1. Check the details omitted in 4.4.3, 4.4a.12, 4.4a.13, 4.5.4, 4.6.2 and 4.6.3.

4.4.3 Check that $\vdash_K \Box(p \supset \Box(q \supset q))$.

$$\begin{gathered}
\neg\Box(p \supset \Box(q \supset q)), 0 \\
\Diamond\neg(p \supset \Box(q \supset q)), 0 \\
0r1 \\
\neg(p \supset \Box(q \supset q)), 1 \\
p, 1 \\
\neg\Box(q \supset q), 1 \\
\Diamond\neg(q \supset q), 1 \\
1r2 \\
\neg(q \supset q), 2 \\
q, 2 \\
\neg q \\
\otimes
\end{gathered}$$

4.4a.12 Check that $\neg\Diamond(\Box p \wedge \Diamond\neg p)$ is not logically valid (in L).

$\nvdash_L \neg\Diamond(\Box p \wedge \Diamond\neg p)$

$$\begin{gathered}
\neg\neg\Diamond(\Box p \wedge \Diamond\neg p), 0 \\
\Diamond(\Box p \wedge \Diamond\neg p), 0 \\
0r1 \\
\Box p \wedge \Diamond\neg p, 1 \\
\Box p, 1 \\
\Diamond\neg p, 1
\end{gathered}$$

Countermodel such that:

$$W = \{w_0, w_1\}; N = \{w_0\}; v_{w_1}(\Box p) = 1, v_{w_1}(\Diamond\neg p) = 1$$

$$w_0 \quad \rightarrow \quad \boxed{w_1} \quad \begin{array}{c} \Box p \\ \Diamond\neg p \end{array}$$

4.4a.13 Check that $\neg\Diamond(\Box p \wedge \Diamond\neg p)$ is valid (in N).

$\vdash_N \neg\Diamond(\Box p \wedge \Diamond\neg p)$

$$\begin{gathered}\neg\neg\Diamond(\Box p \wedge \Diamond\neg p), 0\\ \Diamond(\Box p \wedge \Diamond\neg p), 0\\ 0r1\\ \Box p \wedge \Diamond\neg p, 1\\ \Box p, 1\\ \Diamond\neg p, 1\\ 1r2\\ \neg p, 2\\ p, 2\\ \otimes\end{gathered}$$

4.5.4 Check that the following are invalid, and show that they are invalid in all the normal and non-normal logics we have looked at.

$B \nvDash_{K_\upsilon} A \multimap B$

$q \nvDash_{K_\upsilon} \Box(p \supset q)$

$$\begin{gathered}q, 0\\ \neg\Box(p \supset q), 0\\ \Diamond\neg(p \supset q), 0\\ \neg(p \supset q), 1\\ p, 1\\ \neg q, 1\end{gathered}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

$$v_{w_0}(q) = 1, v_{w_1}(p) = 1, v_{w_1}(q) = 0$$

$\neg A \nvDash_{K_\upsilon} A \multimap B$

$\neg p \nvDash_{K_\upsilon} \neg\Box(p \supset q)$

$$\begin{gathered}\neg p, 0\\ \neg\Box(p \supset q), 0\\ \Diamond\neg(p \supset q), 0\\ \neg(p \supset q), 1\\ p, 1\\ \neg q, 1\end{gathered}$$

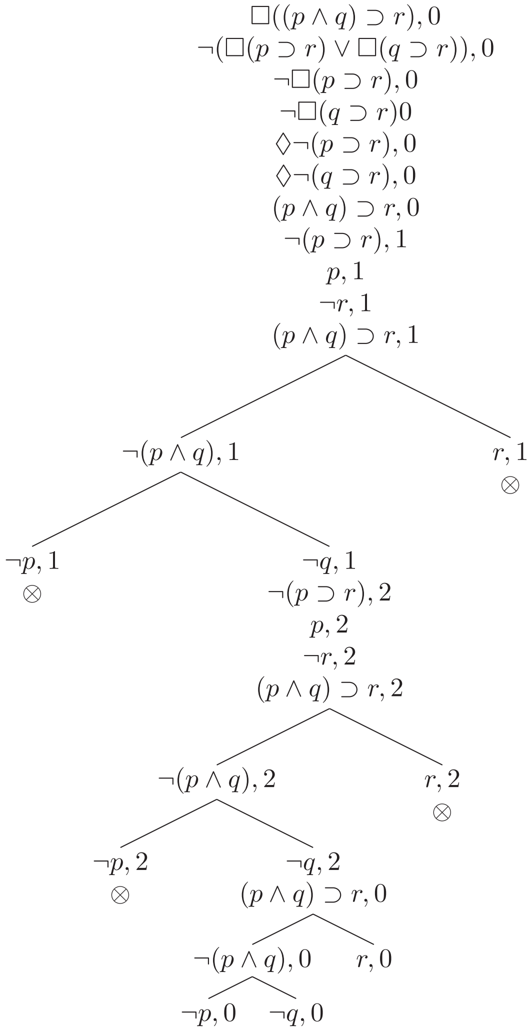
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

$$v_{w_0}(p) = 0, v_{w_1}(p) = 1, v_{w_1}(q) = 0$$

$(A \wedge B) \dashv\!\!\!\!\exists C \nvDash_{K_\upsilon} (A \dashv\!\!\!\!\exists C) \vee (B \dashv\!\!\!\!\exists C)$

$\Box((p \wedge q) \supset r) \nvDash_{K_\upsilon} \Box(p \supset r) \vee \Box(q \supset r)$



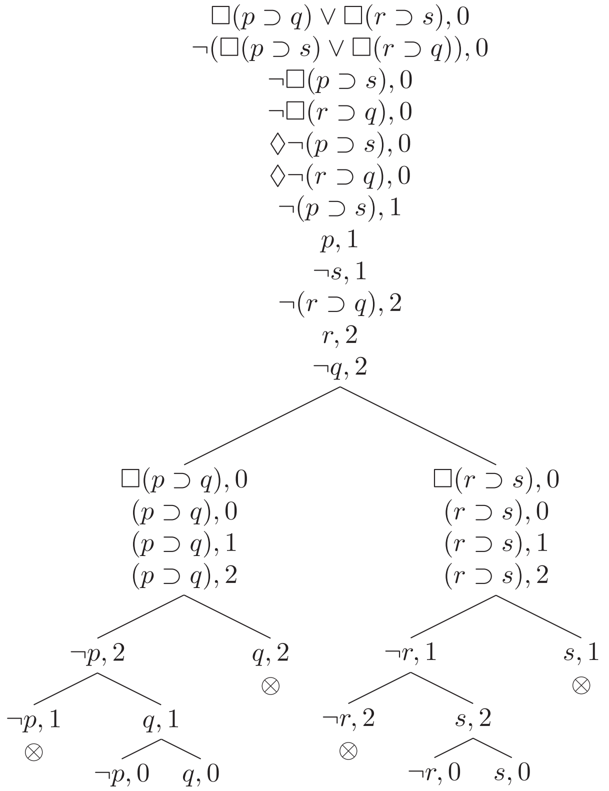
The following interpretation, taken from the left-most open branch, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$v_{w_0}(p) = 0, v_{w_1}(p) = 1, v_{w_1}(q) = 1, v_{w_1}(r) = 0, v_{w_2}(p) = 0, v_{w_2}(q) = 0, v_{w_2}(r) = 0$$

$(A \mathbin{-\!\!\!\exists} B) \vee (C \mathbin{-\!\!\!\exists} D) \nvDash_{K_v} (A \mathbin{-\!\!\!\exists} D) \vee (C \mathbin{-\!\!\!\exists} B)$

$\Box(p \supset q) \vee \Box(r \supset s) \nvDash_{K_v} \Box(p \supset s) \vee \Box(r \supset q)$



The following interpretation, taken from the left-most open branch, shows this inference to be invalid:

$$v_{w_0}(p) = 0, v_{w_1}(p) = 1, v_{w_1}(s) = 0, v_{w_2}(p) = 0, v_{w_2}(q) = 0, v_{w_2}(r) = 1$$

$\neg(A \strictif B) \nvDash_{K_\upsilon} A$

$\neg\Box(p \supset q) \nvDash_{K_\upsilon} p$

$$\begin{array}{c}
\neg\Box(p \supset q), 0 \\
\neg p, 0 \\
\Diamond\neg(p \supset q), 0 \\
\neg(p \supset q), 1 \\
p, 1 \\
\neg q, 1
\end{array}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

$$v_{w_0}(p) = 0, v_{w_1}(p) = 1, v_{w_1}(q) = 0$$

Since the preceeding are invalid in the strongest logic we have dealt with K_υ, they must be invalid in every logic we have dealt with.

■

4.6.2 Show that *modus ponens* for $\strictif$ holds in K_υ and L_υ.

$A, A \strictif B \vdash_{L_\upsilon} B$
$A, \Box(A \supset B) \vdash_{L_\upsilon} B$

$$\begin{array}{c}
A, 0 \\
\Box(A \supset B), 0 \\
\neg B, 0 \\
A \supset B, 0 \\
\swarrow \quad \searrow \\
\neg A, 0 \qquad B, 0 \\
\otimes \qquad\quad \otimes
\end{array}$$

K_υ is an extension of L_υ therefore $A, A \strictif B \vdash_{K_\upsilon} B$ also holds.

