Solutions

Louis Barson
Kyoto University

March 27, 2009

1. Verify the claims made about intuitionist validity, left as exercises in 6.6.

The intuitionist conditional validates the paradoxes of the material conditional.

$q \vDash p \supset q$

$$\begin{array}{c} q, +0 \\ p \supset q, -0 \\ 0r0,\ 0r1,\ 1r1 \\ p, +1 \\ q, -1 \\ q, +1 \\ \otimes \end{array}$$

$\neg p \vDash p \supset q$

$$\begin{array}{c} \neg p, +0 \\ p \supset q, -0 \\ 0r0,\ 0r1,\ 1r1 \\ p, +1 \\ q, -1 \\ p, -1 \\ \otimes \end{array}$$

The intuitionist conditional does not validate the the more damaging paradoxes of 1.9:

$(p \wedge q) \sqsupset s \nvDash (p \sqsupset s) \vee (q \sqsupset s)$

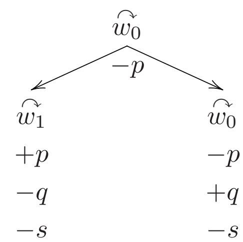
$$(p \wedge q) \sqsupset s, +0$$
$$(p \sqsupset s) \vee (q \sqsupset s), -0$$
$$0r0$$
$$p \sqsupset s, -0$$
$$q \sqsupset s, -0$$
$$0r1,\ 1r1$$
$$p, +1$$
$$s, -1$$
$$0r2, 2r2$$
$$q, +2$$
$$s, -2$$

- $p \wedge q, -1$
  - $p, -1$
    $\otimes$
  - $q, -1$
    - $p \wedge q, -2$
      - $p, -2$
        - $p \wedge q, -0$
          - $p, -0$
          - $q, -0$
        - $s + 0$
      - $q, -2$
        $\otimes$
    - $s, +2$
      $\otimes$
- $s, +1$
  $\otimes$

Counter-model from the left-most open branch such that:

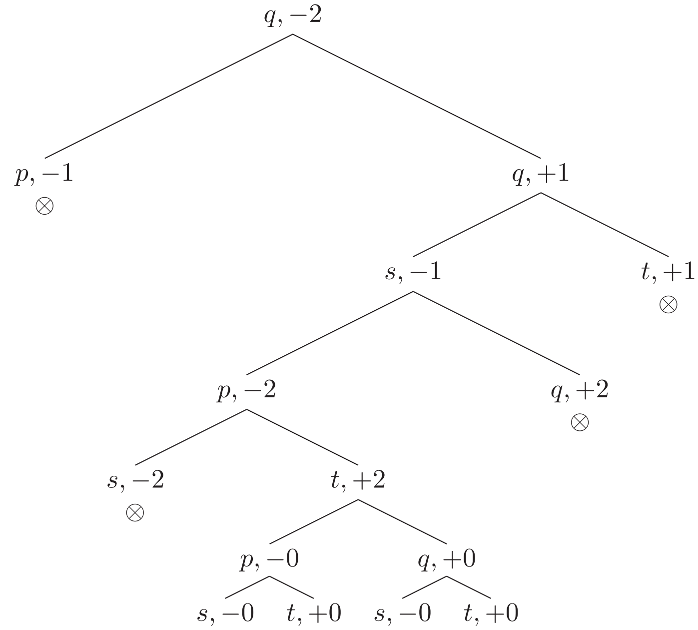
$$W = \{w_0, w_1, w_2\}$$
$$w_0Rw_0, w_1Rw_1, w_2Rw_2, w_0Rw_1, w_0Rw_2$$
$$v_{w_0}(p) = 0, v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_1}(s) = 0, v_{w_2}(p) = 0, v_{w_2}(q) = 1, v_{w_2}(s) = 0$$

This can be represented in the following picture:



$(p \supset q) \wedge (s \supset t) \nvDash (p \supset t) \vee (s \supset q)$

$$(p \supset q) \wedge (s \supset t), +0$$
$$(p \supset t) \vee (s \supset q), -0$$
$$0r0$$
$$p \supset q, +0$$
$$s \supset t, +0$$
$$p \supset t, -0$$
$$s \supset q, -0$$
$$0r1,\ 1r1$$
$$p, +1$$
$$t, -1$$
$$0r2,\ 2r2$$
$$s, +2$$
$$q, -2$$



