Solutions

Louis Barson
Kyoto University

April 7, 2009

1. Check all the details omitted in 7.5.2.

I will first check the table on page 124. For this it will be useful to have a list of the characteristic features of the logics treated in this chapter:

K_3: $\mathcal{D} = (1)$; i thought of as *neither true nor false*.

$f_\supset$	1	i	0
1	1	i	0
i	1	i	i
0	1	1	1

Ł$_3$: $\mathcal{D} = (1)$; i thought of as *neither true nor false*.

$f_\supset$	1	i	0
1	1	i	0
i	1	1	i
0	1	1	1

LP: $\mathcal{D} = (1, i)$; i thought of as *both true and false*.

$f_\supset$	1	i	0
1	1	i	0
i	1	i	i
0	1	1	1

RM_3: $\mathcal{D} = (1, i)$; i thought of as *both true and false*.

$f_\supset$	1	i	0
1	1	0	0
i	1	i	0
0	1	1	1

(1) $q \models_{K_3} p \supset q$

Suppose the conclusion is undesignated. Then by the truth table for $\supset_{K_3}$, there are four possibilities for p and q: $(1, i), (1, 0), (i, i)$, and $(i, 0)$. In all cases, the premise q is undesignated.

(1) $q \models_{\text{Ł}_3} p \supset q$

Suppose the conclusion is undesignated. Then by the truth table for $\supset_{\text{Ł}_3}$, there are three possibilities for p and q: $(1, i), (1, 0)$, and $(i, 0)$. In all cases, the premise q is undesignated.

(1) $q \models_{LP} p \supset q$

Suppose the conclusion is undesignated. Then the truth-value of $p \supset q$ is 0. So the truth value of q is 0.

(1) $q \not\models_{RM_3} p \supset q$

In the case where $v(p) = 1$, and $v(q) = i$, the premise is designated and conclusion undesignated, therefore the inference is invalid.

(2) $\neg p \models_{K_3} p \supset q$

Suppose the conclusion is undesignated. Then by the truth table for $\supset_{K_3}$, there are four possibilities for p and q: $(1, i), (1, 0), (i, i)$, and $(i, 0)$. In all cases, the premise $\neg p$ is undesignated.

(2) $\neg p \models_{\text{Ł}_3} p \supset q$

Suppose the conclusion is undesignated. Then by the truth table for $\supset_{\text{Ł}_3}$, there are three possibilities for p and q: $(1, i), (1, 0)$, and $(i, 0)$. In all cases, the premise $\neg p$ is undesignated.

(2) $\neg p \models_{LP} p \supset q$

Suppose the conclusion is undesignated. Then the truth-value of $p \supset q$ is 0. So the truth value of p is 1, and the truth value of $\neg p$ is 0.

(2) $\neg p \not\models_{RM_3} p \supset q$

In the case where $v(p) = i$, and $v(q) = 0$, the premise is designated and conclusion undesignated, therefore the inference is invalid.

(3) $(p \wedge q) \supset r \vDash_{K_3} (p \supset q) \vee (q \supset r)$

p	q	r	$(p \wedge q)$	$\supset r$	$(p \supset q)$	$\vee$	$(q \supset r)$
1	1	1	1	**1**	1	**1**	1
1	1	i	1	i	1	**1**	i
1	1	0	1	**0**	1	**1**	0
1	i	1	i	**1**	i	**1**	1
1	i	i	i	i	i	i	i
1	i	0	i	i	i	i	i
1	0	1	0	**1**	0	**1**	1
1	0	i	0	**1**	0	**1**	1
1	0	0	0	**1**	0	**1**	1
i	1	1	i	**1**	1	**1**	1
i	1	i	i	i	1	**1**	i
i	1	0	i	i	1	**1**	0
i	i	1	i	**1**	i	**1**	1
i	i	i	i	i	i	i	i
i	i	0	i	i	i	i	i
i	0	1	0	**1**	i	**1**	1
i	0	i	0	**1**	i	**1**	1
i	0	0	0	**1**	i	**1**	1
0	1	1	0	**1**	1	**1**	1
0	1	i	0	**1**	1	**1**	i
0	1	0	0	**1**	1	**1**	0
0	i	1	0	**1**	1	**1**	1
0	i	i	0	**1**	1	**1**	i
0	i	0	0	**1**	1	**1**	i
0	0	1	0	**1**	1	**1**	1
0	0	i	0	**1**	1	**1**	1
0	0	0	0	**1**	1	**1**	1

Looking at the bolded columns, we can see that there is no interpretation where the premise is designated and conclusion undesignated. Therefore the inference is valid.