■

4.6.3 Show that $\Box B \vDash A \strictif B$, $\neg\Diamond A \vDash A \strictif B$, $\vDash A \strictif (B \vee \neg B)$ and $\vDash (A \wedge \neg A) \strictif B$ hold in all modal logics.

The weakest modal logic we have met so far is L. I will show that they hold in L and this will imply that they hold in all logics we have dealt with so far.

$\Box B \vdash_L A \strictif B$

$\Box B \vdash_L \Box(A \supset B)$

$$\begin{array}{c} \Box B, 0 \\ \neg\Box(A \supset B), 0 \\ \Diamond\neg(A \supset B), 0 \\ 0r1 \\ \neg(A \supset B), 1 \\ B, 1 \\ A, 1 \\ \neg B, 1 \\ \otimes \end{array}$$

By Soundness: $\Box B \vDash_L A \strictif B$

$\blacksquare$

$\neg\Diamond A \vdash_L A \strictif B$

$\neg\Diamond A \vdash_L \Box(A \supset B)$

$$\begin{array}{c} \neg\Diamond A, 0 \\ \neg\Box(A \supset B), 0 \\ \Diamond\neg(A \supset B), 0 \\ \Box\neg A, 0 \\ 0r1 \\ \neg(A \supset B), 1 \\ \neg A, 1 \\ A, 1 \\ \otimes \end{array}$$

By Soundness: $\neg\Diamond A \vDash_L A \strictif B$

$\blacksquare$

$\vdash_L A \mathrel{-\!\!3} (B \vee \neg B)$

$\vdash_L \Box(A \supset (B \vee \neg B)$

$$\begin{array}{c}
\neg\Box(A \supset (B \vee \neg B)), 0 \\
\Diamond\neg(A \supset (B \vee \neg B)), 0 \\
0r1 \\
\neg(A \supset (B \vee \neg B)), 1 \\
A, 1 \\
\neg(B \vee \neg B), 1 \\
\neg B, 1 \\
\neg\neg B, 1 \\
\otimes
\end{array}$$

By Soundness: $\vDash_L A \mathrel{-\!\!3} (B \vee \neg B)$

■

$\vdash_L (A \wedge \neg A) \mathrel{-\!\!3} B$

$\vdash_L \Box((A \wedge \neg A) \supset B)$

$$\begin{array}{c}
\neg\Box((A \wedge \neg A) \supset B), 0 \\
\Diamond\neg((A \wedge \neg A) \supset B), 0 \\
\neg((A \wedge \neg A) \supset B), 1 \\
A \wedge \neg A, 1 \\
\neg B, 1 \\
A, 1 \\
\neg A, 1 \\
\otimes
\end{array}$$

By Soundness: $\vDash_L (A \wedge \neg A) \mathrel{-\!\!3} B$

■

2. Show the following for N:

(a) $\vdash A \mathrel{-\!\!3} A$

$\vdash \Box(A \supset A)$

$$\begin{array}{c}
\neg\Box(A \supset A), 0 \\
\Diamond\neg(A \supset A), 0 \\
0r1 \\
\neg(A \supset A), 1 \\
A, 1 \\
\neg A, 1 \\
\otimes
\end{array}$$

(b) $\vdash ((A \strictif B) \wedge (B \strictif C)) \strictif (A \strictif C)$

$\vdash \Box((\Box(A \supset B) \wedge \Box(B \supset C)) \supset \Box(A \supset C))$

$$\begin{array}{c}
\neg\Box((\Box(A \supset B) \wedge \Box(B \supset C)) \supset \Box(A \supset C)), 0 \\
\Diamond\neg((\Box(A \supset B) \wedge \Box(B \supset C)) \supset \Box(A \supset C)), 0 \\
0\text{r}1 \\
\neg((\Box(A \supset B) \wedge \Box(B \supset C)) \supset \Box(A \supset C)), 1 \\
\Box(A \supset B) \wedge \Box(B \supset C), 1 \\
\neg\Box(A \supset C), 1 \\
\Box(A \supset B), 1 \\
\Box(B \supset C), 1 \\
\Diamond\neg(A \supset C), 1 \\
1\text{r}2 \\
\neg(A \supset C), 2 \\
A \supset B, 2 \\
B \supset C, 2 \\
A, 2 \\
\neg C, 2 \\
\begin{array}{cc}
\begin{array}{c} \neg A, 2 \\ \otimes \end{array} &
\begin{array}{c} B, 2 \\ \begin{array}{cc} \begin{array}{c} \neg B, 2 \\ \otimes \end{array} & \begin{array}{c} C, 2 \\ \otimes \end{array} \end{array} \end{array}
\end{array}
\end{array}$$

(c) $\vdash (A \strictif B) \strictif (\neg B \strictif \neg A)$

$\vdash \Box(\Box(A \supset B) \supset \Box(\neg B \supset \neg A))$

$$\begin{array}{c}
\neg\Box(\Box(A \supset B) \supset \Box(\neg B \supset \neg A)), 0 \\
\Diamond\neg(\Box(A \supset B) \supset \Box(\neg B \supset \neg A)), 0 \\
0\text{r}1 \\
\neg(\Box(A \supset B) \supset \Box(\neg B \supset \neg A)), 1 \\
\Box(A \supset B), 1 \\
\neg\Box(\neg B \supset \neg A), 1 \\
\Diamond\neg(\neg B \supset \neg A), 1 \\
1\text{r}2 \\
\neg(\neg B \supset \neg A), 2 \\
A \supset B, 2 \\
\neg B, 2 \\
\neg\neg A, 2 \\
\begin{array}{cc}
\begin{array}{c} \neg A, 2 \\ \otimes \end{array} &
\begin{array}{c} B, 2 \\ \otimes \end{array}
\end{array}
\end{array}$$

(d) $\vdash \Box\neg A \supset \Box\neg(A \wedge B)$

$\vdash \Box\neg A \supset \Box\neg(A \wedge B)$

$$\begin{gathered}\neg(\Box\neg A \supset \Box\neg(A \wedge B)), 0 \\ \Box\neg A, 0 \\ \neg\Box\neg(A \wedge B), 0 \\ \Diamond\neg\neg(A \wedge B), 0 \\ 0r1 \\ \neg\neg(A \wedge B), 1 \\ \neg A, 1 \\ A \wedge B, 1 \\ A, 1 \\ \otimes\end{gathered}$$

3. Show the following for N. Specify a counter-model and draw a picture of it.

(a) $\nvdash \Box p \supset p$

$$\begin{gathered}\neg(\Box p \supset p), 0 \\ \Box p, 0 \\ \neg p, 0\end{gathered}$$

Counter-model such that,

$$W = N = \{w_0\}$$

$$v_{w_0}(p) = 0$$

This can be represented in the following picture:

$$\begin{gathered}w_0 \\ \neg p\end{gathered}$$

(b) $\nvdash \Box p \supset \Box\Box p$

$$\begin{gathered}\neg(\Box p \supset \Box\Box p), 0 \\ \Box p, 0 \\ \neg\Box\Box p, 0 \\ \Diamond\neg\Box p, 0 \\ 0r1 \\ \neg\Box p, 1 \\ p, 1 \\ \Diamond\neg p, 1\end{gathered}$$

Counter-model such that,

$$W = \{w_0, w_1\}; N = \{w_0\}; w_0 R w_1; v_{w_1}(p) = 1$$

This can be represented in the following picture:

$$\begin{array}{ccc} w_0 & \rightarrow & \boxed{w_1} \\ & & p \end{array}$$

(c) $\nvDash \neg(p -\!\!\!\!\exists\, p) -\!\!\!\!\exists\, q$

$\nvDash \Box(\neg\Box(p \supset q) \supset q)$

$$\begin{array}{c} \neg\Box(\neg\Box(p \supset q) \supset q), 0 \\ \Diamond\neg(\neg\Box(p \supset q) \supset q), 0 \\ 0\text{r}1 \\ \neg(\neg\Box(p \supset q) \supset q), 1 \\ \neg\Box(p \supset p), 1 \\ \neg q, 1 \\ \Diamond\neg(p \supset p), 1 \end{array}$$

Counter-model such that,

$$W = \{w_0, w_1\}; N = \{w_0\}; w_0 R w_1; v_{w_1}(q) = 0$$

This can be represented in the following picture:

$$\begin{array}{ccc} w_0 & \rightarrow & \boxed{w_1} \\ & & \neg q \end{array}$$

(d) $\nvDash \Box(p -\!\!\!\!\exists\, p)$

$\nvDash \Box\Box(p \supset p)$

$$\begin{array}{c} \neg\Box\Box(p \supset p), 0 \\ \Diamond\neg\Box(p \supset p), 0 \\ 0\text{r}1 \\ \neg\Box(p \supset p), 1 \\ \Diamond\neg(p \supset p), 1 \end{array}$$