Counter-model from the left-most open branch such that:

$$W = \{w_0, w_1, w_2\}$$
$$w_0 R w_0, w_1 R w_1, w_2 R w_2, w_0 R w_1, w_0 R w_2$$
$$v_{w_0}(p) = 0, v_{w_0}(s) = 0, v_{w_1}(p) = 1, v_{w_1}(q) = 1, v_{w_1}(s) = 0$$
$$v_{w_1}(t) = 0, v_{w_2}(p) = 0, v_{w_2}(q) = 0, v_{w_2}(s) = 1, v_{w_2}(t) = 1$$

This can be represented in the following picture:

$$\begin{array}{ccc} & \overset{\curvearrowright}{w_0} & \\ \swarrow & -p & \searrow \\ \overset{\curvearrowright}{w_1} & -s & \overset{\curvearrowright}{w_2} \\ +p & & -p \\ +q & & -q \\ -s & & +s \\ -t & & +t \end{array}$$

$\rightharpoondown (p \sqsupset q) \not\models p$

$$\begin{array}{c} \rightharpoondown (p \sqsupset q), +0 \\ p, -0 \\ 0r0 \\ p \sqsupset q, -0 \\ 0r1,\ 1r1 \\ p, +1 \\ q, -1 \end{array}$$

Counter-model such that:

$$W = \{w_0, w_1\}$$

$$w_0 R w_0, w_1 R w_1, w_0 R w_1$$

$$v_{w_0}(p) = 0, v_{w_1}(p) = 1, v_{w_1}(q) = 0$$

This can be represented in the following picture:

$$\begin{array}{ccc} \overset{\curvearrowright}{w_0} & \longrightarrow & \overset{\curvearrowright}{w_1} \\ -p & & +p \\ & & -q \end{array}$$

The intuitionist conditional is not suitable as an account of a conditional with an enthymematic *ceteris paribus* clause:

$p \supset q \models \neg q \supset \neg p$

$$
\begin{array}{c}
p \supset q, +0 \\
\neg q \supset \neg p, -0 \\
0\mathrm{r}0,\ 0\mathrm{r}1,\ 1\mathrm{r}1 \\
\neg q, +1 \\
\neg p, -1 \\
\begin{array}{cc}
\begin{array}{c}
p, -1 \\
1\mathrm{r}2, 2\mathrm{r}2,\ 0\mathrm{r}2 \\
p, +2 \\
q, -2 \\
\begin{array}{cc}
p, -2 & q, +2 \\
\otimes & \otimes
\end{array}
\end{array}
&
\begin{array}{c}
q, +1 \\
q, -1 \\
\otimes
\end{array}
\end{array}
\end{array}
$$

$p \supset q, q \supset s \models p \supset s$

$$
\begin{array}{c}
p \supset q, +0 \\
q \supset s, +0 \\
p \supset s, -0 \\
0\mathrm{r}0,\ 0\mathrm{r}1,\ 1\mathrm{r}1 \\
p, +1 \\
s, -1 \\
\begin{array}{cc}
\begin{array}{c}
p, -1 \\
\otimes
\end{array}
&
\begin{array}{c}
q, +1 \\
\begin{array}{cc}
q, -1 & s, +1 \\
\otimes & \otimes
\end{array}
\end{array}
\end{array}
\end{array}
$$

$p \supset s \models (p \wedge q) \supset s$

$$
\begin{array}{c}
p \supset s, +0 \\
(p \wedge q) \supset s, -0 \\
0\mathrm{r}0,\ 0\mathrm{r}1,\ 1\mathrm{r}1 \\
p \wedge q, +1 \\
s, -1 \\
p, +1 \\
q, +1 \\
\begin{array}{cc}
p, -1 & s, +1 \\
\otimes & \otimes
\end{array}
\end{array}
$$

The intuitionist conditional validates the strict paradox:

$\models (p\wedge \rightharpoondown p) \sqsupset q$

$$\begin{array}{c}
(p\wedge \rightharpoondown p) \sqsupset q, -0 \\
\text{0r0, 0r1, 1r1} \\
p\wedge \rightharpoondown p, +1 \\
q, -1 \\
p, +1 \\
\rightharpoondown p, +1 \\
p, -1 \\
\otimes
\end{array}$$

However it does not validate the classical instance, because $q\vee \rightharpoondown q$ is not a logical truth:

$\not\models p \sqsupset (q\vee \rightharpoondown q)$

$$\begin{array}{c}
p \sqsupset (q\vee \rightharpoondown q), -0 \\
\text{0r0, 0r1, 1r1} \\
p, +1 \\
q\vee \rightharpoondown q, -1 \\
q, -1 \\
\rightharpoondown q, -1 \\
\text{1r2, 2r2, 0r2} \\
q, +2 \\
p, +2
\end{array}$$