(3) $(p \wedge q) \supset r \vDash_{\text{Ł}_3} (p \supset q) \vee (q \supset r)$

p	q	r	$(p \wedge q) \supset r$		$(p \supset q)$	$\vee$	$(q \supset r)$
1	1	1	1	**1**	1	**1**	1
1	1	i	1	i	1	**1**	i
1	1	0	1	**0**	1	**1**	0
1	i	1	i	**1**	i	**1**	1
1	i	i	i	**1**	i	i	1
1	i	0	i	i	i	i	i
1	0	1	0	**1**	0	**1**	1
1	0	i	0	**1**	0	**1**	1
1	0	0	0	**1**	0	**1**	1
i	1	1	i	**1**	1	**1**	1
i	1	i	i	**1**	1	**1**	i
i	1	0	i	i	1	**1**	0
i	i	1	i	**1**	1	**1**	1
i	i	i	i	**1**	1	**1**	1
i	i	0	i	i	1	**1**	i
i	0	1	0	**1**	i	**1**	1
i	0	i	0	**1**	i	**1**	1
i	0	0	0	**1**	i	**1**	1
0	1	1	0	**1**	1	**1**	1
0	1	i	0	**1**	1	**1**	i
0	1	0	0	**1**	1	**1**	0
0	i	1	0	**1**	1	**1**	1
0	i	i	0	**1**	1	**1**	1
0	i	0	0	**1**	1	**1**	i
0	0	1	0	**1**	1	**1**	1
0	0	i	0	**1**	1	**1**	1
0	0	0	0	**1**	1	**1**	1

Looking at the bolded columns, we can see that there is no interpretation where the premise is designated and conclusion undesignated. Therefore the inference is valid.

(3) $(p \wedge q) \supset r \vDash_{LP} (p \supset q) \vee (q \supset r)$

p	q	r	$(p \wedge q)$	$\supset r$	$(p \supset q)$	$\vee$	$(q \supset r)$
1	1	1	1	**1**	1	**1**	1
1	1	i	1	i	1	**1**	i
1	1	0	1	**0**	1	**1**	0
1	i	1	i	**1**	i	**1**	1
1	i	i	i	i	i	i	i
1	i	0	i	i	i	i	i
1	0	1	0	**1**	0	**1**	1
1	0	i	0	**1**	0	**1**	1
1	0	0	0	**1**	0	**1**	1
i	1	1	i	**1**	1	**1**	1
i	1	i	i	i	1	**1**	i
i	1	0	i	i	1	**1**	0
i	i	1	i	**1**	i	**1**	1
i	i	i	i	i	i	i	i
i	i	0	i	i	i	i	i
i	0	1	0	**1**	i	**1**	1
i	0	i	0	**1**	i	**1**	1
i	0	0	0	**1**	i	**1**	1
0	1	1	0	**1**	1	**1**	1
0	1	i	0	**1**	1	**1**	i
0	1	0	0	**1**	1	**1**	0
0	i	1	0	**1**	1	**1**	1
0	i	i	0	**1**	1	**1**	i
0	i	0	0	**1**	1	**1**	i
0	0	1	0	**1**	1	**1**	1
0	0	i	0	**1**	1	**1**	1
0	0	0	0	**1**	1	**1**	1

Looking at the bolded columns, we can see that there is no interpretation where the premise is designated and conclusion undesignated. Therefore the inference is valid.

(3) $(p \wedge q) \supset r \vDash_{RM_3} (p \supset q) \vee (q \supset r)$

p	q	r	$(p \wedge q)$	$\supset r$	$(p \supset q)$	$\vee$	$(q \supset r)$
1	1	1	1	**1**	1	**1**	1
1	1	i	1	**0**	1	**1**	0
1	1	0	1	**0**	1	**1**	0
1	i	1	i	**1**	0	**1**	1
1	i	i	i	i	0	i	i
1	i	0	i	**0**	0	**0**	0
1	0	1	0	**1**	0	**1**	1
1	0	i	0	**1**	0	**1**	1
1	0	0	0	**1**	0	**1**	1
i	1	1	i	**1**	1	**1**	1
i	1	i	i	i	1	**1**	0
i	1	0	i	**0**	1	**1**	0
i	i	1	i	**1**	i	**1**	1
i	i	i	i	i	i	i	i
i	i	0	i	**0**	i	i	0
i	0	1	0	**1**	0	**1**	1
i	0	i	0	**1**	0	**1**	1
i	0	0	0	**1**	0	**1**	1
0	1	1	0	**1**	1	**1**	1
0	1	i	0	**1**	1	**1**	0
0	1	0	0	**1**	1	**1**	0
0	i	1	0	**1**	1	**1**	1
0	i	i	0	**1**	1	**1**	i
0	i	0	0	**1**	1	**1**	0
0	0	1	0	**1**	1	**1**	1
0	0	i	0	**1**	1	**1**	1
0	0	0	0	**1**	1	**1**	1

Looking at the bolded columns, we can see that there is no interpretation where the premise is designated and conclusion undesignated. Therefore the inference is valid.