Counter-model such that,

$$W = \{w_0, w_1\}; N = \{w_0\}; w_0 R w_1$$

This can be represented in the following picture:

$$\begin{tikzcd} w_0 \arrow[r] & \boxed{w_1} \end{tikzcd}$$

(e)$\nvdash (p ⥽ q) ⥽ (\Box p ⥽ \Box q)$

$\nvdash \Box(\Box(p \supset q) \supset \Box(\Box p \supset \Box q))$

$$\neg\Box(\Box(p \supset q) \supset \Box(\Box p \supset \Box q)), 0$$
$$\Diamond\neg(\Box(p \supset q) \supset \Box(\Box p \supset \Box q)), 0$$
$$0r1$$
$$\neg(\Box(p \supset q) \supset \Box(\Box p \supset \Box q)), 1$$
$$\Box(p \supset q), 1$$
$$\neg\Box(\Box p \supset \Box q), 1$$
$$\Diamond\neg(\Box p \supset \Box q), 1$$
$$1r2$$
$$\neg(\Box p \supset \Box q), 2$$
$$\Box p, 2$$
$$\neg\Box q, 2$$
$$p \supset q, 2$$
$$\begin{array}{cc} \neg p, 2 & q, 2 \end{array}$$

Counter-model from the left-most open branch such that,

$$W = N = \{w_0, w_1, w_2\}; w_0 R w_1, w_1 R w_2; v_{w_2}(p) = 0$$

This can be represented in the following picture:

$$\begin{tikzcd} w_0 \arrow[r] & w_1 \arrow[r] & w_2 \\ & & \neg p \end{tikzcd}$$

(f) $\nvdash \Box\Box p \strictif (\Box q \strictif \Box\Box q)$

$\nvdash \Box(\Box\Box p \supset \Box(\Box q \supset \Box\Box q))$

$$\begin{gathered}
\neg\Box(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 0 \\
\Diamond\neg(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 0 \\
0r1 \\
\neg(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 1 \\
\Box\Box p, 1 \\
\neg\Box(\Box q \supset \Box\Box q), 1 \\
\Diamond\neg(\Box q \supset \Box\Box q), 1 \\
1r2 \\
\neg(\Box q \supset \Box\Box q), 2 \\
\Box q, 2 \\
\neg\Box\Box q, 2 \\
\Diamond\neg\Box q, 2 \\
\Box p, 2 \\
2r3 \\
\neg\Box q, 3 \\
q, 3 \\
\Diamond\neg q, 3 \\
p, 3
\end{gathered}$$

Counter-model such that,

$$W = \{w_0, w_1, w_2, w_3\}; N = \{w_0, w_1, w_2\}; w_0Rw_1, w_1Rw_2, w_2Rw_3; v_{w_3}(q) = 1, v_{w_3}(p) = 1$$

This can be represented in the following picture:

$$\begin{tikzcd}
w_0 \arrow[r] & w_1 \arrow[r] & w_2 \arrow[r] & \boxed{w_3} \\
& & & p, q
\end{tikzcd}$$

(g) $\nvdash \Diamond\Diamond p$

$$\begin{gathered}
\neg\Diamond\Diamond p, 0 \\
\Box\neg\Diamond p, 0
\end{gathered}$$

Counter-model such that,

$$W = N = \{w_0\}$$

This can be represented in the following picture:

$$w_0$$

(h) $\nvdash \Box\Box(p \vee \neg p)$

$$\begin{gathered}
\neg\Box\Box(p \vee \neg p), 0 \\
\Diamond\neg\Box(p \vee \neg p), 0 \\
0\mathrm{r}1 \\
\neg\Box(p \vee \neg p), 1 \\
\Diamond\neg(p \vee \neg p), 1
\end{gathered}$$

Counter-model such that,

$$W = \{w_0, w_1\}; N = \{w_0\}; w_0 R w_1$$

This can be represented in the following picture:

$$w_0 \qquad \rightarrow \qquad \boxed{w_1}$$

4. Which of the above (in problem 3) hold in $S2$ ($N\rho$)? Which hold in $S3$ ($N\rho\tau$)?

Only (a) is valid in $S2$:
(a) $\vdash_{N\rho} \Box p \supset p$

$$\begin{gathered}
\neg(\Box p \supset p), 0 \\
0\mathrm{r}0 \\
\Box p, 0 \\
\neg p, 0 \\
p, 0 \\
\otimes
\end{gathered}$$

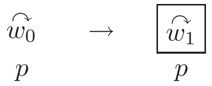
(b) $\nvdash_{N\rho} \Box p \supset \Box\Box p$

$$\begin{gathered}
\neg(\Box p \supset \Box\Box p), 0 \\
0\mathrm{r}0 \\
\Box p, 0 \\
\neg\Box\Box p, 0 \\
\Diamond\neg\Box p, 0 \\
p, 0 \\
0\mathrm{r}1,\ 1\mathrm{r}1 \\
\neg\Box p, 1 \\
p, 1 \\
\Diamond\neg p, 1
\end{gathered}$$

Counter-model such that,

$$W = \{w_0, w_1\}; N = \{w_0\}; w_0Rw_0, w_0Rw_1, w_1Rw_1; v_{w_0}(p) = 1, v_{w_1}(p) = 1$$

This can be represented in the following picture:



(c) $\nvDash_{N\rho} \neg(p \strictif p) \strictif q$

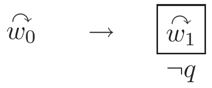
$\nvDash_{N\rho} \Box(\neg\Box(p \supset p) \supset q)$

$$\begin{array}{c}
\neg\Box(\neg\Box(p \supset p) \supset q), 0 \\
\text{0r0} \\
\Diamond\neg(\neg\Box(p \supset p) \supset q), 0 \\
\text{0r1, 1r1} \\
\neg(\neg\Box(p \supset p) \supset q), 1 \\
\neg\Box(p \supset p), 1 \\
\neg q, 1 \\
\Diamond\neg(p \supset p), 1
\end{array}$$

Counter-model such that,

$$W = \{w_0, w_1\}; N = \{w_0\}; w_0Rw_0, w_0Rw_1w_1Rw_1; v_{w_1}(q) = 0$$

This can be represented in the following picture:



(d) $\nvDash_{N\rho} \Box(p \strictif p)$

$\nvDash_{N\rho} \Box\Box(p \supset p)$

$$\begin{array}{c}
\neg\Box\Box(p \supset p), 0 \\
\text{0r0} \\
\Diamond\neg\Box(p \supset p), 0 \\
\text{0r1,1r1} \\
\neg\Box(p \supset p), 1 \\
\Diamond\neg(p \supset p), 1
\end{array}$$

Counter-model such that,

$$W = \{w_0, w_1\}; N = \{w_0\}; w_0 R w_1, w_0 R w_1, w_1 R w_1$$

This can be represented in the following picture:

$$\overset{\frown}{w_0} \qquad \to \qquad \boxed{\overset{\frown}{w_1}}$$

(e) $\nvdash_{N\rho} (p ⥽ q) ⥽ (\Box p ⥽ \Box q)$

$\nvdash_{N\rho} \Box(\Box(p \supset q) \supset \Box(\Box p \supset \Box q))$

$$\neg\Box(\Box(p \supset q) \supset \Box(\Box p \supset \Box q)), 0$$
$$0r0$$
$$\Diamond\neg(\Box(p \supset q) \supset \Box(\Box p \supset \Box q)), 0$$
$$0r1,\ 1r1$$
$$\neg(\Box(p \supset q) \supset \Box(\Box p \supset \Box q)), 1$$
$$\Box(p \supset q), 1$$
$$\neg\Box(\Box p \supset \Box q), 1$$
$$\Diamond\neg(\Box p \supset \Box q), 1$$
$$p \supset q, 1$$
$$1r2,\ 2r2$$
$$\neg(\Box p \supset \Box q), 2$$
$$\Box p, 2$$
$$\neg\Box q, 2$$
$$p \supset q, 2$$
$$\Diamond\neg q, 2$$
$$2r3,\ 3r3$$
$$\neg q, 3$$
$$p, 3$$

$$\begin{array}{cccc} \multicolumn{2}{c}{\neg p, 1} & \multicolumn{2}{c}{q, 1} \\ \neg p, 2 & q, 2 & \neg p, 2 & q, 2 \\ p, 2 & p, 2 & p, 2 & p, 2 \\ \otimes & & \otimes & \end{array}$$

Counter-model from left-most open branch such that,

$$W = \{w_0, w_1, w_2, w_3\}; N = \{w_1, w_2\}$$
$$w_0 R w_0, w_0 R w_1, w_1 R w_1, w_1 R w_2, w_2 R w_2, w_2 R w_3, w_3 R w_3$$
$$v_{w_1}(p) = 0, v_{w_2}(p) = 1, v_{w_3}(p) = 1, v_{w_2}(q) = 1, v_{w_3}(q) = 0$$

This can be represented in the following picture:

$$\begin{array}{ccccccc}
\overset{\frown}{w_0} & \rightarrow & \overset{\frown}{w_1} & \rightarrow & \overset{\frown}{w_2} & \rightarrow & \overset{\frown}{w_3} \\
 & & \neg p & & p & & p, \neg q
\end{array}$$

(f) $\nvdash_{N\rho} \Box\Box p ⥽ (\Box q ⥽ \Box\Box q)$

$\nvdash_{N\rho} \Box(\Box\Box p \supset \Box(\Box q \supset \Box\Box q))$

$$\begin{array}{c}
\neg\Box(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 0 \\
0\text{r}0 \\
\Diamond\neg(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 0 \\
0\text{r}1,\ 1\text{r}1 \\
\neg(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 1 \\
\Box\Box p, 1 \\
\neg\Box(\Box q \supset \Box\Box q), 1 \\
\Diamond\neg(\Box q \supset \Box\Box q), 1 \\
\Box p, 1 \\
p, 1 \\
1\text{r}2,\ 2\text{r}2 \\
\neg(\Box q \supset \Box\Box q), 2 \\
\Box q, 2 \\
\neg\Box\Box q, 2 \\
\Box p, 2 \\
\Diamond\neg\Box q, 2 \\
p, 2 \\
q, 2 \\
2\text{r}3,\ 3\text{r}3 \\
\neg\Box q, 3 \\
p, 3 \\
\Diamond\neg q, 3 \\
q, 3
\end{array}$$