Counter-model such that:

$$W = \{w_0, w_1, w_2\}$$

$$w_0Rw_0, w_1Rw_1, , w_2Rw_2, w_0Rw_1, w_1Rw_2, w_0Rw_2$$

$$v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_2}(p) = 1, v_{w_2}(q) = 1$$

This can be represented in the following picture:

$$\begin{tikzcd}
\overset{\frown}{w_0} \arrow[r] \arrow[rr, bend right] & \overset{\frown}{w_1} \arrow[r] & \overset{\frown}{w_2} \\
& +p & +p \\
& -q & +q
\end{tikzcd}$$

2. Show that in an intuitionist interpretation, $\neg\neg A$ is true at a world, w iff for all w' such that wRw', there is a w'' such that $w'Rw''$ and A is true at w''.

$$\begin{array}{lll}
\neg\neg A \text{ is true at world } w & \text{iff} & v_w(\neg\neg A) = 1 \\
& \text{iff} & \text{for all } w' \text{ such that } wRw',\ v_{w'}(\neg A) = 0 \\
& \text{iff} & \text{for all } w' \text{ such that } wRw', \text{ there is a } w'' \text{ such that} \\
& & w'Rw'', \text{ and } v_{w''}(A) = 1
\end{array}$$

■

3. Show the following in intuitionist logic:

(a) $\vdash (p \wedge (\neg p \vee q)) \supset q$

$$\begin{array}{c}
(p \wedge (\neg p \vee q)) \supset q, -0 \\
0r0 \\
p \wedge (\neg p \vee q), +0 \\
q, -0 \\
p, +0 \\
\neg p \vee q, +0 \\
\begin{array}{cc}
\neg p, +0 & q, +0 \\
p, -0 & \otimes \\
\otimes &
\end{array}
\end{array}$$

(b) $\vdash \neg (p \wedge \neg p)$

$$\begin{array}{c}
\neg (p \wedge \neg p), -0 \\
0r0,\ 0r1,\ 1r1 \\
p \wedge \neg p, +1 \\
p, +1 \\
\neg p, +1 \\
p, -1 \\
\otimes
\end{array}$$

(c) $\neg p \vee q \vdash p \supset q$

$$\begin{array}{c}
\neg p \vee q, +0 \\
p \supset q, -0 \\
0r0,\ 0r1,\ 1r1 \\
p, +1 \\
q, -1 \\
\begin{array}{cc}
\neg p, +0 & q, +0 \\
p, -1 & q, +1 \\
\otimes & \otimes
\end{array}
\end{array}$$

(d) $\rightharpoondown (p \vee q) \vdash \rightharpoondown p \wedge \rightharpoondown q$

$$
\begin{array}{c}
\rightharpoondown (p \vee q), +0 \\
\rightharpoondown p \wedge \rightharpoondown q, -0 \\
0\text{r}0 \\
p \vee q, -0 \\
p, -0 \\
q, -0 \\
\begin{array}{cc}
\rightharpoondown p, -0 & \rightharpoondown q, -0 \\
0\text{r}1,\ 1\text{r}1 & 0\text{r}1,\ 1\text{r}1 \\
p, +1 & p, +1 \\
p \vee q, -1 & p \vee q, -1 \\
p, -1 & p, -1 \\
q, -1 & q, -1 \\
\otimes & \otimes
\end{array}
\end{array}
$$

(e) $\rightharpoondown p \wedge \rightharpoondown q \vdash \rightharpoondown (p \vee q)$

$$
\begin{array}{c}
\rightharpoondown p \wedge \rightharpoondown q, +0 \\
\rightharpoondown (p \vee q), -0 \\
0\text{r}0 \\
\rightharpoondown p, +0 \\
\rightharpoondown q, +0 \\
0\text{r}1,\ 1\text{r}1 \\
p \vee q, +1 \\
p, -1 \\
q, -1 \\
\begin{array}{cc}
p, +1 & q, +1 \\
\otimes & \otimes
\end{array}
\end{array}
$$

(f) $\rightharpoondown p \vee \rightharpoondown q \vdash \rightharpoondown (p \wedge q)$

$$
\begin{array}{c}
\rightharpoondown p \vee \rightharpoondown q, +0 \\
\rightharpoondown (p \wedge q), -0 \\
0\text{r}0,\ 0\text{r}1,\ 1\text{r}1 \\
p \wedge q, +1 \\
p, +1 \\
q, +1 \\
\begin{array}{cc}
\rightharpoondown p, +0 & \rightharpoondown q, +0 \\
p, -1 & q, -1 \\
\otimes & \otimes
\end{array}
\end{array}
$$