(4) $(p \supset q) \wedge (r \supset s) \vDash_{K_3} (p \supset s) \vee (r \supset q)$

Suppose the conclusion is undesignated. Then it takes either the value 0 or the value i.

If the truth-value of the conclusion $(p \supset s) \vee (r \supset q)$ is 0, then the truth value of both $p \supset s$ and $r \supset q$ is also 0. Looking at the truth-table for $\supset_{K_3}$ we can see that this only happens when, as in the classical case, the antecedent is true, and consequent false. Thus p and r take 1, s and q take 0. But then the truth value of $p \supset q$ is 0, and the conjunct $(p \supset q) \wedge (r \supset s)$ is also: the premise is undesignated.

If the conclusion takes the value i, looking at the truth-table for $\vee$, we can see there are three possibilities: $(i, i), (0, i)$, or $(i, 0)$.

$v(p \supset s) = i$, $v(r \supset q) = i$

By the truth-table for $\supset_{K_3}$, there are three possibilities for each conditional: $(1, i), (i, i)$, and $(i, 0)$. The nine combinations are listed below.

p	$\supset$	s	r	$\supset$	q
1		i	1		i
1		i	i		i
1		i	i		**0**
i		i	1		i
i		i	i		i
i		i	i		**0**
i		0	1		i
i		0	i		i
i		0	i		**0**

Looking at the bolded columns for p and q, we can see that on any of these interpretations, $p \supset q$ will come out as i or 0, i.e. undesignated. Thus the conjunct will also: the premise is undesignated on all these interpretations.

$v(p \supset s) = 0$, $v(r \supset q) = i$

$v(p) = 1, v(s) = 0$. There are three possibilities for r and q: $(1, i), (i, i)$, and $(i, 0)$. Thus there are three possibilities for $p \supset q$, $(i, i, 0)$ respectively. In these interpretations also, $p \supset q$ is undesignated, so the conjunctive premise, is undesignated.

$v(p \supset s) = i$, $v(p \supset s) = 0$

$v(r) = 1$, $v(q) = 0$. There are three possibilities for p and s: $(1, i), (i, i)$, and $(i, 0)$.Thus there are three possibilities for $p \supset q$, $(0, i, i)$ respectively. In these interpretations also, $p \supset q$ is undesignated, so the conjunct, i.e. the premise, is undesignated.

There is no interpretation where the conclusion is undesignated, and premise designated, therefore the inference is valid.

(4) $(p \supset q) \wedge (r \supset s) \vDash_{Ł_3} (p \supset s) \vee (r \supset q)$

Suppose the conclusion is undesignated. Then it takes either the value 0 or the value i.

If the truth-value of the conclusion $(p \supset s) \vee (r \supset q)$ is 0, then both $p \supset s$ and $r \supset q$ are also 0. Looking at the truth-table for $\supset_{Ł_3}$ we can see that this only happens when, as in the classical case, the antecedent is true, and consequent false. Thus p and r take 1, s and q take 0. But then the truth value of $p \supset q$ is 0, and the conjunct $(p \supset q) \wedge (r \supset s)$ is also: the premise is undesignated.

If the conclusion takes the value i, looking at the truth-table for $\vee$, we can see there are three possibilities: $(i, i), (0, i)$, or $(i, 0)$.

$v(p \supset s) = i$, $v(r \supset q) = i$

By the truth-table for $\supset_{Ł_3}$, there are two possibilities for each conditional: $(1, i)$ and $(i, 0)$. The four combinations are listed below.

p	$\supset$	s	r	$\supset$	q
1		i	1		i
1		i	i		**0**
i		0	1		i
i		0	i		**0**

Looking at the bolded columns for p and q, we can see that on any of these interpretations, $p \supset q$ will come out as i or 0, i.e. undesignated. Thus the conjunct will also: the premise is undesignated.

$v(p \supset s) = 0$, $v(r \supset q) = i$

$v(p) = 1, v(s) = 0$. There are two possibilities for r and q: $(1, i)$ and $(i, 0)$. Thus there are two possibilities for $p \supset q$: $(i, 0)$. $p \supset q$ is undesignated, so the premise, is undesignated.

