1. Complete the details left open in 5.2.1, 5.4.3, 5.5.4, 5.5.8, 5.6.11, 5.7.2, 5.7.6 and 5.8.8.

5.2.1: Check that Antecedent strengthening, Transitivity, and Contraposition are valid in classical logic.

$A \supset B \vdash (A \wedge C) \supset B$

$$\begin{array}{c} A \supset B \\ \neg((A \wedge C) \supset B) \\ A \wedge C \\ \neg B \\ A \\ C \\ \begin{array}{cc} \neg A & B \\ \otimes & \otimes \end{array} \end{array}$$

$A \supset B, B \supset C \vdash A \supset C$

$$\begin{array}{c} A \supset B \\ B \supset C \\ \neg(A \supset C) \\ A \\ \neg C \\ \begin{array}{cc} \neg A & B \\ \otimes & \begin{array}{cc} \neg B & C \\ \otimes & \otimes \end{array} \end{array} \end{array}$$

$A \supset B, C \vdash \neg B \supset \neg A$

$$\begin{array}{c} A \supset B \\ \neg(\neg B \supset \neg A) \\ \neg B \\ \neg\neg A \\ A \\ \begin{array}{cc} \neg A & B \\ \otimes & \otimes \end{array} \end{array}$$

5.4.3 Check that the second and third arguments of 5.2.2 are invalid.

$p > q, q > r \nvdash_C p > r$

$$\begin{array}{c} p > q, 0 \\ q > r, 0 \\ \neg(p > r), 0 \\ 0\mathrm{r}_p 1 \\ \neg r, 1 \\ q, 1 \end{array}$$

Counter-model such that:

$$W = \{w_0, w_1\}; w_0 R_p w_1; v_{w_1}(q) = 1, v_{w_1}(r) = 0$$

This can be represented in the following picture:

$$\begin{array}{ccc} w_0 & \xrightarrow{p} & w_1 \\ & & q, \neg r \end{array}$$

$p > q \nvdash_C \neg q > \neg p$

$$\begin{array}{c} p > q, 0 \\ \neg(\neg q > \neg p), 0 \\ 0\mathrm{r}_{\neg q} 1 \\ \neg\neg p, 1 \\ p, 1 \end{array}$$

Counter-model such that:

$$W = \{w_0, w_1\}; w_0 R_{\neg q} w_1; v_{w_1}(p) = 1$$

This can be represented in the following picture:

$$\begin{array}{ccc} w_0 & \xrightarrow{\neg q} & w_1 \\ & & p \end{array}$$

5.5.4: Check that modus ponens for $>$ fails in C.

$p, p > q \nvdash_C q$

$$\begin{array}{c} p, 0 \\ p > q, 0 \\ \neg q, 0 \end{array}$$

Counter-model such that:

$$W = \{w_0\}; v_{w_0}(p) = 1, v_{w_0}(q) = 0$$

This can be represented in the following picture:

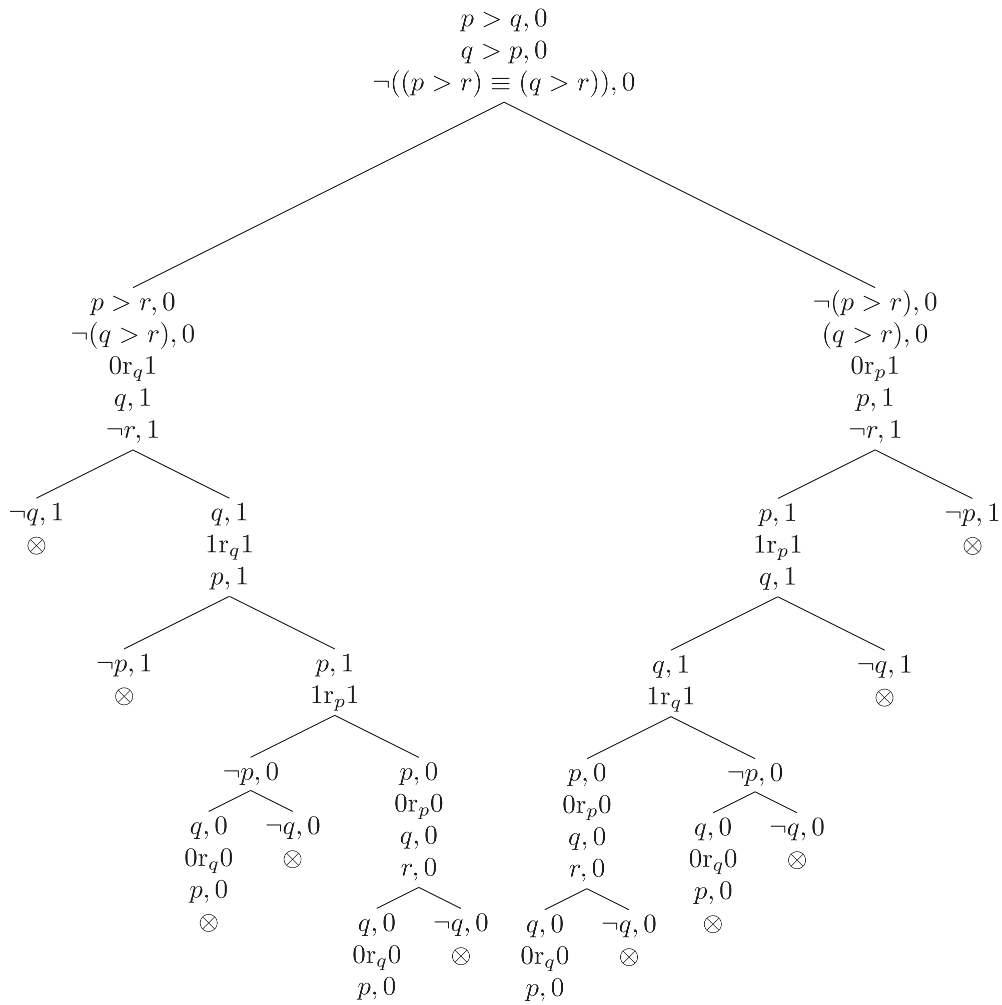
$$\begin{array}{c} w_0 \\ p, \neg q \end{array}$$