(g) $p \supset (p \supset q) \vdash p \supset q$

$$
\begin{array}{c}
p \supset (p \supset q), +0 \\
p \supset q, -0 \\
0r0,\ 0r1,\ 1r1 \\
p, +1 \\
q, -1 \\
\begin{array}{cc}
\begin{array}{c} p, -1 \\ \otimes \end{array} &
\begin{array}{c} p \supset q, +1 \\
\begin{array}{cc}
\begin{array}{c} p, -1 \\ \otimes \end{array} & \begin{array}{c} q, +1 \\ \otimes \end{array}
\end{array}
\end{array}
\end{array}
\end{array}
$$

(h) $\vdash \neg\neg (p \vee \neg p)$

$$
\begin{array}{c}
\neg\neg (p \vee \neg p), -0 \\
0r0,\ 0r1,\ 1r1 \\
\neg (p \vee \neg p), +1 \\
p \vee \neg p, -1 \\
p, -1 \\
\neg p, -1 \\
1r2,\ 2r2,\ 0r2 \\
p, +2 \\
p \vee \neg p, -2 \\
p, -2 \\
\neg p, -2 \\
\otimes
\end{array}
$$

4. Either by using tableaux, or by constructing counter-models directly, show each of the following. In each case, define the interpretation and draw a picture of it. (For simplicity, omit the extra arrows required by transitivity. Take them as read.) Check that the interpretation works.

(a) $\nvDash p \vee \neg p$

$$
\begin{array}{c}
p \vee \neg p, -0 \\
0r0 \\
p, -0 \\
\neg p - 0 \\
0r1,\ 1r1 \\
p, +1
\end{array}
$$

Counter-model such that:

$$W = \{w_0, w_1\}; w_0 R w_0, w_1 R w_1, w_0 R w_1; v_{w_0}(p) = 0, v_{w_1}(p) = 1$$

This can be represented in the following picture:

$$\begin{array}{ccc} \overset{\curvearrowright}{w_0} & \longrightarrow & \overset{\curvearrowright}{w_1} \\ -p & & +p \end{array}$$

p is false at w_0, and $\neg p$ is false at w_0 because p is true at w_1, therefore $p \vee \neg p$ is also false at w_0.

(b) $\neg p \sqsupset p \nvDash p$

Counter-model such that:

$$W = \{w_0, w_1\}$$

$$w_0 R w_0, w_1 R w_1, w_0 R w_1$$

$$v_{w_0}(p) = 0, v_{w_1}(p) = 1$$

This can be represented in the following picture:

$$\begin{array}{ccc} \overset{\curvearrowright}{w_0} & \longrightarrow & \overset{\curvearrowright}{w_1} \\ -p & & +p \end{array}$$

At w_1, p is true, therefore $\neg p$ is false at w_0, making the premise $\neg p \sqsupset p$ true at w_0. The conclusion is false at w_0.

(c) $\neg (p \wedge q) \nvDash \neg p \vee \neg q$

Counter-model such that:

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_0, w_1 R w_1, w_2 R w_2, w_0 R w_1, w_0 R w_2$$

$$v_{w_0}(p) = 0, v_{w_1}(p) = 0, v_{w_1}(q) = 1, v_{w_2}(p) = 1, v_{w_2}(q) = 0$$

This can be represented in the following picture:

$$\begin{array}{ccc} & \overset{\frown}{w_0} & \\ \swarrow & -p & \searrow \\ \overset{\frown}{w_1} & & \overset{\frown}{w_2} \\ -p & & +p \\ +q & & -q \end{array}$$

At all three worlds, $p \wedge q$ is false, so the premise, $\rightharpoondown (p \wedge q)$, is true at w_0. $\rightharpoondown p$ is false at w_0 because p is true at w_2. $\rightharpoondown q$ is false at w_0 because q is true at w_1. Therefore $\rightharpoondown p \vee \rightharpoondown q$ is false at w_0.

(d) $\rightharpoondown p \sqsupset \rightharpoondown q \nvDash q \sqsupset p$

Counter-model such that:

$$W = \{w_0, w_1\}$$

$$w_0 R w_0, w_1 R w_1, w_0 R w_1$$

$$v_{w_0}(p) = 0, v_{w_0}(q) = 1, v_{w_1}(p) = 1, v_{w_1}(q) = 0$$

This can be represented in the following picture:

$$\begin{array}{ccc} \overset{\frown}{w_0} & \longrightarrow & \overset{\frown}{w_1} \\ -p & & +p \\ +q & & -q \end{array}$$

The premise states that for all worlds, if $\rightharpoondown p$, then $\rightharpoondown q$. Because $\rightharpoondown p$ is not true at any world, the premise is true. At w_0, q is true, and p is false, so the conclusion $q \sqsupset p$ is false.