$v(p \supset s) = i$, $v(p \supset s) = 0$

$v(r) = 1$, $v(q) = 0$. There are two possibilities for p and s: $(1, i)$ and $(i, 0)$.Thus there are two possibilities for $p \supset q$: $(0, i)$. In these interpretations also, $p \supset q$ is undesignated, so the conjunct, i.e. the premise, is undesignated.

There is no interpretation where the conclusion is undesignated, and premise designated, therefore the inference is valid.

(4) $(p \supset q) \wedge (r \supset s) \vDash_{LP} (p \supset s) \vee (r \supset q)$

Suppose the conclusion to be undesignated. Then it takes the value 0.

If the conclusion $(p \supset s) \vee (r \supset q)$ takes the value 0, then both $p \supset s$ and $r \supset q$ take the value 0. Looking at the truth-table for $\supset$ we can see that this only happens in the classical case. Thus p and r take 1, s and q take 0. But then $p \supset q$ takes 0, and the conjunct takes 0: the premise is undesignated.

There is no interpretation where the conclusion is undesignated, and premise designated, therefore the inference is valid.

(4) $(p \supset q) \wedge (r \supset s) \vDash_{RM_3} (p \supset s) \vee (r \supset q)$

Suppose the conclusion to be undesignated. Then it takes the value 0.

If the conclusion $(p \supset s) \vee (r \supset q)$ takes the value 0, then both $p \supset s$ and $r \supset q$ take the value 0. Looking at the truth-table for $\supset_{RM_3}$ we can see that there are three possibilities for each conditional: $(1, i), (1, 0)$, and $(i, 0)$. The nine combinations are listed below.

p	$\supset$	s	r	$\supset$	q
1		i	1		i
1		i	1		**0**
1		i	i		**0**
1		0	1		i
1		0	1		**0**
1		0	i		**0**
i		**0**	**1**		i
i		0	1		**0**
i		0	i		**0**

Looking at the bolded columns for p and q, we can see that on any of these interpretations (except line 7), $p \supset q$ comes out as i or 0, i.e. undesignated. Thus the conjunct does also: the premise is undesignated. On line 7, $r \supset s$ is undesignated, so again the premise is undesignated.

(5) $\neg(p \supset q) \vDash_{K_3} p$

Suppose the premise is designated. Then $\neg(p \supset q)$ takes the value 1. Then $p \supset q$ takes 0. This only happens in the classical case, where p takes 1, and q takes 0. But since p takes 1, the conclusion is designated.

(5) $\neg(p \supset q) \vDash_{Ł_3} p$

Suppose the premise is designated. Then $\neg(p \supset q)$ takes the value 1. Then $p \supset q$ takes 0. This only happens in the classical case, where p takes 1, and q takes 0. But since p takes 1, the conclusion is designated.

(5) $\neg(p \supset q) \vDash_{LP} p$

Suppose the conclusion is undesignated. Then p takes the value 0. Then $p \supset q$ takes 1, or i. So the premise $\neg(p \supset q)$ takes 0 or i, and is undesignated.

(5) $\neg(p \supset q) \vDash_{RM_3} p$

Suppose the conclusion is undesignated. Then p takes the value 0. Then $p \supset q$ takes 1, or i. So the premise $\neg(p \supset q)$ takes 0 or i, and is undesignated.

(6) $p \supset r \vDash_{K_3} (p \wedge q) \supset r$

Suppose the conclusion is undesignated. Then $(p \wedge q) \supset r$ takes the value 0 or i.

If it takes 0, then $p \wedge q$ takes 1, and r takes 0. Hence p takes 1, making $p \supset r$ false: the premise is undesignated.

If it takes i, then there are three possibilities: $(1, i), (i, i)$, and $(i, 0)$.

For the first case, $p \wedge q$ is true, so p and q take 1. $v(r) = i$, so $v(p \supset r) = i$: the premise is undesignated.

For the second and third case, $v(p \wedge q) = i$, so one of p and q is i (the other is i or 1). In either case, the antecedent of the premise is i, and the consequent is i or 0 respectively. In either case the premise is undesignated.

(6) $p \supset r \vDash_{Ł_3} (p \wedge q) \supset r$

Suppose the conclusion is undesignated. Then $(p \wedge q) \supset r$ takes the value 0 or i.

If it takes 0, then $p \wedge q$ takes 1, and r takes 0. Hence p takes 1, making $p \supset r$ false: the premise is undesignated.

If it takes i, then there are two possibilities: $(1, i)$ and $(i, 0)$.