5.5.8 Construct a simple interpretation showing that $p > r \nvdash_{C+} p > (r \wedge q)$.

$$\begin{array}{c} \overset{p}{\curvearrowright} \\ w_0 \\ p, r, \neg q \end{array}$$

$p > r$ is true at w_0: w_0 is the only p world, and r is true there. $p > (r \wedge q)$ is false at w_0, because w_0 is a p world, and $\neg q$, and hence $r \wedge q$ is false there. This countermodel satisfies conditions (1) and (2), because at all p-worlds p is true, and vice versa.

5.6.11: Check that $p > q, q > p \nvdash_{C^+} (p > r) \equiv (q > r)$



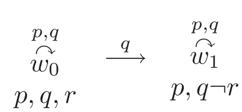
Counter-model from left-most open branch such that:

$$W = \{w_0, w_1\}$$

$$w_0 R_p w_0, w_0 R_q w_0, w_1 R_p w_1, w_1 R_q w_1, w_0 R_q w_1; \text{for all other } A, f_A(w) = [A]$$

$$v_{w_0}(p) = 1, v_{w_0}(q) = 1, v_{w_0}(r) = 1, v_{w_1}(p) = 1, v_{w_1}(q) = 1, v_{w_1}(r) = 0$$

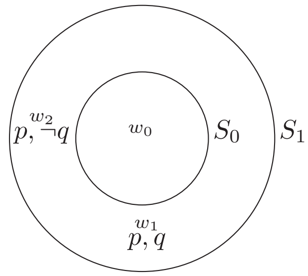
This can be represented in the following picture:



5.7.2 Show that *Conditional Excluded Middle* is not valid in S.

$\nvDash_S (p > q) \vee (p > \neg q)$

Since there is no tableaux system for S, I will simply draw a similarity sphere diagram:

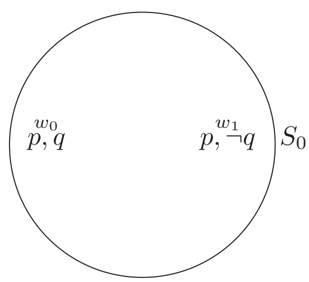


In the above, there are two most similar worlds to w_0 where p is true. w_2 makes q false, meaning that $p > q$ is false at w_0, and w_1 makes q true, meaning $p > \neg q$ is false at w_0. Thus the disjunction is false at w_0, and the inference is invalid in S.

5.7.6 Check that $A \wedge B \vDash A > B$ does not hold in S.

$p \wedge q \nvDash_S p > q$

Since there is no tableaux system for S, I will draw a similarity sphere diagram directly:



In the above, $p \wedge q$ is true at w_0. However, in S, w_0 is not guaranteed to be a unique world most similar to itself. Perhaps counter-intuitively, w_1 is a world equally similar to w_0, except that q is false there. Therefore $p > q$, is false at w_0 and the inference is invalid.

5.8.8 Check that $\Box B \vDash A > B$, $\Box\neg A \vDash A > B$ and $\vDash (A \wedge \neg A) > B$ in C^+

$\Box B \vdash_{C^+} A > B$

$$\begin{array}{c} \Box B, 0 \\ \neg(A > B), 0 \\ 0\mathrm{r}_A 1 \\ A, 1 \\ \neg B, 1 \\ B, 1 \\ \otimes \end{array}$$

$\Box\neg A \vdash_{C^+} A > B$

$$\begin{array}{c} \Box\neg A, 0 \\ \neg(A > B), 0 \\ 0\mathrm{r}_A 1 \\ A, 1 \\ \neg B, 1 \\ \neg A, 1 \\ \otimes \end{array}$$

$\vdash_{C^+} (A \wedge \neg A) > B$

$$\begin{array}{c} \neg(A \wedge \neg A) > B, 0 \\ 0r_{A\wedge\neg A}1 \\ A \wedge \neg A, 1 \\ \neg B, 1 \\ A, 1 \\ \neg A, 1 \\ \otimes \end{array}$$