(e) $p \sqsupset (q \vee r) \nvDash (p \sqsupset q) \vee (p \sqsupset r)$

$p \sqsupset (q \vee r), +0$
$(p \sqsupset q) \vee (p \sqsupset r), -0$
0r0
$p \sqsupset q, -0$
$p \sqsupset r, -0$
0r1, 1r1
$p, +1$
$q, -1$
0r2, 2r2
$p, +2$
$r, -2$

$p, -1$ | $q \vee r, +1$
$\otimes$

$q, +1$ | $r, +1$
$\otimes$

$p, -2$ | $q \vee r, +2$
$\otimes$

$q, +2$ | $r, +2$
$\otimes$

$p, -0$ | $q \vee r, +0$

$q, +0$ | $r, +0$

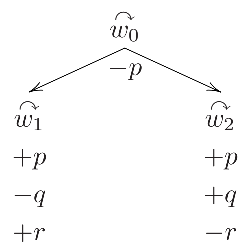
Counter-model from the open left-most branch such that:

$$W = \{w_0, w_1, w_2\}$$

$$w_0Rw_0, w_1Rw_1, w_2Rw_2, w_0Rw_1, w_0Rw_2$$

$$v_{w_0}(p) = 0, v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_1}(r) = 1, v_{w_2}(p) = 1, v_{w_2}(q) = 1, v_{w_2}(r) = 0$$

This can be represented in the following picture:



The premise $p \sqsupset (q \vee r)$ is true at w_0, because in all p-worlds, (w_1, and w_2), either q or r is true. However, neither $p \sqsupset q$ nor $p \sqsupset r$ is true at w_0. For the former, p is true but q is false at w_1, for the latter p is true but r false at w_2. Therefore the conclusion, $(p \sqsupset q) \vee (p \sqsupset r)$, is false.

5. Show that if $\vDash A \vee B$ then $\vDash A$ or $\vDash B$. (Hint: take counter-models for A and B; let A fail in the first at w_A, and B fail in the second at w_B. Construct a counter-model for $A \vee B$ by putting the two together in an appropriate way, adding a new world, w, such that wRw_A and wRw_B.) Show that it is not the case that if $\vDash \neg (A \wedge B)$ then $\vDash \neg A$ or $\vDash \neg B$. (Hint: consider the formula $\neg (p \wedge \neg p)$.)

"Show that if $\vDash A \vee B$ then $\vDash A$ or $\vDash B$"

Contrapositive proof. Suppose that $\nvDash A$ and $\nvDash B$. Then there is an interpretation, I_A, and a world, w_A, in the interpretation, where A is not true. Similarly for I_B and w_B. We can assume that the worlds of the two interpretations are distinct.

Now let us construct the interpretation, I, whose worlds are those of I_A and I_B, which relate to each other exactly as they do in those interpretations, and where the truth values at each world are the same as those in those interpretations. In addition, there is one new world, w, such that w relates to itself, w_A, w_B, and all the worlds that w_A and w_B relate to. Finally, for any parameter, p, if $\nu_{w_A}(p) = 0$ then $\nu_w(p) = 0$, and if $\nu_{w_B}(p) = 0$ then $\nu_w(p) = 0$.

This is an intuitionist interpretation. Every formula has the same truth value at w_A in I and I_A, since we have not done anything to change these. Similarly, every formula has the same truth value at w_B in I and I_B. Now suppose that $\vDash A \vee B$. Then $A \vee B$ is true at w, so either A or B is true at w. If it is A, then by heredity, A is true at w_A, which it is not; similarly for B. Hence, $\nvDash A \vee B$.

"Show that it is not the case that if $\vDash \neg (A \wedge B)$ then $\vDash \neg A$ or $\vDash \neg B$."

Let A be p, and B be $\neg p$. $\vDash \neg (p \wedge \neg p)$ (shown in a tableau in 3(b)), but $\nvDash \neg p$ ($W = \{w_p\}$; w_pRw_p; $v_{w_p}(p) = 0$) and $\nvDash \neg\neg p$ ($W = \{w_p\}$; w_pRw_p;

$v_{w_p}(p) = 1)$

6. Show that in intuitionist logic $\nvDash (p \sqsupset q) \vee (q \sqsupset p)$. Show that this is valid in LC. (Hint: suppose that it is not, and argue by *reductio*.)