For the first case, $p \wedge q$ is true, so p and q take 1. $v(r) = i$, so $v(p \supset r) = i$: the premise is undesignated.

For the second case, $v(p \wedge q) = i$, so one of p and q is i (the other is i or 1). In either case, the antecedent of the premise is i, and the consequent is 0, so the premise is undesignated.

(6) $p \supset r \vDash_{LP} (p \wedge q) \supset r$

Suppose the conclusion is undesignated. Then $(p \wedge q) \supset r$ takes the value 0. Accordingly $p \wedge q$ takes 1, and r takes 0. Hence p takes 1, making $p \supset r$ false: the premise is undesignated.

(6) $p \supset r \vDash_{RM_3} (p \wedge q) \supset r$

Suppose the conclusion is undesignated. Then $(p \wedge q) \supset r$ takes the value 0. Looking at the truth-table for RM_3, there are three possibilities, $(1, i), (1, 0)$, and $(i, 0)$.

$\boxed{v(p \wedge q) = 1, v(r) = i}$

$v(p \wedge q) = 1$, so $v(p) = 1$. Therefore $v(p \supset r) = 0$: the premise is undesignated.

$\boxed{v(p \wedge q) = 1, v(r) = 0}$

$v(p \wedge q) = 1$, so $v(p) = 1$. Therefore $v(p \supset r) = 0$: the premise is undesignated.

$\boxed{v(p \wedge q) = i, v(r) = 0}$

$v(p \wedge q) = i$, so $v(p) = i$ or 1. Therefore $v(p \supset r) = 0$: the premise is undesignated.

(7) $p \supset q, q \supset r \vDash_{K_3} p \supset r$

Suppose the conclusion is undesignated. Then $v(p \supset r) = 0$, or $v(p \supset r) = i$.

If $v(p \supset r) = 0$, then $v(p) = 1, v(r) = 0$. If $v(q) = 0$ then $v(p \supset q) = 0$. If $v(q) = 1$ then $v(q \supset r) = 0$. If $v(q) = i$, then $v(p \supset q) = i$ In either case, the premise is undesignated.

If $v(p \supset r) = i$ then there are three possibilities:$(1, i), (i, i)$, and $(i, 0)$

$\boxed{v(p) = 1, v(r) = i}$

If $v(q) = 0$, then $v(p \supset q) = 0$. If $v(q) = 1$ then $v(q \supset r) = i$. If $v(q) = i$, then $v(p \supset q) = i$ In either case, the premise is undesignated.

$\boxed{v(p) = i, v(r) = i}$

If $v(q) = 0$, then $v(p \supset q) = i$. If $v(q) = 1$ then $v(q \supset r) = i$. If $v(q) = i$, then $v(p \supset q) = i$ In either case, the premise is undesignated.

$\boxed{v(p) = i, v(r) = 0}$

If $v(q) = 0$, then $v(p \supset q) = i$. If $v(q) = 1$ then $v(q \supset r) = 0$. If $v(q) = i$, then $v(p \supset q) = i$ In either case, the premise is undesignated.

(7) $p \supset q, q \supset r \vDash_{Ł_3} p \supset r$

Suppose the conclusion is undesignated. Then $v(p \supset r) = 0$, or $v(p \supset r) = i$.

If $v(p \supset r) = 0$, then $v(p) = 1, v(r) = 0$. If $v(q) = 0$ then $v(p \supset q) = 0$. If $v(q) = 1$ then $v(q \supset r) = 0$. If $v(q) = i$, then $v(p \supset q) = i$ In either case, the premise is undesignated.

If $v(p \supset r) = i$ then there are two possibilities:$(1, i)$ and $(i, 0)$

$\boxed{v(p) = 1, v(r) = i}$

If $v(q) = 0$, then $v(p \supset q) = 0$. If $v(q) = 1$ then $v(q \supset r) = i$. If $v(q) = i$, then $v(p \supset q) = i$ In either case, the premise is undesignated.

$\boxed{v(p) = i, v(r) = 0}$

If $v(q) = 0$, then $v(p \supset q) = i$. If $v(q) = 1$ then $v(q \supset r) = 0$. If $v(q) = i$, then $v(q \supset r) = i$ In either case, the premise is undesignated.

(7) $p \supset q, q \supset r \nvDash_{LP} p \supset r$

$v(p) = 1, v(r) = 0, v(q) = i$. On this valuation, the premises both take i, and hence are designated, but the conclusion is undesignated, showing that the inference is invalid.

(7) $p \supset q, q \supset r \vDash_{RM_3} p \supset r$

Suppose the conclusion is undesignated. Then $v(p \supset r) = 0$.