Counter-model such that,

$$W = \{w_0, w_1, w_2, w_3\}; N = \{w_0, w_1, w_2\}$$

$$w_0Rw_0, w_0Rw_1, w_1Rw_1, w_1Rw_2, w_2Rw_2, w_2Rw_3, w_3Rw_3$$

$$v_{w_1}(p) = 1, v_{w_2}(p) = 1, v_{w_3}(p) = 1, v_{w_2}(q) = 1, v_{w_3}(q) = 1$$

This can be represented in the following picture:

$$\begin{array}{ccccccc} \overset{\curvearrowright}{w_0} & \rightarrow & \overset{\curvearrowright}{w_1} & \rightarrow & \overset{\curvearrowright}{w_2} & \rightarrow & \boxed{\overset{\curvearrowright}{w_3}} \\ & & p & & p, q & & p, q \end{array}$$

(g) $\nvdash_{N\rho} \Diamond\Diamond p$

$$\begin{array}{c} \neg\Diamond\Diamond p, 0 \\ 0\mathrm{r}0 \\ \Box\neg\Diamond p, 0 \\ \neg\Diamond p, 0 \\ \Box\neg p, 0 \\ \neg p, 0 \end{array}$$

Counter-model such that,

$$W = N = \{w_0\}; w_0 R w_0; v_{w_0}(p) = 0$$

This can be represented in the following picture:

$$\begin{array}{c} \overset{\curvearrowright}{w_0} \\ \neg p \end{array}$$

(h) $\nvdash_{N\rho} \Box\Box(p \vee \neg p)$

$$\begin{array}{c} \neg\Box\Box(p \vee \neg p), 0 \\ 0\mathrm{r}0 \\ \Diamond\neg\Box(p \vee \neg p), 0 \\ 0\mathrm{r}1,\ 1\mathrm{r}1 \\ \neg\Box(p \vee \neg p), 1 \\ \Diamond\neg(p \vee \neg p), 1 \end{array}$$

Counter-model such that,

$$W = \{w_0, w_1\}; N = \{w_0\}; w_0 R w_0, w_0 R w_1, w_1 R w_1$$

This can be represented in the following picture:

$$\begin{array}{ccc} \overset{\curvearrowright}{w_0} & \rightarrow & \boxed{\overset{\curvearrowright}{w_1}} \end{array}$$

$\blacksquare$

Only (a) and (f) are valid in $N\rho\tau$:

(a) $\vdash_{N\rho\tau} \Box p \supset p$

$$\begin{array}{c}
\neg(\Box p \supset p), 0 \\
0\text{r}0 \\
\Box p, 0 \\
\neg p, 0 \\
p, 0 \\
\otimes
\end{array}$$

(b) $\nvdash_{N\rho\tau} \Box p \supset \Box\Box p$

$$\begin{array}{c}
\neg(\Box p \supset \Box\Box p), 0 \\
0\text{r}0 \\
\Box p, 0 \\
\neg\Box\Box p, 0 \\
\Diamond\neg\Box p, 0 \\
p, 0 \\
0\text{r}1,\ 1\text{r}1 \\
\neg\Box p, 1 \\
p, 1 \\
\Diamond\neg p, 1
\end{array}$$

Counter-model such that,

$$W = \{w_0, w_1\}; N = \{w_0\}; w_0Rw_0, w_0Rw_1, w_1Rw_1; v_{w_0}(p) = 1, v_{w_1}(p) = 1$$

This can be represented in the following picture:

$$\begin{array}{ccc}
\overset{\curvearrowright}{w_0} & \rightarrow & \boxed{\overset{\curvearrowright}{w_1}} \\
p & & p
\end{array}$$

(c) $\nvdash_{N\rho\tau} \neg(p \strictif p) \strictif q$

$\nvdash_{N\rho\tau} \Box(\neg\Box(p \supset p) \supset q)$

$$\begin{array}{c} \neg\Box(\neg\Box(p \supset p) \supset q), 0 \\ 0\mathrm{r}0 \\ \Diamond\neg(\neg\Box(p \supset p) \supset q), 0 \\ 0\mathrm{r}1,\ 1\mathrm{r}1 \\ \neg(\neg\Box(p \supset p) \supset q), 1 \\ \neg\Box(p \supset p), 1 \\ \neg q, 1 \\ \Diamond\neg(p \supset p), 1 \end{array}$$

Counter-model such that,

$$W = \{w_0, w_1\}; N = \{w_0\}; w_0Rw_0, w_0Rw_1w_1Rw_1; v_{w_1}(q) = 0$$

This can be represented in the following picture:

$$\overset{\frown}{w_0} \qquad \rightarrow \qquad \underset{\neg q}{\boxed{\overset{\frown}{w_1}}}$$

(d) $\nvdash_{N\rho\tau} \Box(p \strictif p)$

$\nvdash_{N\rho\tau} \Box\Box(p \supset p)$

$$\begin{array}{c} \neg\Box\Box(p \supset p), 0 \\ 0\mathrm{r}0 \\ \Diamond\neg\Box(p \supset p), 0 \\ 0\mathrm{r}1,1\mathrm{r}1 \\ \neg\Box(p \supset p), 1 \\ \Diamond\neg(p \supset p), 1 \end{array}$$

Counter-model such that,

$$W = \{w_0, w_1\}; N = \{w_0\}; w_0Rw_1, w_0Rw_1, w_1Rw_1$$

This can be represented in the following picture:

$$\overset{\frown}{w_0} \qquad \rightarrow \qquad \boxed{\overset{\frown}{w_1}}$$

(e) $\vdash_{N\rho\tau} (p \strictif q) \strictif (\Box p \strictif \Box q)$

$\vdash_{N\rho\tau} \Box(\Box(p \supset q) \supset \Box(\Box p \supset \Box q))$

$$\begin{gathered}
\neg\Box(\Box(p \supset q) \supset \Box(\Box p \supset \Box q)), 0 \\
0\text{r}0 \\
\Diamond\neg(\Box(p \supset q) \supset \Box(\Box p \supset \Box q)), 0 \\
0\text{r}1,\ 1\text{r}1 \\
\neg(\Box(p \supset q) \supset \Box(\Box p \supset \Box q)), 1 \\
\Box(p \supset q), 1 \\
\neg\Box(\Box p \supset \Box q), 1 \\
p \supset q, 1 \\
\Diamond\neg(\Box p \supset \Box q), 1 \\
1\text{r}2,\ 2\text{r}2,\ 0\text{r}2 \\
\neg(\Box p \supset \Box q), 2 \\
\Box p, 2 \\
\neg\Box q, 2 \\
\Diamond\neg q, 2 \\
2\text{r}3,\ 3\text{r}3,\ 0\text{r}3,\ 1\text{r}3 \\
\neg q, 3 \\
p, 3 \\
p \supset q, 3 \\
\begin{array}{cc}
\diagup & \diagdown \\
\neg p, 3 & q, 3 \\
\otimes & \otimes
\end{array}
\end{gathered}$$

(f) $\vdash_{N\rho\tau} \Box\Box p \strictif (\Box q \strictif \Box\Box q)$

$\vdash_{N\rho\tau} \Box(\Box\Box p \supset \Box(\Box q \supset \Box\Box q))$

$$\begin{array}{c}
\neg\Box(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 0 \\
\text{0r0} \\
\Diamond\neg(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 0 \\
\text{0r1, 1r1} \\
\neg(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 1 \\
\Box\Box p, 1 \\
\neg\Box(\Box q \supset \Box\Box q), 1 \\
\Diamond\neg(\Box q \supset \Box\Box q), 1 \\
\Box p, 1 \\
p, 1 \\
\text{1r2, 2r2, 0r2} \\
\neg(\Box q \supset \Box\Box q), 2 \\
\Box q, 2 \\
\neg\Box\Box q, 2 \\
\Box p, 2 \\
\Diamond\neg\Box q, 2 \\
p, 2 \\
q, 2 \\
\text{2r3, 3r3, 1r3, 0r3} \\
\neg\Box q, 3 \\
p, 3 \\
\Diamond\neg q, 3 \\
\Box p, 3 \\
q, 3 \\
\text{3r4, 4r4, 2r4, 1r4, 0r4} \\
\neg q, 4 \\
q, 4 \\
\otimes
\end{array}$$

(g) $\nvdash_{N\rho\tau} \Diamond\Diamond p$

$$\begin{array}{c}
\neg\Diamond\Diamond p, 0 \\
\text{0r0} \\
\Box\neg\Diamond p, 0 \\
\neg\Diamond p, 0 \\
\Box\neg p, 0 \\
\neg p, 0
\end{array}$$

Counter-model such that,

$$W = N = \{w_0\}; w_0 R w_0; v_{w_0}(p) = 0$$

This can be represented in the following picture:

$$\overset{\frown}{w_0}$$
$$\neg p$$

(h) $\nvdash_{N\rho\tau} \Box\Box(p \vee \neg p)$

$$\neg\Box\Box(p \vee \neg p), 0$$
$$0r0$$
$$\Diamond\neg\Box(p \vee \neg p), 0$$
$$0r1,\ 1r1$$
$$\neg\Box(p \vee \neg p), 1$$
$$\Diamond\neg(p \vee \neg p), 1$$

Counter-model such that,

$$W = \{w_0, w_1\}; N = \{w_0\}; w_0Rw_0, w_0Rw_1, w_1Rw_1$$

This can be represented in the following picture:

$$\overset{\frown}{w_0} \qquad \rightarrow \qquad \boxed{\overset{\frown}{w_1}}$$

■

5. Repeat 3.10, problem 7, with N instead of K. (Beware: in $N\tau$, $\Box p \supset \Box\Box p$ is *not* valid. A little ingenuity is required here.)