$\nvDash (p \sqsupset q) \vee (q \sqsupset p)$

$$(p \sqsupset q) \vee (q \sqsupset p), -0$$
$$0r0$$
$$p \sqsupset q, -0$$
$$q \sqsupset p, -0$$
$$0r1,\ 1r1$$
$$p, +1$$
$$q, -1$$
$$0r2,\ 2r2$$
$$q, +2$$
$$p, -2$$

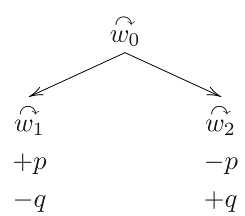
Counter-model such that:

$$W = \{w_0, w_1, w_2\}$$

$$w_0Rw_0, w_1Rw_1, w_2Rw_2, w_0Rw_1, w_0Rw_2$$

$$v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_1}(p) = 1, v_{w_2}(p) = 0, v_{w_2}(q) = 1$$

This can be represented in the following picture:



$\vDash_{LC} (p \sqsupset q) \vee (q \sqsupset p)$

If this were invalid, there would be a counter-model for it satisfying the restriction of LC. The LC restriction states that every world in an interpretation is related to, or related from, every other world. In other words, there can be no two unrelated worlds. However, any counter-model for the above requires there to be at least two worlds which are not related to each other:

Any counter-model I for the above must at the very least consist of a world w_i such that w_0Rw_i and $v_{w_i}(p) = 1, v_{w_i}(q) = 0$ (in order to make $p \sqsupset q$ false at w_0), and a world w_j such that w_0Rw_j and $v_{w_j}(p) = 0, v_{w_j}(q) = 1$ (in order to make $q \sqsupset p$ false at w_0. I cannot be an IC interpretation, because by the IC restriction, either $w_i R w_j$, $w_j R w_i$, or $w_i = w_j$. In the first case, by the heredity condition, $p, \multimap p$ at w_j. In the second case, by the heredity condition, $q, \multimap q$ at w_i. In the third case, $p, \multimap p, q \multimap q$ at w_i and w_j.

We will find a sound and complete tableaux rule for LC in question 10. We could also use this rule to show the above via a tableau:

$$
\begin{array}{c}
(p \sqsupset q) \vee (q \sqsupset p), -0 \\
0r0 \\
p \sqsupset q, -0 \\
q \sqsupset p, -0 \\
0r1,\ 1r1 \\
p, +1 \\
q, -1 \\
0r2,\ 2r2 \\
q, +2 \\
p, -2
\end{array}
$$

$$
\begin{array}{ccc}
1r2 & 1{=}2 & 2r1 \\
p, +2 & p, +2 & q, +1 \\
\otimes & \otimes & \otimes
\end{array}
$$

The tableau splits into three because w_1 and w_2 must be related or identical be by LC

8. *Consider the following tableau rule:

$$
\begin{array}{c}
p, -j \\
\text{irj} \\
\downarrow \\
p, -i
\end{array}
$$

Show that if this rule is added to tableaux for intuitionist logic, they are still sound. Use the completeness of intuitionist tableaux to infer that the rule is redundant.

All we need to do is check the new case for this rule in the Soundness Lemma.

Let b be any branch of a tableau, and $I =< W, R, v >$ be any intuitionist interpretation. We need to show that if I is faithful to b, and this rule is applied, than I is faithful to the extension produced.

Suppose that $p, -j$ and irj are on b, and that we apply the rule to get p, i. p is false at $f(j)$ and $f(i) R f(j)$. If p were true at $f(i)$, then by the heredity rule,

p would be true at $f(j)$. This is not the case, so p is false at $f(i)$, as required.

"Use the completeness of intuitionist tableaux to infer that the rule is redundant."

Suppose the tableau with the new rule closes. Then since the new tableaux are sound, the inference is valid. Hence by the Completeness Theorem for the usual tableaux, the usual tableau for this inference closes. Hence, the extra rule is redundant.

9. *Call a *strong intuitionist interpretation* one where R satisfies the additional condition: for all x, $y \epsilon W$, if xRy and yRx then $x = y$. (This makes R a partial order.) If an inference is intuitionistically valid, it is obviously truth-preserving in all worlds of all strong intuitionist interpretations. Show the converse. (Hint: Consider the interpretation induced by an open branch of a tableau for an invalid inference.)

Show that if an inference is truth-preserving in all worlds of all strong Intuitionist interpretations, it is intuitionistically valid.

Contrapositive proof: Suppose that an inference is not intuitionistically valid. Consider the tableau showing this to be so, and the interpretation induced by any open branch. If i is any world on the branch, iri is on the branch. If j is any other number such that irj is on the branch, it is introduced by the $-$ rule for $\neg$ or $\sqsupset$. Moreover, none of the rules for r will then produce a line of the form jri (as the rule for σ would). Hence, the interpretation is a strong intuitionist interpretation.