2. Show that the following are true in C:

(a) $\Box(A \equiv B) \vdash (C > A) \equiv (C > B)$

$$
\begin{array}{c}
\Box(A \equiv B), 0 \\
\neg((C > A) \equiv (C > B)), 0 \\
\begin{array}{cc}
\begin{array}{c}
C > A, 0 \\
\neg(C > B), 0 \\
0 r_C 1 \\
\neg B, 1 \\
A, 1 \\
A \equiv B, 1 \\
\begin{array}{cc}
\begin{array}{c} A, 1 \\ B, 1 \\ \otimes \end{array} &
\begin{array}{c} \neg A, 1 \\ \neg B \\ \otimes \end{array}
\end{array}
\end{array} &
\begin{array}{c}
\neg(C > A), 0 \\
(C > B), 0 \\
0 r_C 1 \\
\neg A, 1 \\
B, 1 \\
A \equiv B, 1 \\
\begin{array}{cc}
\begin{array}{c} A, 1 \\ B, 1 \\ \otimes \end{array} &
\begin{array}{c} \neg A, 1 \\ \neg B \\ \otimes \end{array}
\end{array}
\end{array}
\end{array}
\end{array}
$$

(b) $A > (B \wedge C) \vdash (A > B) \wedge (A > C)$

$$
\begin{array}{c}
A > (B \wedge C), 0 \\
\neg((A > B) \wedge (A > C)), 0 \\
\begin{array}{cc}
\begin{array}{c}
\neg(A > B), 0 \\
0 r_A 1 \\
\neg B, 1 \\
B \wedge C, 1 \\
B, 1 \\
C, 1 \\
\otimes
\end{array} &
\begin{array}{c}
\neg(A > C), 0 \\
0 r_A 1 \\
\neg C, 1 \\
B \wedge C, 1 \\
B, 1 \\
C, 1 \\
\otimes
\end{array}
\end{array}
\end{array}
$$

(c) $(A > B) \wedge (A > C) \vdash A > (B \wedge C)$

$$
\begin{array}{c}
(A > B) \wedge (A > C), 0 \\
\neg(A > (B \wedge C)), 0 \\
A > B, 0 \\
A > C, 0 \\
0 r_A 1 \\
\neg(B \wedge C), 1 \\
B, 1 \\
C, 1 \\
\begin{array}{cc}
\begin{array}{c} \neg B, 1 \\ \otimes \end{array} &
\begin{array}{c} \neg C \\ \otimes \end{array}
\end{array}
\end{array}
$$

(d) $A > (B \supset C) \vdash (A > B) \supset (A > C)$

$$\begin{array}{c} A > (B \supset C), 0 \\ \neg((A > B) \supset (A > C)), 0 \\ A > B, 0 \\ \neg(A > C), 0 \\ 0\mathrm{r}_A 1 \\ \neg C, 1 \\ B, 1 \\ B \supset C, 1 \\ \begin{array}{cc} \neg B, 1 & C \\ \otimes & \otimes \end{array} \end{array}$$

(e) $\vdash A > (B \vee \neg B)$

$$\begin{array}{c} \neg(A > (B \vee \neg B)), 0 \\ 0\mathrm{r}_A 1 \\ \neg(B \vee \neg B), 1 \\ \neg B, 1 \\ \neg\neg B, 1 \\ \otimes \end{array}$$

3. Show that the following are false in C, but true in C^+. Specify a C counter-model.

(a) $\vdash p > p$

$\nvdash_C p > p$

$$\begin{array}{c} \neg(p > p), 0 \\ 0\mathrm{r}_p 1 \\ \neg p, 1 \end{array}$$

Countermodel such that:

$$W = \{w_0, w_1\}; w_0 R_p w_1; v_{w_1}(p) = 0$$

$\vdash_{C^+} p > p$

$$\begin{array}{c} \neg(p > p), 0 \\ 0\mathrm{r}_p 1 \\ p, 1 \\ \neg p, 1 \\ \otimes \end{array}$$

(b) $p, p > q \vdash q$

$\nvdash_C p, p > q \vdash q$

$$\begin{array}{c} p, 0 \\ p > q, 0 \\ \neg q, 0 \end{array}$$

Countermodel such that:

$$W = \{w_0\}; v_{w_0}(p) = 1, v_{w_0}(q) = 0$$

$\vdash_{C+} p, p > q \vdash q$

$$\begin{array}{c} p, 0 \\ p > q, 0 \\ \neg q, 0 \\ \begin{array}{cc} p, 0 & \neg p, 0 \\ 0\mathrm{r}_p 0 & \otimes \\ q, 0 & \\ \otimes & \end{array} \end{array}$$

(c) $p \strictif q \vdash p > q$

$p \strictif q \nvdash_C p > q$

$\Box(p \supset q) \nvdash_C p > q$

$$\begin{array}{c} \Box(p \supset q), 0 \\ \neg(p > q), 0 \\ 0\mathrm{r}_p 1 \\ \neg q, 1 \\ p \supset q, 1 \\ \begin{array}{cc} \neg p, 1 & q, 1 \\ & \otimes \end{array} \end{array}$$