(a) If R is reflexive (ρ), it is extendable (η). Hence, if truth is preserved at all worlds of all η-interpretations, it is preserved at all worlds of all ρ-interpretations. Consequently, the system N_ρ is an extension of the system N_η. Find an inference demonstrating that it is a proper extension.

$$\boxed{\Box A \supset A}$$

$\vdash_{N_\rho} (\Box A \supset A)$

$$\neg(\Box A \supset A), 0$$
$$0r0$$
$$\Box A, 0$$
$$\neg A, 0$$
$$A, 0$$
$$\otimes$$

$\nvdash_{N_\eta} (\Box A \supset A)$

$$\begin{array}{c} \neg(\Box A \supset A), 0 \\ \Box A, 0 \\ \neg A, 0 \\ 0\text{r}1 \\ A, 1 \\ 1\text{r}2 \\ \vdots \end{array}$$

$$W = \{w_0, w_1, w_2...\}; N = \{w_0\}$$

$$w_0 R w_1, w_1 R w_2...$$

$$v_{w_0}(p) = 0, v_{w_1}(p) = 1$$

$$\begin{tikzcd} w_0 \arrow[r] & \boxed{w_1} \arrow[r] & \boxed{w_2} \arrow[r] & ... \\ \neg p & p & & \end{tikzcd}$$

■

This interpretation shows that it is not the case that if truth is preserved at all worlds of all ρ-interpretations, it is preserved at all worlds of all η-interpretations: i.e. that ρ is a proper extension of η.

(b) Show that none of the systems N_ρ, N_σ and N_τ is an extension of any of the others (i.e., for each pair, find an inference that is valid in one but not the other, and then vice versa).(Hint: see 3.5.10.)

$$\boxed{\text{There is at least one inference valid in } N_\rho \text{ that is not valid in } N_\sigma \text{ or } N_\tau}$$

$\vdash_{N_\rho} \Box p \supset p$

$$\begin{array}{c} \neg(\Box p \supset p), 0 \\ 0\text{r}0 \\ \Box p, 0 \\ \neg p, 0 \\ p, 0 \\ \otimes \end{array}$$

The same tableau shows this inference to be invalid in both N_σ and N_τ:

$\nvdash_{N_\sigma} \Box p \supset p$

$\nvdash_{N_\tau} \Box p \supset p$

$$\begin{gathered} \neg(\Box p \supset p), 0 \\ \Box p, 0 \\ \neg p, 0 \end{gathered}$$

$$W = N = \{w_0\}$$

$$v_{w_0}(p) = 0$$

$$\begin{gathered} w_0 \\ \neg p \end{gathered}$$

■

There is at least one inference valid in N_σ that is not valid in N_ρ or N_τ

$\vdash_{N_\sigma} p \supset \Box\Diamond p$

$$\begin{gathered} \neg(p \supset \Box\Diamond p), 0 \\ p, 0 \\ \neg\Box\Diamond p, 0 \\ \Diamond\neg\Diamond p, 0 \\ 0\text{r}1,\ 1\text{r}0 \\ \neg\Diamond p, 1 \\ \Box\neg p, 1 \\ \neg p, 0 \\ \otimes \end{gathered}$$

$\nvdash_{N_\rho} p \supset \Box\Diamond p$

$$\begin{gathered} \neg(p \supset \Box\Diamond p), 0 \\ 0\text{r}0 \\ p, 0 \\ \neg\Box\Diamond p, 0 \\ \Diamond\neg\Diamond p, 0 \\ 0\text{r}1,\ 1\text{r}1 \\ \neg\Diamond p, 1 \\ \Box\neg p, 1 \\ \neg p, 1 \end{gathered}$$

$$W = N = \{w_0, w_1\}$$

$$w_0 R w_1, w_0 R w_0, w_1 R w_1$$

$$v_{w_0}(p) = 1, v_{w_1}(p) = 0$$

$$\begin{array}{ccc} \overset{\curvearrowright}{w_0} & \longrightarrow & \overset{\curvearrowright}{w_1} \\ p & & \neg p \end{array}$$

$\nvdash_{N_\tau} p \supset \Box\Diamond p$

$$\begin{array}{c} \neg(p \supset \Box\Diamond p), 0 \\ p, 0 \\ \neg\Box\Diamond p, 0 \\ \Diamond\neg\Diamond p, 0 \\ 0\mathrm{r}1 \\ \neg\Diamond p, 1 \\ \Box\neg p, 1 \end{array}$$

$$W = N = \{w_0, w_1\}$$

$$w_0 R w_1$$

$$v_{w_0}(p) = 1$$

$$\begin{array}{ccc} w_0 & \longrightarrow & w_1 \\ p & & \end{array}$$

■

$$\boxed{\text{There is at least one inference valid in } N_\tau \text{ that is not valid in } N_\rho \text{ or } N_\sigma}$$

$\vdash_{N\tau} \Box\Box p \mathbin{-\!\!3} (\Box q \mathbin{-\!\!3} \Box\Box q)$

$\vdash_{N\tau} \Box(\Box\Box p \supset \Box(\Box q \supset \Box\Box q))$

$$\begin{gathered}
\neg\Box(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 0 \\
\Diamond\neg(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 0 \\
\text{0r1} \\
\neg(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 1 \\
\Box\Box p, 1 \\
\neg\Box(\Box q \supset \Box\Box q), 1 \\
\Diamond\neg(\Box q \supset \Box\Box q), 1 \\
\text{1r2, 0r2} \\
\neg(\Box q \supset \Box\Box q), 2 \\
\Box q, 2 \\
\neg\Box\Box q, 2 \\
\Box p, 2 \\
\Diamond\neg\Box q, 2 \\
\text{2r3, 1r3, 0r3} \\
\neg\Box q, 3 \\
p, 3 \\
\Diamond\neg q, 3 \\
\Box p, 3 \\
q, 3 \\
\text{3r4, 2r4, 1r4, 0r4} \\
\neg q, 4 \\
q, 4 \\
\otimes
\end{gathered}$$

$\nvdash_{N\rho} \Box\Box p \strictif (\Box q \strictif \Box\Box q)$

$\nvdash_{N\rho} \Box(\Box\Box p \supset \Box(\Box q \supset \Box\Box q))$

$$\begin{array}{c}
\neg\Box(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 0 \\
0\mathrm{r}0 \\
\Diamond\neg(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 0 \\
0\mathrm{r}1,\ 1\mathrm{r}1 \\
\neg(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 1 \\
\Box\Box p, 1 \\
\neg\Box(\Box q \supset \Box\Box q), 1 \\
\Diamond\neg(\Box q \supset \Box\Box q), 1 \\
\Box p, 1 \\
p, 1 \\
1\mathrm{r}2,\ 2\mathrm{r}2 \\
\neg(\Box q \supset \Box\Box q), 2 \\
\Box q, 2 \\
\neg\Box\Box q, 2 \\
\Box p, 2 \\
\Diamond\neg\Box q, 2 \\
p, 2 \\
q, 2 \\
2\mathrm{r}3,\ 3\mathrm{r}3 \\
\neg\Box q, 3 \\
p, 3 \\
\Diamond\neg q, 3 \\
q, 3
\end{array}$$

Counter-model such that,

$$W = \{w_0, w_1, w_2, w_3\}; N = \{w_0, w_1, w_2\}$$

$$w_0Rw_0, w_0Rw_1, w_1Rw_1, w_1Rw_2, w_2Rw_2, w_2Rw_3, w_3Rw_3$$

$$v_{w_1}(p) = 1, v_{w_2}(p) = 1, v_{w_3}(p) = 1, v_{w_2}(q) = 1, v_{w_3}(q) = 1$$

This can be represented in the following picture:

$$\begin{array}{ccccccc}
\overset{\curvearrowright}{w_0} & \to & \overset{\curvearrowright}{w_1} & \to & \overset{\curvearrowright}{w_2} & \to & \boxed{\overset{\curvearrowright}{w_3}} \\
 & & p & & p, q & & p, q
\end{array}$$

$\nvdash_{N\sigma} \Box\Box p \mathrel{-\!\!\!3} (\Box q \mathrel{-\!\!\!3} \Box\Box q)$