I satisfies the heredity rule, for all $w \epsilon W$, if $v_w(p) = 1$ and wRw' then $v'_w(p) = 1$. The cases at which I and SI might differ are those when wRw' and $w'Rw$ and hence $w = w'$ in SI obtains. But in these cases, in I, by the heredity rule, w and w' have all parameters in common. So there is no difference between the two interpretations in terms of which parameters are true at which worlds, and so for inference α, if I shows α to be invalid then SI shows α to be invalid.

10. * Construct a tableau system for LC. (Hint: look at 3.6b.) Prove that this is sound and complete with respect to the semantics.

To do this, we will need to add the restriction to our validity conditions, and to add a new tableaux rule.

The LC restriction is: for all $w_1, w_2 \epsilon W$, $w_1 R w_2$ or $w_2 R w_1$, or $w_1 = w_2$.

The new tableaux rule could be:

$$\begin{array}{ccc} & \text{for any } i \text{ and } j \text{ on the branch} & \\ \swarrow & \downarrow & \searrow \\ \text{irj} & \text{i = j} & \text{jri} \end{array}$$

To supplement this we will need a pair of rules for =. These will be the same as in 3.6b.5:

Where $\alpha(i)$ is a line of the tableau containing an 'i', and $\alpha(j)$ is the same, with 'j' replacing 'i'.

$$\begin{array}{cc} \alpha(i) & \alpha(i) \\ i = j & j = i \\ \downarrow & \downarrow \\ \alpha(j) & \alpha(j) \end{array}$$

Thus:

if $\alpha(i)$ is A, i, $\alpha(j)$ is A, j
if $\alpha(i)$ is kri, $\alpha(j)$ is krj
if $\alpha(i)$ is $i = k$, $\alpha(j)$ is $j = k$

Now we have to modify the proofs of 6.7.

In particular,

Soundness:

We must add a new clause to the definition of faithfulness:

If $i = j$ is on b then $f(i)$ is $f(j)$.

And now check that the Soundness Lemma still works given the new rules:

Suppose that i and j are on b, and that we apply the rule to get three branches of the form irj, $i = j$ and jri respectively. Since I is faithful to the branch w_i and w_j are in W, therefore by the LC restriction, either $f(i)Rf(j)$, $f(i)$ is $f(j)$, or $f(j)Rf(i)$. In the first case, the interpretation is faithful to the left-hand branch, in the second to the middle branch, and in the third it is faithful to the right-hand branch, as required.

Suppose that $\alpha(i)$, and $i = j$ are on b, and that we apply the rule to get $\alpha(j)$. Since f shows I to be faithful to b, $f(i) = f(j)$. If $\alpha(i)$ is A, i, then A is true at $f(i)$. Hence A is true at $f(j)$ as required. If $\alpha(i)$ is kri, then $f(k)Rf(i)$ and so $f(k)Rf(j)$, as required. If $\alpha(i)$ is $i = k$, then, because $f(j) = f(k)$, $j = k$ as required.

Completeness:

We must define the interpretation as in 3.7:

Given a completed open branch of a tableau, b, let I be the set of world numbers that occur on b. Define a relation on I as follows, $i \sim j$ iff:

$i = j$ or '$i = j$' occurs on b, or '$j = i$' occurs on b

$\sim$ is obviously reflexive and symmetric. By the $=$ rule, it is also transitive. Therefore it is an equivalence relation. Let $[i]$ be the equivalence class of i. The induced interpretation is $< W, R, v >$, where $W = \{w_{[i]} : i\epsilon I\}; w_{[i]}Rw_{[j]}$ iff irj is on b; $v_{w_{[i]}}(p) = 1$ if p, i is on b, and $v_{w_{[i]}}(p) = 0$ if $\neg p, i$ is on b.

The Completeness Lemma becomes:

if A, i is on b then A is true at $w_{[i]}$
if $\neg A, i$ is on b then A is false at $w_{[i]}$

All the cases for the Completeness Lemma will be rephrased in terms of equivalence classes, as was done in 3.7.

We must also show that the interpretation induced by an LC tableau branch is of the appropriate kind:

Suppose that $w_{[i]}$ and $w_{[j]}$ are in W. Because b is complete, irj, jri or $i = j$ are on b. Now by the LC validity condition, either $w_{[i]}Rw_{[j]}$, $w_{[j]}Rw_{[i]}$, or $i \sim j$. In the first two cases we have what we need. In the last case, $[i] = [j]$, so $w_{[i]} = w_{[k]}$, so we have what we need there as well.