Countermodel such that:

$$W = \{w_0, w_1\}; w_0 R_p w_1; v_{w_1}(p) = 0, v_{w_1}(q) = 0$$

$p \strictif q \vdash_{C^+} p > q$

$\Box(p \supset q) \vdash_{C^+} p > q$

$$
\begin{array}{c}
\Box(p \supset q), 0 \\
\neg(p > q), 0 \\
0\mathrm{r}_p 1 \\
p, 1 \\
\neg q, 1 \\
p \supset q, 1 \\
\begin{array}{cc}
\neg p, 1 & q, 1 \\
\otimes & \otimes
\end{array}
\end{array}
$$

(d) $p \wedge \neg q \vdash \neg(p > q)$

$p \wedge \neg q \nvdash_C \neg(p > q)$

$$
\begin{array}{c}
p \wedge \neg q, 0 \\
\neg\neg(p > q), 0 \\
p > q, 0 \\
p, 0 \\
\neg q, 0
\end{array}
$$

Countermodel such that:

$$W = \{w_0\}; v_{w_0}(p) = 1, v_{w_0}(q) = 0$$

$p \wedge \neg q \vdash_{C^+} \neg(p > q)$

$$
\begin{array}{c}
p \wedge \neg q, 0 \\
\neg\neg(p > q), 0 \\
p > q, 0 \\
p, 0 \\
\neg q, 0 \\
\begin{array}{cc}
p, 0 & \neg p, 0 \\
0\mathrm{r}_p 0 & \otimes \\
q, 0 & \\
\otimes &
\end{array}
\end{array}
$$

4. Show that the following are false in C^+. Specify a counter-model, either by constructing a tableau, or directly.

(a) $p > q \vDash (p \wedge r) > q$

$p > q \nvDash_{C^+} (p \wedge r) > q$

- $p > q, 0$
- $\neg((p \wedge r) > q), 0$
- $0r_{p\wedge r}1$
- $p \wedge r, 1$
- $\neg q, 1$
- $p, 1$
- $r, 1$
  - $\neg p, 1$
    - $\otimes$
  - $p, 1$
    - $1\mathrm{r}_p 1$
    - $p \wedge r, 1$
      - $1r_{p\wedge r}1$
      - $p, 1$
      - $r, 1$
        - $\neg p, 0$
          - $p \wedge r, 0$
            - $0r_{p\wedge r}0$
            - $p, 0$
            - $\otimes$
          - $\neg(p \wedge r), 0$
            - $\neg p, 0$
            - $\neg r, 0$
              - $\otimes$
        - $p, 0$
          - $0r_p0$
          - $q, 0$
            - $p \wedge r, 0$
              - $0_{p\wedge r}0$
              - $p, 0$
              - $r, 0$
            - $\neg(p \wedge r), 0$
              - $\neg p, 0$
                - $\otimes$
              - $\neg r, 0$
    - $\neg(p \wedge r), 1$
      - $\neg p, 1$
        - $\otimes$
      - $\neg r, 1$
        - $\otimes$

Countermodel from the left-most open branch such that:

$$W = \{w_0, w_1\}$$
$$w_0 R_{p\wedge r} w_1, w_1 R_p w_1, w_1 R_{p\wedge r} w_1; \text{for all other } A, f_A(w) = [A]$$
$$v_{w_0}(p) = 0, v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_1}(r) = 1$$

This can be represented in the following picture:

$$\begin{array}{ccc} w_0 & \xrightarrow{p\wedge r} & \overset{p,p\wedge r}{\overset{\curvearrowright}{w_1}} \\ \neg p & & p,\neg q, r \end{array}$$

(b) $p > q \models \neg q > \neg p$

$p > q \nvdash_{C+} \neg q > \neg p$

$p > q, 0$
$\neg(\neg q > \neg p), 0$
$0r_{\neg q}1$
$\neg q, 1$
$\neg\neg p, 1$
$p, 1$

- $p, 1$; $1\mathrm{r}_p 1$
  - $\neg q, 1$; $1r_{\neg q}1$
    - $p, 0$; $0\mathrm{r}_p 0$; $q, 0$
      - $\neg q, 0$; $0r_{\neg q}0$; $\otimes$
      - $\neg\neg q, 0$; $q, 0$
    - $\neg p, 0$
      - $\neg q, 0$; $0r_{\neg q}0$; $\otimes$
      - $\neg\neg q, 0$; $q, 0$
  - $\neg\neg q, 1$; $\otimes$
- $\neg p, 1$; $\otimes$

Counter-model from the left-most open branch such that:

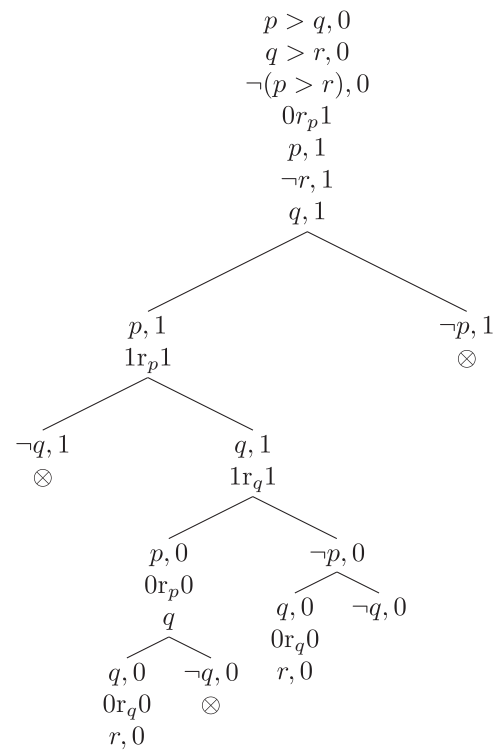
$$W = \{w_0, w_1\}$$
$$w_0 R_p w_0, w_0 R_{\neg q} w_1, w_1 R_p w_1, w_1 R_{\neg q} w_1; \text{for all other } A, f_A(w) = [A]$$
$$v_{w_0}(p) = 1, v_{w_0}(q) = 0, v_{w_1}(p) = 1, v_{w_1}(q) = 0$$