Looking at the truth-table for $\supset_{RM_3}$, there are three possibilities: $(1, i), (1, 0)$, and $(i, 0)$

$\boxed{v(p) = 1, v(r) = i}$

If $v(q) = 0$, then $v(p \supset q) = 0$. If $v(q) = 1$ then $v(q \supset r) = 0$. If $v(q) = i$, then $v(p \supset q) = 0$ In either case, the premise is undesignated.

$\boxed{v(p) = 1, v(r) = 0}$

If $v(q) = 0$, then $v(p \supset q) = 0$. If $v(q) = 1$ then $v(q \supset r) = 0$. If $v(q) = i$, then $v(q \supset r) = 0$ In either case, the premise is undesignated.

$\boxed{v(p) = i, v(r) = 0}$

If $v(q) = 0$, then $v(p \supset q) = 0$. If $v(q) = 1$ then $v(q \supset r) = 0$. If $v(q) = i$, then $v(q \supset r) = 0$ In either case, the premise is undesignated.

(8) $p \supset q \models_{K_3} \neg q \supset \neg p$

Suppose the conclusion is undesignated. Then $v(\neg q \supset \neg p)$ is either 0 or i.

If $v(\neg q \supset \neg p) = 0$, then $v(\neg q) = 1$ and $v(\neg p) = 0$. So $v(q) = 0$ and $v(p) = 1$. But then $v(p \supset q) = 0$.

If $v(\neg q \supset \neg p) = i$ then there are three possibilities: $(1, i), (i, i)$, and $(i, 0)$

$\boxed{v(\neg q) = 1, v(\neg p) = i}$

$v(q) = 0$, and $v(p) = 1$, so $v(p \supset q) = i$; the premise is undesignated.

$\boxed{v(\neg q) = i, v(\neg p) = i}$

$v(q) = i$, and $v(p) = i$. Thus $v(p \supset q) = i$; the premise is undesignated.

$\boxed{v(\neg q) = i, v(\neg p) = 0}$

$v(q) = i$, and $v(p) = 1$. Thus $v(p \supset q) = i$; the premise is undesignated.

(8) $p \supset q \models_{Ł_3} \neg q \supset \neg p$

Suppose the conclusion is undesignated. Then $v(\neg q \supset \neg p)$ is either 0 or i.

If $v(\neg q \supset \neg p) = 0$, then $v(\neg q) = 1$ and $v(\neg p) = 0$. So $v(q) = 0$ and $v(p) = 1$. But then $v(p \supset q) = 0$.

If $v(\neg q \supset \neg p) = i$ then there are two possibilities: $(1, i)$ and $(i, 0)$

$\boxed{v(\neg q) = 1, v(\neg p) = i}$

$v(q) = 0$, and $v(p) = 1$, so $v(p \supset q) = i$; the premise is undesignated.

$v(\neg q) = i, v(\neg p) = 0$

$v(q) = i$, and $v(p) = 1$. Thus $v(p \supset q) = i$; the premise is undesignated.

(8) $p \supset q \models_{LP} \neg q \supset \neg p$

Suppose the conclusion is undesignated. Then $v(\neg q \supset \neg p) = 0$.

If $v(\neg q \supset \neg p) = 0$, then $v(\neg q) = 1$ and $v(\neg p) = 0$. So $v(q) = 0$ and $v(p) = 1$. But then $v(p \supset q) = 0$; the premise is undesignated.

(8) $p \supset q \models_{RM_3} \neg q \supset \neg p$

Suppose the conclusion is undesignated. Then $v(\neg q \supset \neg p) = 0$.

There are three possibilities: $(1, i), (1, 0)$, and $(i, 0)$

$v(\neg q) = 1, v(\neg p) = i$

$v(q) = 0$ and $v(p) = i$. So $v(p \supset q) = 0$; the premise is undesignated.

$v(\neg q) = 1, v(\neg p) = 0$

$v(q) = 0$ and $v(p) = 1$. So $v(p \supset q) = 0$; the premise is undesignated.

$v(\neg q) = i, v(\neg p) = 0$

$v(q) = i$ and $v(p) = 1$. So $v(p \supset q) = 0$; the premise is undesignated.

(9) $\not\models_{K_3} p \supset (q \vee \neg q)$

$v(p) = 1, v(q) = i$ — On this interpretation the truth-value of the conditional is i, which is not designated in K_3.

(9) $\not\models_{Ł_3} p \supset (q \vee \neg q)$

$v(p) = 1, v(q) = i$ — On this interpretation the truth-value of the conditional is i, which is not designated in $Ł_3$.