■

11. * The McKinsey-Tarski *translation* is a map, M, from the sentences of intuitionist propositional logic in to the language of $K\rho\tau$, defined, by recursion, thus:

$$\begin{array}{ccc} p^M & = & \Box p \\ (A \wedge B)^M & = & A^M \wedge B^M \\ (A \vee B)^M & = & A^M \vee B^M \\ (A \sqsupset B)^M & = & \Box(A^M \supset B^M) \\ (\neg A)^M & = & \Box\neg A^M \end{array}$$

Given an intuitionist interpretation (which is also, of course, a $K\rho\tau$ interpretation), show by recursion on the construction of sentences that A is true at a world, w, iff A^M is true at w. Let $\Sigma^M = \{A^M : A \epsilon \Sigma\}$. Infer that if $\Sigma^M \vDash_{K\rho\tau} A^M$, then $\Sigma \vDash_I A$. Suppose that $\Sigma^M \nvDash_{K\rho\tau} A^M$ (and hence that $\Sigma^M \nvdash_{K\rho\tau} A^M$), and consider the interpretation induced by an open branch of the tableau. Show that this satisfies the heredity condition, and hence infer the converse.

"Given an intuitionist interpretation (which is also, of course, a $K\rho\tau$ interpretation), show by recursion on the construction of sentences that A is true at a world, w, iff A^M is true at w."

$\boxed{A = p}$

p is true at w in I	iff	for all w' such that wRw', $v'_w(p) = 1$ (By Heredity)
	iff	$\Box p$ is true at w
	iff	A^M is true at w

$\boxed{A = B \wedge C}$

$B \wedge C$ is true at w in I	iff	$v_w(B) = 1$ and $v_w(C) = 1$
	iff	B^M and C^M are true at w
	iff	A^M is true at w

$\boxed{A = B \vee C}$

$B \vee C$ is true at w in I	iff	$v_w(B) = 1$ or $v_w(C) = 1$
	iff	B^M or C^M are true at w
	iff	A^M is true at w

$\boxed{A = B \sqsupset C}$

$B \sqsupset C$ is true at w in I	iff	for all w' such that wRw', $v_{w'}(B) = 0$ or $v_{w'}(C) = 1$
	iff	for all w' such that wRw', B^M is false or C^M is true at w
	iff	A^M is true at w

$\boxed{A = \neg B}$

$\neg B$ is true at w in I	iff	for all w' such that wRw', $v_{w'}(B) = 0$
	iff	for all w' such that wRw', B^M is false at w
	iff	A^M is true at w

"Let $\Sigma^M = \{A^M : A \epsilon \Sigma\}$. Infer that if $\Sigma^M \vDash_{K\rho\tau} A^M$, then $\Sigma \vDash_I A$. "

Contrapositive proof: Suppose that $\Sigma \nvDash_I A$. Then $\Sigma \nvdash_I A$, and there is an intuitionist interpretation I induced by an open branch of a tableau showing this. We have the fact that A is true at a world, w, iff A^M is true at w (shown above). We also have the fact that the members of Σ^M are true at w iff the members of Σ are true at w (by the first result, construction, and the condition directly above). Finally it is a fact that R in I satisfies the $\rho\tau$ condition

(This follows from the definition of R in intuitionism). Therefore I shows that $\Sigma^M \not\models_{K\rho\tau} A^M$.

"Suppose that $\Sigma^M \not\models_{K\rho\tau} A^M$ (and hence that $\Sigma^M \nvdash_{K\rho\tau} A^M$), and consider the interpretation induced by an open branch of the tableau. Show that this satisfies the heredity condition, and hence infer that if $\Sigma \models_I A$, then $\Sigma^M \models_{K\rho\tau} A^M$."

Contrapositive proof: Suppose that $\Sigma^M \nvdash_{K_{\rho\tau}} A$. Consider the interpretation induced by an open branch, I. Provided that this respects the heredity condition, it is an intuitionist interpretation, and so $\Sigma^M \nvdash_I A$. So suppose that, in this, p is true at w_i. Then p, i occurs on the branch. (For this to work, the induced interpretation must be such that $\nu_{w_i}(p) = 1$ iff p, i occurs on the branch. See 2.9.5). Now, the only way for p, i to occur in the branch is for it to be obtained from lines of the form $\Box p, j$ and jri. (Given the nature of an M-T translation, there are no lines of the form $\Diamond p, i$.) So suppose that $w_i R w_k$. Then irk is on the branch, as is jrk, by the transitivity rule. Hence, p, k is on the branch, and $\nu_{w_k}(p) = 1$. The result follows by the Soundness and Completeness theorems for $K_{\rho\tau}$ and I.