This can be represented in the following picture:

$$\begin{array}{ccc} \overset{p}{\overset{\curvearrowright}{w_0}} & \xrightarrow{\neg q} & \overset{p,\neg q}{\overset{\curvearrowright}{w_1}} \\ p, q & & p,\neg q \end{array}$$

(c) $p > q, q > r \vDash p > r$

$p > q, q > r \nvdash_{C^+} p > r$



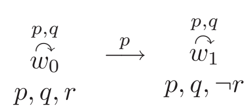
Countermodel from left-most open branch such that:

$$W = \{w_0, w_1\}$$

$$w_0R_pw_0, w_0R_qw_0, w_0R_pw_1, w_1R_pw_1, w_1R_qw_1; \text{for all other } A, f_A(w) = [A]$$

$$v_{w_0}(p) = 1, v_{w_0}(q) = 1, v_{w_0}(r) = 1, v_{w_1}(p) = 1, v_{w_1}(q) = 1, v_{w_1}(r) = 0$$

This can be represented in the following picture:



5. Show that the following fail in C, but hold provided we add the condition on f indicated.

(a) $(p \vee q) > r \vDash (p > r) \wedge (q > r)$

$f_p(w) \cup f_q(w) \subseteq f_{p \vee q}(w)$

$$(p \vee q) > r \nvdash_C (p > r) \wedge (q > r)$$

$$\begin{array}{c}
(p \vee q) > r, 0 \\
\neg((p > r) \wedge (q > r)), 0 \\
\begin{array}{cc}
\swarrow & \searrow \\
\neg(p > r), 0 & \neg(q > r), 0 \\
0\mathrm{r}_p 1 & 0\mathrm{r}_q 1 \\
\neg r, 1 & \neg r, 1
\end{array}
\end{array}$$

$$\begin{array}{c}
(p \vee q) > r \vdash_C (p > r) \wedge (q > r) \\
\text{with the condition} \\
f_p(w) \cup f_q(w) \subseteq f_{p \vee q}(w)
\end{array}$$

Suppose that $(p \vee q) > r$ is true at a world of an interpretation, w. Then $f_{p \vee q} \subseteq [r]$. By the condition, $f_p \subseteq [r]$ and $f_q \subseteq [r]$. So $p > r$, $q > r$, and so their conjunction, are true at w.

Although strictly it requires us to show the soundness of the condition before proceeding, this can also be shown using a tableau.

$$\begin{array}{c}
(p \vee q) > r, 0 \\
\neg((p > r) \wedge (q > r)), 0 \\
\begin{array}{cc}
\swarrow & \searrow \\
\neg(p > r), 0 & \neg(q > r), 0 \\
0\mathrm{r}_p 1 & 0\mathrm{r}_q 1 \\
\neg r, 1 & \neg r, 1 \\
0 r_{p \vee q} 1 & 0 r_{p \vee q} 1 \\
r, 1 & r, 1 \\
\otimes & \otimes
\end{array}
\end{array}$$

(b) $(p > r) \wedge (q > r) \vDash (p \vee q) > r$
$f_{p\vee q}(w) \subseteq f_p(w) \cup f_q(w)$

$$(p > r) \wedge (q > r) \nvdash_C (p \vee q) > r$$

$$\begin{array}{c} (p > r) \wedge (q > r), 0 \\ \neg((p \vee q) > r), 0 \\ p > r, 0 \\ q > r, 0 \\ 0r_{p\vee q}1 \\ \neg r, 1 \end{array}$$

$$(p > r) \wedge (q > r) \vdash_C (p \vee q) > r$$
with the condition
$$f_{p\vee q}(w) \subseteq f_p(w) \cup f_q(w)$$

Suppose that $(p > r) \wedge (q > r)$ is true at a world of an interpretation, w. Then $f_p(w) \subseteq [r]$ and $f_q(w) \subseteq [r]$. So, $f_p(w) \cup f_q(w) \subseteq [r]$, and, by the condition $f_{p\vee q}(w) \subseteq [r]$, as required.