(9) $\models_{LP} p \supset (q \vee \neg q)$

Suppose the conclusion is undesignated, then $v(q \vee \neg q) = 0$ But the only way this could come about were if $v(q) = 0$ and $v(\neg q) = 0$, which cannot be. Since there is no interpretation showing the conclusion to be undesignated, the inference is valid.

(9) $\not\models_{RM_3} p \supset (q \vee \neg q)$

$v(p) = 1, v(q) = i$ — On this interpretation the truth-value of the conditional is 0

(10)$\not\models_{K_3} (p \wedge \neg p) \supset q$

$v(p) = i, v(q) = 0$ — On this interpretation the truth-value of the conditional is i, which is not designated in K_3.

(10)$\not\models_{Ł_3} (p \wedge \neg p) \supset q$

$v(p) = i, v(q) = 0$ — On this interpretation the truth-value of the conditional is i, which is not designated in $Ł_3$.

(10)$\models_{LP} (p \wedge \neg p) \supset q$

Suppose the conclusion is undesignated, then $v((p \wedge \neg p) \supset q) = 0$. Therefore $v(p \wedge \neg p) = 1$, which is impossible.

(10)$\not\models_{RM_3} (p \wedge \neg p) \supset q$

$v(p) = i, v(q) = 0$ — On this interpretation the truth-value of the conditional is 0.

2. Call a many-valued logic in the language of the classical propositional calculus *normal* if, amongst its truth values are two, 1 and 0, such that 1 is designated, 0 is not, and for every truth function corresponding to a connective, the output for these inputs is the same as the classical output. ($K_3, Ł_3, LP$ and RM_3 are all normal.) Show that every normal many-valued logic is a sub-logic of classical logic (i.e., that every inference valid in the logic is valid in classical logic).

Contrapositive proof: Take an inference which is invalid in classical logic: $\Sigma \not\models A$. Consider a classical interpretation showing this to be invalid, I. Because all formulae in Σ and A are truth-functional, and all inputs in classical logic are 1 or 0, there is a normal interpretation I^N which agrees with I on its assignment of truth-values. I^N shows $\Sigma \not\models A$ in all normal many-valued logics.

■

3. Observe that in K_3 if an interpretation assigns the value i to a propositional parameter that occurs in a formula, then it assigns that value to the formula itself. Infer that there are no logical truths in K_3. Are there any logical truths in Ł$_3$?

A logical truth is a sentence that is designated on all interpretations. There are no such sentences in K_3:

As observed in the question, for all formulas, there is a interpretation that assigns i to all propositional parameters in the formula, and hence assigns the value to the whole formula. (This can be seen by a quick look at the truth-tables for K_3.) Since i is not designated, all sentences in K_3 are undesignated on the interpretation which assigns i to all parameters. Therefore there are no logical truths in K_3.

There are logical truths in Ł$_3$. For instance $\models_{Ł_3} A \supset A$, as can be shown by a short truth-table:

A	$A \supset A$
1	**1**
i	**1**
0	**1**

■

4. Let v_1 and v_2 be any interpretations of K_3 or LP. Write $v_1 \preceq v_2$ to mean that for every propositional parameter p:

if $v_1(p) = 1$, then $v_2(p) = 1$; and if $v_1(p) = 0$, then $v_2(p) = 0$

Show by induction on the way that formulas are constructed, that if $v_1 \preceq v_2$, then the displayed condition is true for all formulas. Does the result hold for Ł$_3$ and RM_3?

For both K_3 and LP:

The atomic case requires no argument.

$\boxed{\neg A}$

If $v_1(\neg A) = 1$, then $v_1(A) = 0$. So $v_2(A) = 0$. But then $v_2(\neg A) = 1$

$\boxed{A \wedge B}$

If $v_1(A \wedge B) = 1$, then $v_1(A) = 1$ and $v_1(B) = 1$. So $v_2(A) = 1$ and $v_2(B) = 1$. But then $v_2(A \wedge B) = 1$.

If $v_1(A \wedge B) = 0$, then $v_1(A) = 0$ or $v_1(B) = 0$. So $v_2(A) = 0$ or $v_2(B) = 0$. But then $v_2(A \wedge B) = 0$.

$\boxed{A \vee B}$

If $v_1(A \vee B) = 1$, then $v_1(A) = 1$ or $v_1(B) = 1$. So $v_2(A) = 1$ or $v_2(B) = 1$. But then $v_2(A \vee B) = 1$.

If $v_1(A \vee B) = 0$, then $v_1(A) = 0$ and $v_1(B) = 0$. So $v_2(A) = 0$ and $v_2(B) = 0$. But then $v_2(A \vee B) = 0$.