$\nvdash_{N\sigma} \Box(\Box\Box p \supset \Box(\Box q \supset \Box\Box q))$

$$\begin{gathered}
\neg\Box(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 0 \\
\Diamond\neg(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 0 \\
0\text{r}1,\ 1\text{r}0 \\
\neg(\Box\Box p \supset \Box(\Box q \supset \Box\Box q))\Box\Box q)), 1 \\
\Box\Box p, 1 \\
\Box p, 0 \\
p, 1 \\
\neg\Box(\Box q \supset \Box\Box q), 1 \\
\Diamond\neg(\Box q \supset \Box\Box q), 1 \\
1\text{r}2,\ 2\text{r}1 \\
\neg(\Box q \supset \Box\Box q), 2 \\
\Box q, 2 \\
q, 1 \\
\neg\Box\Box q, 2 \\
\Box p, 2 \\
p, 1 \\
\Diamond\neg\Box q, 2 \\
2\text{r}3,\ 3\text{r}2 \\
\neg\Box q, 3 \\
p, 3 \\
\Diamond\neg q, 3 \\
q, 3
\end{gathered}$$

Counter-model such that,

$$W = \{w_0, w_1, w_2, w_3\}; N = \{w_0, w_1, w_2\}$$

$$w_0 R w_1, w_1 R w_0, w_1 R w_2, w_2 R w_1, w_2 R w_3, w_3 R w_2$$

$$v_{w_1}(p) = 1, v_{w_1}(q) = 1, v_{w_3}(p) = 1, v_{w_3}(q) = 1$$

This can be represented in the following picture:

$$\begin{array}{ccccccc}
w_0 & \leftrightarrows & w_1 & \leftrightarrows & w_2 & \leftrightarrows & \boxed{w_3} \\
 & & p, q & & & & p, q
\end{array}$$

■

(c) By combining the individual conditions, we obtain the systems $N_{\rho\sigma}$, $N_{\rho\tau}$, $N_{\sigma\tau}$, $N_{\sigma\eta}$, and $N_{\tau\eta}$. $N_{\rho\sigma}$ is an extension of N_ρ and N_σ. Show that it is a proper extension of each of these. Do the same for the other four binary systems. Show that $N_{\rho\sigma}$is a proper extension of $N_{\eta\sigma}$, and that $N_{\rho\tau}$ is a proper extension of $N_{\eta\tau}$. Show that none of the other binary systems is an extension of any other.

In the previous exercise, we found inferences which were valid in only one of the unary systems N_ρ, N_σ and N_τ.

The corresponding inference for N_η is

$\vdash_{N_\eta} \Box p \supset \Diamond p$

$$\begin{gathered}
\neg(\Box p \supset \Diamond p), 0 \\
\Box p, 0 \\
\neg\Diamond p, 0 \\
\Box\neg p, 0 \\
0r1 \\
\neg p, 1 \\
p, 1 \\
\otimes
\end{gathered}$$

The same tableau shows this inference to be invalid in both N_σ and N_τ:

$\nvdash_{N_\sigma} \Box p \supset \Diamond p$

$\nvdash_{N_\tau} \Box p \supset \Diamond p$

$$\begin{gathered}
\neg(\Box p \supset \Diamond p), 0 \\
\Box p, 0 \\
\neg\Diamond p, 0 \\
\Box\neg p, 0
\end{gathered}$$

$$W = N = \{w_0\}$$

$$w_0$$

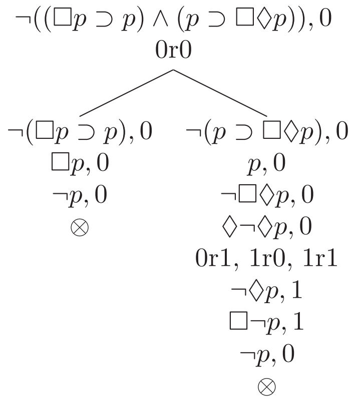
This inference is valid in N_ρ, because N_ρ is an extension of N_η.

We now have characteristic inferences for all four unary systems. Back to the question

"$N_{\rho\sigma}$ is an extension of N_ρ and N_σ. Show that it is a proper extension of each of these. Do the same for the other four binary systems."

The conjunction of the sentences found in the previous exercise will do the trick:

$\vdash_{N_{\rho\sigma}} (\Box p \supset p) \wedge (p \supset \Box\Diamond p)$



The inference is a conjunction of two sentences, both of which have been seen in the last exercise to be invalid in the two unary systems for N. Thus, each of the unary systems will make the conjunction invalid.

A similar inference can be generated for all binary systems, from the conjunction of 'characteristic' sentences found in the last exercise (and above) for its constitutive unary systems. The inference will be valid in the relevant binary system, and invalid in the constitutive unary systems, showing that all the binary systems are proper extensions of their constitutive unary systems. (Binary systems which include η will not be extensions of the same systems with η substituted for ρ - but this is only relevent in the next part of the question.)

"Show that $N_{\rho\sigma}$is a proper extension of $N_{\eta\sigma}$, and that $N_{\rho\tau}$ is a proper extension of $N_{\eta\tau}$."

$\Box p \supset p$ is, as we have seen, a tautology in N_ρ. It is also a tautology in $N_{\rho\sigma}$ because $N_{\rho\sigma}$ is an extension of N_ρ. It is invalid in $N_{\eta\sigma}$, as can be seen in the following tree diagram:

$\nvdash_{N_{\eta\sigma}} \Box p \supset p$

$$\begin{array}{c}
\neg(\Box p \supset p), 0 \\
\Box p, 0 \\
\neg p, 0 \\
0\text{r}1,\ 1\text{r}0 \\
p, 1 \\
1\text{r}2,\ 2\text{r}1 \\
\vdots
\end{array}$$

$$W = \{w_0, w_1, w_2 ...\}; N = \{w_0\}$$

$$w_0Rw_1, w_1Rw_0, w_1Rw_2, w_2Rw_1 ...$$

$$v_{w_0}(p) = 0, v_{w_1}(p) = 1$$

$$\begin{array}{ccccc}
w_0 & \leftrightarrows & \boxed{w_1} & \leftrightarrows & ... \\
\neg p & & p & &
\end{array}$$

Since there is an inference which is valid in $N_{\rho\sigma}$ and invalid in $N_{\eta\sigma}$, $N_{\rho\sigma}$ is a proper extension of $N_{\eta\sigma}$.

■

The same is true of $N_{\rho\tau}$ and $N_{\eta\tau}$:

$\vdash_{N_{\rho\tau}} (\Box p \supset p)$, because $N_{\rho\tau}$ is an extension of N_ρ

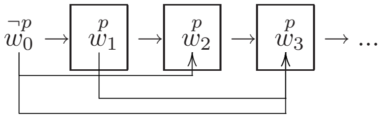
$N_{\eta\tau} \nvdash \Box p \supset p$

$$\begin{array}{c}
\neg(\Box p \supset p), 0 \\
\Box p, 0 \\
\neg p, 0 \\
0\text{r}1 \\
p, 1 \\
1\text{r}2,\ 0\text{r}2 \\
p, 2 \\
2\text{r}3,\ 0\text{r}3,\ 1\text{r}3 \\
\vdots
\end{array}$$

$$W = \{w_0, w_1, w_2, w_3 ...\}; N = \{w_0\}$$

$$w_0Rw_1, w_1Rw_2, w_0Rw_2, w_2Rw_3, w_0R2_3, w_1Rw_3 ...$$

$$v_{w_0}(p) = 0, v_{w_1}(p) = 1, v_{w_2}(p) = 1, v_{w_3}(p) = 1 ...$$



■

Thus the conjunction of the two ‘characteristic’ sentences for the unary systems shows that $N_{\rho\sigma}$is a proper extension of $N_{\eta\sigma}$, and that $N_{\rho\tau}$ is a proper extension of $N_{\eta\tau}$.

Back to the question for the last time:

“Show that none of the other binary systems is an extension of any other.”

There are 5 binary systems for N, $N_{\rho\sigma}$, $N_{\rho\tau}$, $N_{\sigma\tau}$, $N_{\sigma\eta}$, and $N_{\tau\eta}$. We showed that $N_{\rho\sigma}$is a proper extension of $N_{\eta\sigma}$, and that $N_{\rho\tau}$ is a proper extension of $N_{\eta\tau}$ in the last part of the question. Accordingly, if a system is not an extension of $N_{\rho\sigma}$, it is not an extension of $N_{\eta\sigma}$, and if a system is not an extension of $N_{\rho\tau}$, it is not an extension of $N_{\eta\tau}$. Therefore, there are only three systems that need to be shown to be mutually exclusive: $N_{\sigma\tau}$, $N_{\rho\sigma}$, and $N_{\rho\tau}$. To show this, it will suffice to find an inference that is valid in one, but not in the other, for each pair.

$\boxed{N_{\sigma\tau}}$

$\vdash_{N\sigma\tau} \Box\Box p \mathrel{-\!\!3} (\Box q \mathrel{-\!\!3} \Box\Box q)$ (Because this inference is valid in N_τ, as we have seen, and $N_{\sigma\tau}$ is an extension of N_τ)

$\nvdash_{N_{\rho\sigma}} \Box\Box p \mathrel{-\!\!3} (\Box q \mathrel{-\!\!3} \Box\Box q)$

$\nvdash_{N_{\rho\sigma}} \Box(\Box\Box p \supset \Box(\Box q \supset \Box\Box q))$

$$\begin{gathered}
\neg\Box(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 0 \\
0\text{r}0 \\
\Diamond\neg(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 0 \\
0\text{r}1,\ 1\text{r}1,\ 1\text{r}0 \\
\neg(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 1 \\
\Box\Box p, 1 \\
\Box p, 0 \\
p, 0 \\
\neg\Box(\Box q \supset \Box\Box q), 1 \\
\Diamond\neg(\Box q \supset \Box\Box q), 1 \\
\Box p, 1 \\
p, 1 \\
1\text{r}2,\ 2\text{r}2,\ 2\text{r}1 \\
\neg(\Box q \supset \Box\Box q), 2 \\
\Box q, 2 \\
\neg\Box\Box q, 2 \\
q, 1 \\
\Box p, 2 \\
\Diamond\neg\Box q, 2 \\
p, 2 \\
q, 2 \\
2\text{r}3,\ 3\text{r}3,\ 3\text{r}2 \\
\neg\Box q, 3 \\
p, 3 \\
\Diamond\neg q, 3 \\
q, 3
\end{gathered}$$

■

$\vdash_{N_{\sigma\tau}} p \supset \Box\Diamond p$ (Because this inference is valid in N_σ, as we have seen, and $N_{\sigma\tau}$ is an extension of N_σ.)