Although strictly it requires us to show the soundness of the condition before proceeding, this can also be shown using a tableau.

$$\begin{array}{c} (p > r) \wedge (q > r)0 \\ \neg((p \vee q) > r), 0 \\ p > r, 0 \\ q > r, 0 \\ 0r_{p\vee q}, 1 \\ \neg r, 1 \end{array}$$

$$\begin{array}{cc} 0\mathrm{r}_p1 & 0\mathrm{r}_q1 \\ r, 1 & r, 1 \\ \otimes & \otimes \end{array}$$

(c) $p > q, q > r \vDash (p \wedge q) > r$
If $f_p(w) \subseteq [q]$, then $f_{p\wedge q}(w) \subseteq f_q(w)$

$$p > q, q > r \nvdash_C (p \wedge q) > r$$

$$\begin{array}{c} p > q, 0 \\ q > r0 \\ \neg((p \wedge q) > r), 0 \\ 0r_{p\wedge q}1 \\ \neg r, 1 \end{array}$$

$$p > q, q > r \vdash_C (p \wedge q) > r$$
with the condition
If $f_p(w) \subseteq [q]$, then $f_{p\wedge q}(w) \subseteq f_q(w)$

Assume $p > q$ and $q > r$ are true at a world of an interpretation, w. Then $f_p(w) \subseteq [q]$ and $f_q(w) \subseteq [r]$. By the condition, $f_{p\wedge q}(w) \subseteq f_q(w)$. So, $f_{p\wedge q}(w)$ is contained in $[r]$. Hence $(p \wedge q) > r$ as required.

Although strictly it requires us to show the soundness of the condition before proceeding, this can also be shown using a tableau.

$$\begin{array}{c}
p > q, 0 \\
q > r0 \\
\neg((p \wedge q) > r), 0 \\
0r_{p\wedge q}1 \\
\neg r, 1 \\
0\mathrm{r}_q 1 \\
r, 1 \\
\otimes
\end{array}$$

6. Show that the following fail in C^+, but hold in S:

(a) $\Diamond p \vDash \neg(p > (q \wedge \neg q))$

$\Diamond p \nvdash_{C^+} \neg(p > (q \wedge \neg q))$

$$\begin{array}{c}
\Diamond p, 0 \\
\neg\neg(p > (q \wedge \neg q)), 0 \\
p > (q \wedge \neg q), 0 \\
0\mathrm{r}1 \\
p, 1
\end{array}$$

$$\begin{array}{cc}
\begin{array}{c}
p, 1 \\
1r_p 1
\end{array} & \begin{array}{c}
\neg p, 1 \\
\otimes
\end{array}
\end{array}$$

$$\begin{array}{cc}
\begin{array}{c}
p, 0 \\
0r_p 0 \\
q \wedge \neg q, 0 \\
q, 0 \\
\neg q, 0 \\
\otimes
\end{array} & \begin{array}{c}
\neg p, 0 \\
\\
\\
\\
\\
\\
\end{array}
\end{array}$$

Counter-model such that:

$W = \{w_0, w_1\}; w_0 R w_1, w_1 R_p w_1;$ for all other $A, f_A(w) = [A]; v_{w_0}(p) = 0, v_{w_1}(p) = 1$

This can be represented in the following picture:

This is a C^+ model because it satisfies conditions (1) and (2): $f_A(w) \subseteq [A]$ and, if $w \epsilon [A]$, then $w \epsilon f_A(w)$.

$\Diamond p \vdash_S \neg(p > (q \wedge \neg q))$

There are no tableaux for S so I will give a direct proof.

Suppose that the premise is true, and the conclusion false. Then at some world, w_0, $\Diamond p$ and $p > (q \wedge \neg q)$ are true. By the first of these, there is a world, w, such that $w \epsilon f_p(w_0)$. By the second, $f_p(w) \subseteq [q \wedge \neg q]$. Hence, $q \wedge \neg q$ is true at w, which is impossible.

(b) $p > q, \neg(p > \neg r) \vDash (p \wedge r) > q$

$p > q, \neg(p > \neg r) \nvDash_{C+} (p \wedge r) > q$

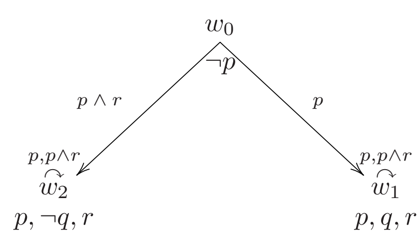
I will simply specify a counter-model:

$$W = \{w_0, w_1, w_2\}$$

$w_0 R_p w_1, w_1 R_p w_1, w_0 R_{p\wedge r} w_2, w_1 R_q w_1, w_1 R_{p\wedge r} 1 w_2 R_p w_2, w_2 R_{p\wedge r} w_2$; for all other $A, f_A(w) = [A]$

$v_{w_0}(p) = 0, v_{w_1}(p) = 1, v_{w_1}(q) = 1, v_{w_1}(r) = 1, v_{w_2}(p) = 1, v_{w_2}(q) = 0, v_{w_2}(r) = 1$

This can be represented in the following picture:



$p > q$ is true at w_0 because at the only p-world in relation to w_0, q is true. $\neg(p > \neg r)$ is true at w_0 because at only p-world in relation to w_0,r is true. $(p \wedge r) > q$ is false at w_0 because at the only $(p \wedge r)$-world in relation to w_0, q is false. Therefore the inference is invalid.

This is a C^+ model because it satisfies conditions (1) and (2): $f_A(w) \subseteq [A]$ and, if $w \epsilon [A]$, then $w \epsilon f_A(w)$.

$p > q, \neg(p > \neg r) \vdash_S (p \wedge r) > q$

I will show that this is valid in S with a direct demonstration.

