\neg$, $A \supset B\rho 0$. Thus $A\rho 1$ and $B\rho 0$, as required.

$\neg(A \supset B), -$

Suppose that we apply the rule for $\neg(A \supset B), -$. Since ρ is faithful to the branch, it is not the case that $\neg(A \supset B)\rho 1$. So, by the rule for $\neg$, it is not the case that $A \supset B\rho 0$. Thus it is not the case that $A\rho 1$ and $B\rho 0$; that is, it is either not the case that $A\rho 1$ or not the case that $\neg B\rho 1$ as required.

Completeness:

The LP closure rule guarantees that the induced interpretation is an RM_3 interpretation, however the four new rules also need corresponding cases in the Completeness Lemma.

$A \supset B, +$

If $A \supset B, +$ occurs on b, then $A, -$ or $\neg B, -$ or $A \wedge \neg A, -$ and $B \wedge \neg B, -$ occur on b. By induction hypothesis, either it is not the case that $A\rho 1$, (in which case $A \supset B\rho 1$) or it is not the case that $B\rho 0$ (in which case $A \supset B\rho 1$), or it is not the case that $A \wedge \neg A\rho 1$ and $B \wedge \neg B\rho 1$, (in which case $A \supset B\rho 1$).

$A \supset B, -$

If $A \supset B, -$, occurs on b then either $A, +$ and $B, -$ or $\neg B, +$ and $\neg A, -$ occur on b. In the first case, by induction hypothesis, $A\rho 1$ and it is not the case that $B\rho 1$. Since this is an RM_3 evaluation, $B\rho 0$. Hence: $A\rho 1$ and $B\rho 0$ and not all of ($A\rho 1$, $A\rho 0$, $B\rho 1$, $B\rho 0$). So it is not the case that $A \supset B\rho 1$. In the second case, by induction hypothesis, $B\rho 0$ and it is not the case that $A\rho 0$. Since this is an RM_3 interpretation, $A\rho 1$. Hence, $A\rho 1$ and $B\rho 0$ and not all of ($A\rho 1$, $A\rho 0$, $B\rho 1$, $B\rho 0$). So it is not the case that $A \supset B\rho 1$.

$\neg(A \supset B), +$

If $\neg(A \supset B), +$ occurs on b, then $A, +$ and $\neg B, +$ occur on b. By induction hypothesis $A\rho 1$ and $B\rho 0$. Hence $A \supset B\rho 0$, as required.

$\neg(A \supset B), -$

If $\neg(A \supset B), -$ appears on b, then either $A, -$ or $\neg B, -$ appear on b. By induction hypothesis, either it is not the case that $A\rho 1$ or it is not the case that $B\rho 0$ In both cases it is not the case that $A \supset B\rho 0$, as required.