$\nvdash_{N_{\rho\tau}} p \supset \Box\Diamond p$

$$\begin{array}{c} \neg(p \supset \Box\Diamond p), 0 \\ 0\text{r}0 \\ p, 0 \\ \neg\Box\Diamond p, 0 \\ \Diamond\neg\Diamond p, 0 \\ 0\text{r}1,\ 1\text{r}1 \\ \neg\Diamond p, 1 \\ \Box\neg p, 1 \\ \neg p, 1 \end{array}$$

■

Next,

$\boxed{N_{\rho\sigma}}$

$\vdash_{N_{\rho\sigma}} p \supset \Box\Diamond p$

$\nvdash_{N_{\rho\tau}} p \supset \Box\Diamond p$

$$\begin{array}{c} \neg(p \supset \Box\Diamond p), 0 \\ 0\text{r}0 \\ p, 0 \\ \neg\Box\Diamond p, 0 \\ \Diamond\neg\Diamond p, 0 \\ 0\text{r}1,\ 1\text{r}1 \\ \neg\Diamond p, 1 \\ \Box\neg p, 1 \\ \neg p, 1 \end{array}$$

■

$\vdash_{N_{\rho\sigma}} \Box p \supset p$

$\nvdash_{N_{\sigma\tau}} \Box p \supset p$

$$\begin{array}{c} \neg(\Box p \supset p) \\ \Box p, 0 \\ \neg p, 0 \end{array}$$

■

Finally

$\boxed{N_{\rho\tau}}$

$\vdash_{N_{\rho\tau}} \Box p \supset p$

$\nvdash_{N_{\sigma\tau}} \Box p \supset p$

$$\begin{gathered}
\neg(\Box p \supset p) \\
\Box p, 0 \\
\neg p, 0
\end{gathered}$$

■

$\vdash_{N_{\rho\tau}} \Box\Box p \strictif (\Box q \strictif \Box\Box q)$

$\nvdash_{N_{\rho\sigma}} \Box\Box p \strictif (\Box q \strictif \Box\Box q)$

$\nvdash_{N_{\rho\sigma}} \Box(\Box\Box p \supset \Box(\Box q \supset \Box\Box q))$

$$\begin{gathered}
\neg\Box(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 0 \\
0\mathrm{r}0 \\
\Diamond\neg(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 0 \\
0\mathrm{r}1,\ 1\mathrm{r}1,\ 1\mathrm{r}0 \\
\neg(\Box\Box p \supset \Box(\Box q \supset \Box\Box q)), 1 \\
\Box\Box p, 1 \\
\Box p, 0 \\
p, 0 \\
\neg\Box(\Box q \supset \Box\Box q), 1 \\
\Diamond\neg(\Box q \supset \Box\Box q), 1 \\
\Box p, 1 \\
p, 1 \\
1\mathrm{r}2,\ 2\mathrm{r}2,\ 2\mathrm{r}1 \\
\neg(\Box q \supset \Box\Box q), 2 \\
\Box q, 2 \\
\neg\Box\Box q, 2 \\
q, 1 \\
\Box p, 2 \\
\Diamond\neg\Box q, 2 \\
p, 2 \\
q, 2 \\
2\mathrm{r}3,\ 3\mathrm{r}3,\ 3\mathrm{r}2 \\
\neg\Box q, 3 \\
p, 3 \\
\Diamond\neg q, 3 \\
q, 3
\end{gathered}$$

■

All three systems have been shown to make inferences valid which the other systems do not. Therefore, apart from those that were mentioned at the beginning of the solution, none of the binary systems is an extension of any another.

(d) Combining three (or four) of the conditions, we obtain only the system $N_{\rho\sigma\tau}$. Show that this is a proper extension of each of the binary systems of the last question.

The solution proceeds in an analogous way to those of section (c).

$\vdash_{N_{\rho\sigma\tau}} \Box\Box p \strictif (\Box q \strictif \Box\Box q)$

(Because $N_{\rho\sigma\tau}$ is an extension of N_{τ})

$\nvdash_{N_{\rho\sigma}} \Box\Box p \strictif (\Box q \strictif \Box\Box q)$ (Shown above)

$\blacksquare$

$\vdash_{N_{\rho\sigma\tau}} \Box p \supset p$

(Because $N_{\rho\sigma\tau}$ is an extension of N_{ρ})

$\nvdash_{N_{\sigma\tau}} \Box p \supset p$ (Shown above)

$\blacksquare$

$\vdash_{N_{\rho\sigma\tau}} p \supset \Box\Diamond p$

(Because $N_{\rho\sigma\tau}$ is an extension of N_{σ})

$\nvdash_{N_{\rho\tau}} p \supset \Box\Diamond p$ (Shown above)

$\blacksquare$

$N_{\rho\sigma\tau}$ is an extension of each of the three binary systems. Further, it makes a theorem valid that the three true binary systems make invalid. Therefore it is a proper extension of each of them.

7. Show that $\vdash \Diamond\Diamond(p \wedge \neg p) \vee \Box(q \strictif q)$, in both $S2$ and $S3$, but that neither disjunct is valid in either $S2$ or $S3$.

$\vdash_{N\rho} \Diamond\Diamond(p \wedge \neg p) \vee \Box(q \strictif q)$
$\vdash_{N\rho} \Diamond\Diamond(p \wedge \neg p) \vee \Box\Box(q \supset q)$

$$\begin{array}{c}
\neg(\Diamond\Diamond(p \wedge \neg p) \vee \Box\Box(q \supset q)), 0 \\
0\mathrm{r}0 \\
\neg\Diamond\Diamond(p \wedge \neg p), 0 \\
\neg\Box\Box(q \supset q), 0 \\
\Box\neg\Diamond(p \wedge \neg p), 0 \\
\Diamond\neg\Box(q \supset q), 0 \\
\neg\Diamond(p \wedge \neg p), 0 \\
\Box\neg(p \wedge \neg p), 0 \\
\neg(p \wedge \neg p), 0 \\
0\mathrm{r}1,\ 1\mathrm{r}1 \\
\neg\Box(q \supset q), 1 \\
\Diamond\neg(q \supset q), 1 \\
\neg\Diamond(p \wedge \neg p), 1 \\
\Box\neg(p \wedge \neg p), 1 \\
\neg(p \wedge \neg p), 1 \\
1\mathrm{r}2,\ 2\mathrm{r}2 \\
\neg(q \supset q), 2 \\
q, 2 \\
\neg q, 2 \\
\otimes
\end{array}$$

We have $\vdash_{N\rho} \Diamond\Diamond(p\wedge\neg p)\vee\Box(q\strictif q)$ which implies $\vdash_{N\rho\tau} \Diamond\Diamond(p\wedge\neg p)\vee\Box(q\strictif q)$. Therefore the above is valid in both $S2$ and $S3$.

$\nvdash_{N\rho\tau} \Diamond\Diamond(p \wedge \neg p)$

$$\begin{array}{c}
\neg\Diamond\Diamond(p \wedge \neg p), 0 \\
0r0 \\
\Box\neg\Diamond(p \wedge \neg p), 0 \\
\neg\Diamond(p \wedge \neg p), 0 \\
\Box\neg(p \wedge \neg p), 0 \\
\neg(p \wedge \neg p), 0 \\
\swarrow \quad \searrow \\
\begin{array}{cc}
\neg p, 0 & \neg\neg p, 0 \\
 & p, 0
\end{array}
\end{array}$$

Counter-model such that:

$$W = N = \{w_0\}; w_0 R w_0; v_{w_0}(p) = 1$$

$\nvdash_{N\rho\tau} \Box(q \mathbin{-\!\!\exists} q)$

$\nvdash_{N\rho\tau} \Box\Box(q \supset q)$

$$\begin{array}{c}
\neg\Box\Box(q \supset q), 0 \\
0r0 \\
\Diamond\neg\Box(q \supset q), 0 \\
0r1,\ 1r1 \\
\neg\Box(q \supset q), 1 \\
\Diamond\neg(q \supset q), 1
\end{array}$$

Counter-model such that:

$$W = \{w_0, w_1\}; N = \{w_0\}; w_0 R w_0, w_0 R w_1, w_1 R w_1$$

$\nvdash_{N\rho\tau} \Diamond\Diamond(p \wedge \neg p)$ and $\nvdash_{N\rho\tau} \Box(q \mathbin{-\!\!\exists} q)$, therefore $\nvdash_{N\rho} \Diamond\Diamond(p \wedge \neg p)$ and $\nvdash_{N\rho} \Box(q \mathbin{-\!\!\exists} q)$

Both disjuncts are invalid in $S2$ and $S3$.