Suppose that at some world of some interpretation, w_0, the premises are true. Then at all the closest worlds where p is true, q is true, and at one of these r is also true. Now, consider the worlds closest to w_0 where $p \wedge r$ is true. There are such things amongst the closest worlds where p is true, and those must be the closest to w_0. At all of these, q is true. Hence, the conclusion is true.

(c) $\Box(p \equiv q) \vDash (p > r) \equiv (q > r)$

$\Box(p \equiv q) \nvdash_{C^+} (p > r) \equiv (q > r)$

I will specify a counter-model directly:

$$W = \{w_0, w_1\}$$
$$w_0 R_p w_0, w_0 R_q w_0, w_0 R_q w_1, w_1 R_p w_1, w_1 R_q w_1$$
$$v_{w_0}(p) = 1, v_{w_0}(q) = 1, v_{w_0}(r) = 1, v_{w_1}(p) = 1, v_{w_1}(q) = 1, v_{w_0}(r) = 0$$
$$\text{for all other } A, f_A(w) = [A]$$

This can be represented in the following picture:

$$\begin{tikzcd} \underset{p,q,r}{w_0} \arrow[loop above, "p,q"] \arrow[r, "q"] & \underset{p,q,\neg r}{w_1} \arrow[loop above, "p,q"] \end{tikzcd}$$

At both worlds, the truth value of p and q are the same, so the premise $\Box(p \equiv q)$ is true at w_0. In w_0, $p > r$ is true (because w_0 is not p-related to w_1). However, $q > r$ is false, because r is false at w_1. This model satisfies (1) and (2), and so is a C_+ interpretation.

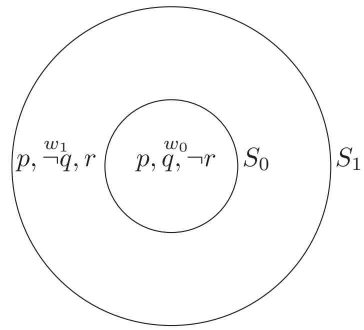
$\Box(p \equiv q) \vdash_S (p > r) \equiv (q > r)$

I will show that this is valid in S with a direct demonstration.

If the premise is true, then the truth value of p and q is the same at every world. So $f_p(w) = f_q(w)$. And so $f_p(w) \subseteq [r]$ iff $f_q(w) \subseteq [r]$.

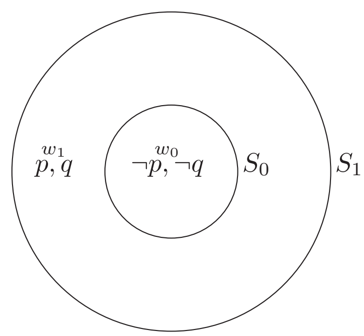
7. By constructing a suitable sphere model, show that the inferences of problem 4 also fail in C_2. Show that the following is also false in C_2: $(p \vee q) > r \models (p > r) \wedge (q > r)$.

(a) $p > q \models (p \wedge r) > q$



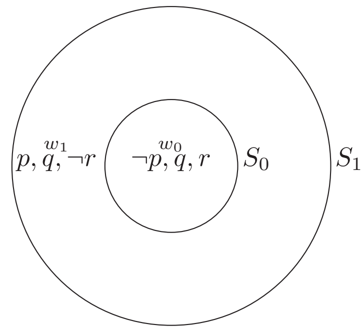
w_0 is the unique closest world at which p is true, and q is true there, so the premise is true. However the closest world at which $p \wedge r$ is true is w_1, and q is false there, so the conclusion is false. Each of S_0 and $S_1 - S_0$ has only one world in it, so the model is a C_2 interpretation.

(b) $p > q \models \neg q > \neg p$



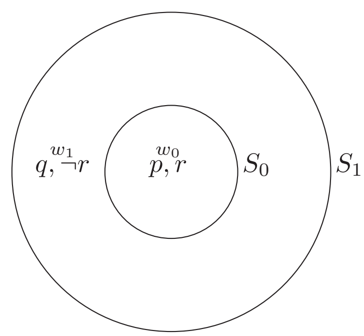
w_1 is the unique closest p-world, and q is true there, so the premise is true. However the closest $\neg q$-world is w_0, and there p is true, so the conclusion is false. Each of S_0 and $S_1 - S_0$ has only one world in it, so the model is a C_2 interpretation.

(c) $p > q, q > r \vDash p > r$



At the unique closest q-world, w_0, r is true, therefore the second premise $q > r$ is true at w_0. At the closest p world, w_2, q is true, therefore the first premise $p > q$ is true. However at the closest p world, w_1 r is false, so the conclusion $p > r$ is false. Each of S_0 and $S_1 - S_0$ has only one world in it, so the model is a C_2 interpretation.

(d) $(p \vee q) > r \vDash (p > r) \wedge (q > r)$



At the closest world where $p \vee q$, w_0, r is true, making the premise true. At the closest world where q, r is false. Therefore $q > r$ is false at w_0, and the conclusion, $(p > r) \wedge (q > r)$ is also. Each of S_0 and $S_1 - S_0$ has only one world in it, so the model is a C_2 interpretation.