$\boxed{A \supset B}$

If $v_1(A \supset B) = 1$, then $v_1(A) = 0$ or $v_1(B) = 1$. So $v_2(A) = 0$ and $v_2(B) = 1$. But then $v_2(A \supset B) = 1$.

If $v_1(A \supset B) = 0$, then $v_1(A) = 1$ and $v_1(B) = 0$. So $v_2(A) = 1$ or $v_2(B) = 0$. But then $v_2(A \supset B) = 0$.

$\boxed{A \equiv B}$

If $v_1(A \equiv B) = 1$, then $v_1(A) = v_1(B) = 1$ or $v_1(A) = v_1(B) = 0$. So $v_2(A) = v_2(B) = 1$ or $v_2(A) = v_2(B) = 0$. But then $v_2(A \equiv B) = 1$.

If $v_1(A \equiv B) = 0$, then $v_1(A) = 1, v_1(B) = 0$ or $v_1(A) = 0, v_1(B) = 1$. So $v_2(A) = 1, v_2(B) = 0$ or $v_2(A) = 0, v_2(B) = 1$. But then $v_2(A \equiv B) = 0$.

■

The result does not hold in Ł$_3$ or RM_3. In both, $\supset$ presents an exception:

For Ł$_3$:

Let $v_1(p) = v_1(q) = i$, and $v_2(p) = 1, v_2(q) = 0$. Then $v_1 \preceq v_2$, but $v_1(p \supset q) = 1$ and $v_2(p \supset q) = 0$.

For RM_3

Let $v_1(p) = 1, v_1(q) = i$, and $v_2(p) = 1, v_2(q) = 1$. Then $v_1 \preceq v_2$, but $v_1(p \supset q) = 0$ and $v_2(p \supset q) = 1$.

■

5. By problem 2, if $\vDash_{LP} A$, then A is a classical logic truth. Use problem 4 to show the converse. (Hint: Suppose that v is an LP interpretation such that $v(A) = 0$. Consider the interpretation, v', which is the same as v except that if $v(p) = i$, $v'(p) = 0$.)

We need to prove that there is no case of a classical logical truth which is invalid in LP. Converse proof: Suppose that v is an LP interpretation such that $\nvDash_{LP} A$. Let v' be any classical interpretation which is obtained by substituting 1 or 0 for every instance of i in v. Then $v \preceq v'$. By problem 4, this means that if $v(A) = 0$, $v'(A) = 0$. So, v' is a classical interpretation which shows that $\nvDash A$.

■

9. *Fill in the details omitted in 7.11.2.

Lemma: For no n is D_{n+1} a logical truth of any modal logic weaker than $K\upsilon$ or of intuitionist logic.

I will first show the result for $K\upsilon$.

D_{n+1} is the disjunction of all sentences of the form $\Box(p_i \supset p_j) \wedge \Box(p_j \supset p_i)$, where $1 \leq i < j \leq n+1$. Since i and j are defined as distinct, there is a $K\upsilon$ interpretation with a world where p_i is true and p_j is false, (or p_i is false and p_j is true), for each pair of i and j. This interpretation will make the disjunction false at every world. To help to visualise this, here is a counter-model for D_1

$\nvDash_{K\upsilon} \Box(p_1 \supset p_2) \wedge \Box(p_2 \supset p_1)$

$W = \{w_0, w_1\}; v_{w_1}(p_1) = 1, v_{w_1}(p_2) = 0$

Clearly the inference is invalid, because w_1 of the interpretation shows the left conjunct to be false.

A similar proof works for intuitionist logic:

D_{n+1} is the disjunction of all sentences of the form $(p_i \sqsupset p_j) \wedge (p_j \sqsupset p_i)$, where $1 \leq i < j \leq n+1$. Since i and j are defined as distinct, we can create an intuitionist interpretation with a world where p_i is false, and p_j is true (or vice versa), for each pair of i and j. However we also need the interpretation to satisfy the heredity condition. Let worlds which contain p_i and p_j be w_{ij}. And let w_0 be such that no parameters are true at w_0, and for all w_{ij}, Rw_0w_{ij}. This interpretation will make the disjunction false at w_0. To help to visualise this, here is a counter-model for D_1:

$\not\models_I \Box(p_1 \supset p_2) \wedge \Box(p_2 \supset p_1)$

$W = \{w_0, w_1\}; Rw_0w_0, Rw_0w_1, Rw_1w_1; v_{w_1}(p_1) = 1, v_{w_1}(p_2) = 0$

The result has been shown for $K\upsilon$ and intuitionist logic, therefore it holds for all weaker logics.

■