8. *Consider an interpretation for N. Call a world *standard* if it is both normal and accesses a non-normal world. A new notion of validity is obtained if we define it in terms of truth preservation at standard worlds. Show that according to this definition of validity, $\Diamond\Diamond A$ is valid. If, in addition, we insist that R be reflexive, or reflexive and transitive, we obtain the non-Lewis systems $S6$ and $S7$, respectively. These are extensions of $S2$ and $S3$, respectively, but, despite the numerology, they are not extensions of $S5$. Design tableau systems for $S6$ and $S7$ and prove them sound and complete.

Under this new notion of validity, an inference is valid iff its truth is preserved at all standard worlds. That is, an inference is invalid iff there is an interpretation which includes a standard world where its premises are made true, and conclusion made false. So, the only relevent worlds for establishing whether an inference is valid or not, are now standard worlds.

Show that $\Diamond\Diamond A$ is valid under this new notion of validity.

Suppose $\Diamond\Diamond A$ were false at standard world w. Because w is standard, it is normal. So in all worlds w is related to, $\Diamond A$ is false. But because w is standard, it accesses a non-normal world, and in that world, by definition,$\Diamond A$ is true. We have a contradiction, showing that the inference is valid.

With regards to tableaux, we are now searching for an interpretation such that w_0 is normal, and is related to a non-normal world. Let w_1 be our non-normal world. In tableaux for $S6$ and $S7$, we will take 1 to be the non-normal world that 0 accesses. The rules are as for $N\rho$, and $N\rho\tau$ respectively, except that in the rules which introduce new worlds, i must now always be greater than 1. Further, there are two additional rules:

Tableaux rule: Add $0r1$ after the premise and negated conclusion.

Closure rule: If a formula $\Box A, 1$ appears on a branch, the branch is closed.

I will now check that the new rules are sound and complete:

Soundness:

The proof is essentially the same as those for the $N\rho$ and $N\rho\tau$ systems, with the following extra steps:

We must add a clause to the definition of faithfulness:

$f(1)$ ϵ W - N

And there is a new case in the Soundness Lemma:

Suppose f is a function which shows I to be faithful to branch section b, and that we apply the $0r1$ rule. Then there is a world w, such that w is non-normal, and w_0Rw. Let f' be the same as f except $f'(1) = w$. f' shows f to be faithful to the extension of the branch.

Completeness:

The proof is essentially the same as those for the $N\rho$ and $N\rho\tau$ systems, but we must check that the interpretation induced is actually a $S6$ or $S7$ interpretation:

Since b is an open complete branch, $0r1$ is on it. By construction and induction hypothesis, we have w_0Rw_1. Further, because of the new closure rule no formula $\Box A, 1$ is on the branch, and so w_1 is non-normal, as required.

9. *Show that $\nvdash_L (\Box p \supset p) \vee \Box q$, but $\vdash_{L\sigma\tau} (\Box p \supset p) \vee \Box q$. Infer that $L\sigma\tau$ is a proper extension of L. By a tableau-theoretic argument, show that $L\rho$ is an extension of $L\sigma\tau$. (Hint: see 4.10.6.) Show that $\nvdash_{L\sigma\tau} \Box p \supset p$, and infer that it is a proper extension.

$\nvdash_L (\Box p \supset p) \vee \Box q$

$$\begin{gathered}
\neg((\Box p \supset p) \vee \Box q), 0 \\
\neg(\Box p \supset p), 0 \\
\neg\Box q, 0 \\
\Box p, 0 \\
\neg p, 0 \\
\Diamond\neg q, 0 \\
0\text{r}1 \\
\neg q, 1 \\
p, 1
\end{gathered}$$

Counter-model such that:

$$W = \{w_0, w_1\}; N = \{w_0\}; w_0Rw_1; v_{w_1}(q) = 0, v_{w_1}(p) = 1$$

$\vdash_{L\sigma\tau} (\Box p \supset p) \vee \Box q$

$$\begin{gathered}
\neg((\Box p \supset p) \vee \Box q), 0 \\
\neg(\Box p \supset p), 0 \\
\neg\Box q, 0 \\
\Box p, 0 \\
\neg p, 0 \\
\Diamond\neg q, 0 \\
0\text{r}1,\ 1\text{r}0,\ 0\text{r}0,\ 1\text{r}1 \\
\neg q, 1 \\
p, 0 \\
\otimes
\end{gathered}$$

Since $L\sigma\tau$ makes an inference valid that L does not, $L\sigma\tau$ is a proper extension of L.

"By a tableau-theoretic argument, show that $L\rho$ is an extension of $L\sigma\tau$. (Hint: see 4.10.6.)"

There is no tableau T such that T is open in $L\rho$, and T is closed in $L\sigma\tau$.

Proof:

Consider a tableau for $L\rho$ in which the rule for σ may be applied. This will have lines of the form $0ri$ and therefore $ir0$, but since the $\Box$ rule is never applied at i (unless $i = 0$, in which cases the rule is redundant because the ρ rule has already been applied), the lines of the form $ir0$ have no effect, and the tableau closes iff it closes without an application of the σ rule. Consider a another tableau for $L\rho$ where the rule for τ may be applied. This will have lines of the form $0ri$, but since no world other than 0 is normal, the $\Diamond$ rule is never applied at i, so we never obtain anything of the form irj. The transitivity rule is never applied, and the tableau closes iff it closed without it. Now consider a tableau for $L\rho$ where both rules may be applied. This will have lines of the form $0ri$ and therefore $ir0$, and so $0r0$ and iri. None of these lines have any further effect.

Therefore, $L\rho$ is an extension of $L\sigma\tau$.

It is a proper extension:

$L\rho \vdash \Box p \supset p$

$$\begin{array}{c} \neg(\Box p \supset p), 0 \\ \text{0r0} \\ \Box p, 0 \\ \neg p, 0 \\ p, 0 \\ \otimes \end{array}$$

$L\sigma\tau \nvdash \Box p \supset p$

$$\begin{array}{c} \neg(\Box p \supset p), 0 \\ \Box p, 0 \\ \neg p, 0 \end{array}$$

10. *What effect does the addition of the constraint η have on L and its other extensions?

$L\eta$ is a proper extension of L:

$\vdash_{L\eta} \Box p \supset \Diamond p$

$$\begin{array}{c}
\neg(\Box p \supset \Diamond p), 0 \\
\Box p, 0 \\
\neg\Diamond p, 0 \\
\Box\neg p, 0 \\
0r1 \\
\neg p, 1 \\
p, 1 \\
\otimes
\end{array}$$

$\nvdash_{L} \Box p \supset \Diamond p$

$$\begin{array}{c}
\neg(\Box p \supset \Diamond p), 0 \\
\Box p, 0 \\
\neg\Diamond p, 0 \\
\Box\neg p, 0
\end{array}$$

$L\rho\eta$ is the same system as $L\rho$, for the same reason that $K\rho\eta$ is the same system as $K\rho$.

$L\sigma\eta$ is the same system as $L\eta$, because $L\sigma$ is the same system as L. (By 4.10.6) Therefore, as shown above, $L\sigma\eta$ is a proper extension of L.

$L\tau\eta$ is the same system as $L\eta$, because $L\tau$ is the same system as L. (By 4.10.6) Therefore, as shown above, $L\sigma\eta$ is a proper extension of L.

$L\rho\sigma\eta$ is the same system as Ł$\rho\eta$ (By 4.10.6), and hence the same system as $L\rho$.

$L\rho\tau\eta$ is the same system as Ł$\rho\eta$ (By 4.10.6), and hence the same system as $L\rho$.

$L\sigma\tau\eta$ is a proper extension of $L\sigma\tau$:

$\vdash_{L\sigma\tau\eta} \Box p \supset \Diamond p$

$$\begin{gathered} \neg(\Box p \supset \Diamond p), 0 \\ \Box p, 0 \\ \neg\Diamond p, 0 \\ \Box\neg p, 0 \\ \text{0r1, 1r0, 0r0, 1r1} \\ \neg p, 1 \\ p, 1 \\ \otimes \end{gathered}$$

$\nvdash_{L\sigma\tau} \Box p \supset \Diamond p$

$$\begin{gathered} \neg(\Box p \supset \Diamond p), 0 \\ \Box p, 0 \\ \neg\Diamond p, 0 \\ \Box\neg p, 0 \end{gathered}$$

$L\rho$ is an extension of $L\sigma\tau\eta$ because it is an extension of $L\sigma\tau$ and $L\eta$. It is a proper extension:

$\vdash_{L\rho} \Box p \supset p$

$$\begin{array}{c} \neg(\Box p \supset p), 0 \\ \text{0r0} \\ \Box p, 0 \\ \neg p, 0 \\ p, 0 \\ \otimes \end{array}$$

$\nvdash_{L\sigma\tau\eta} \Box p \supset p$

$$\begin{array}{c} \neg(\Box p \supset p), 0 \\ \Box p, 0 \\ \neg p, 0 \\ \text{0r1, 1r0} \\ p, 1 \\ \text{1r2, 2r1, 0r2, 2r0} \\ \vdots \end{array}$$

$L\rho\sigma\tau\eta$ is the same system as $L\rho$ by 4.10.6 and the fact that $L\rho\eta$ is the same system as $L\rho$, found above.