8. Determine whether the following hold in each of C_1 and C_2:

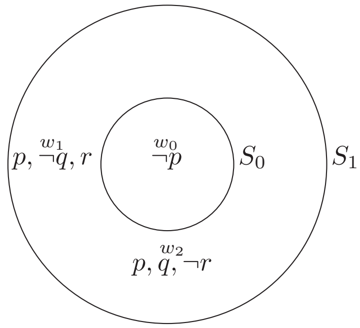
(a) $p > (q \vee r) \vDash (p > q) \vee (p > r)$

This holds in C_2, but not in C_1:

C_2: Let us assume the conclusion is false for a contrapositive proof. $\neg(p > q)$, and $\neg(p > r)$ are true at w_0. Then there is a p-close world where$\neg q$ and a p-close world where $\neg r$, both relative to w_0. In C_2, by (6), because they both contain p, these two worlds are the same. Therefore there is a p-close world relative to

w_0 where: $p, \neg q, \neg r$. But then the premise $p > (q \vee r)$ is false at w_0.

C_1: (7) allows for the possibility of as many worlds as one likes, all equally similar worlds relative to w where p, as long as p is not true in w. Using this we can construct a counter-model.



In every p-close world to w_0, $q \vee r$ is true, however it is not the case that either $p > q$ or $p > r$ is true at w_0: w_1 makes the former false, and w_2 the latter.

This is a C_1 interpretation because S_0 is a singleton (see 5.7.8.).

(b) $p > q, \neg q \vDash \neg q > \neg p$

This holds in both C_1 and C_2.

Let us assume the conclusion is false for a reductio. Then $\neg(\neg q > \neg p)$ is true at w_0. So there is a world in the closest similarity sphere for $\neg q$ where p. But because $\neg q$ is true at w_0, the closest $\neg q$ world must be w_0, by (6) or (7). Which means that, since p is also true there, w_0 is also the closest p world, and so q is true at w_0. Contradiction.

(c) $\Diamond p, p > q \vDash \neg(p > \neg q)$

This holds in both C_1 and C_2.

Let us take the conclusion to be false for reductio. Then $p > \neg q$ and $p > q$ are both true at w_0. So, $f_p(w_0) \subseteq [q]$ and $f_p(w_0) \subseteq [\neg q]$ which is contradictory by either (6) or (7), unless $f_p(w) = \phi$, but by $\Diamond p$ it does not.

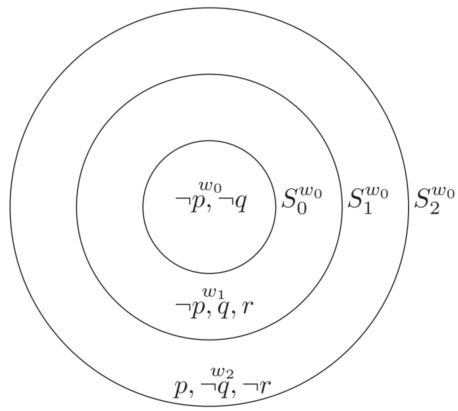
(d) $p > (p > q) \vDash p > q$

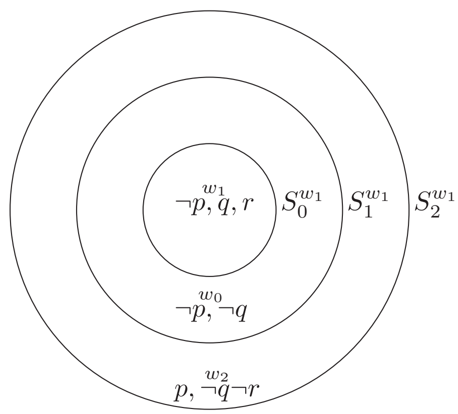
This holds in both C_2 and C_1.

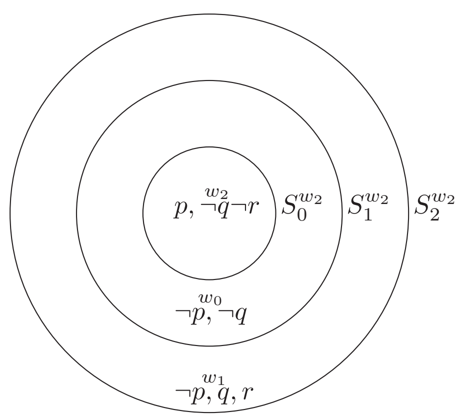
Let us take the conclusion to be false for reductio. Then there is a p-close world to w_0, w', where $\neg q$. The premise asserts that $p > q$ is true at all p-close worlds, so it is true at w'. By (6) or (7) w' is the unique p-close world to itself. Therefore q is true in w'. Contradiction.

(e) $p > (q > r) \models q > (p > r)$

This is not valid in C_2 or C_1.







The premise $p > (q > r)$ is true at w_0 — the closest p world is w_2, and from w_2 the closest q world is w_1, where r is true. The conclusion is false — the closest q world is w_q, and from w_1 the closest p world is w_2, where r is false.

This is a C_1 interpretation because S_0 is a singleton, and a C_2 interpretation because for every other S_i, $S_i - S_{i-1}$ is also a singleton. (see 5.7.8.)