10. By constructing suitable tableaux, determine whether the following are valid in K^t. Where the inference is invalid, specify a countermodel.

(a) $\vdash [F](p \supset q) \supset ([F]p \supset [F]q)$

$$\begin{array}{c}
\neg([F](p \supset q) \supset ([F]p \supset [F]q)), 0 \\
[F](p \supset q), 0 \\
\neg([F]p \supset [F]q), 0 \\
[F]p, 0 \\
\neg[F]q, 0 \\
\langle F\rangle\neg q, 0 \\
0\mathrm{r}1 \\
\neg q, 1 \\
p, 1 \\
p \supset q, 1 \\
\begin{array}{cc}
\neg p, 1 & q, 1 \\
\otimes & \otimes
\end{array}
\end{array}$$

(b) $\vdash \langle F\rangle p \equiv \neg[F]\neg p$

$$\begin{array}{c}
\neg(\langle F\rangle p \equiv \neg[F]\neg p), 0 \\
\begin{array}{cc}
\langle F\rangle p, 0 & \neg\langle F\rangle p, 0 \\
\neg\neg[F]\neg p, 0 & \neg[F]\neg p, 0 \\
[F]\neg p, 0 & [F]\neg p, 0 \\
0\mathrm{r}1 & \langle F\rangle\neg\neg p, 0 \\
p, 1 & 0\mathrm{r}1 \\
\neg p, 1 & \neg\neg p, 1 \\
\otimes & \neg p, 1 \\
 & \otimes
\end{array}
\end{array}$$

(c) $\vdash p \supset [F]\langle P\rangle p$

$$\begin{array}{c}
\neg(p \supset [F]\langle P\rangle p), 0 \\
p, 0 \\
\neg[F]\langle P\rangle p, 0 \\
\langle F\rangle\neg\langle P\rangle p, 0 \\
0\mathrm{r}1 \\
\neg\langle P\rangle p, 1 \\
[P]\neg p, 1 \\
\neg p, 0 \\
\otimes
\end{array}$$

(d) $[F]p \supset [F][F]p \nvdash [P]p \supset [P][P]p$

$$\begin{array}{c}
[F]p \supset [F][F]p, 0 \\
\neg([P]p \supset [P][P]p), 0 \\
[P]p, 0 \\
\neg[P][P]p, 0 \\
\langle P\rangle\neg[P]p, 0 \\
1\mathrm{r}0 \\
\neg[P]p, 1 \\
\langle P\rangle\neg p, 1 \\
p, 1 \\
2\mathrm{r}1 \\
\neg p, 2 \\
\swarrow \qquad \searrow \\
\begin{array}{cc}
\begin{array}{c}
\neg[F]p, 0 \\
\langle F\rangle\neg p, 0 \\
0\mathrm{r}3 \\
\neg p, 3
\end{array} &
\begin{array}{c}
[F][F]p, 0 \\
\\
\\
\\
\end{array}
\end{array}
\end{array}$$

The following interpretation, taken from the left branch, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, w_3\}$$

$$w_1 R w_0, w_2 R w_1, w_0 R w_3$$

$$v_{w_1}(p) = 1, v_{w_2}(p) = 0, v_{w_3}(p) = 0$$

This can be represented in the following diagram:

$$\overset{\neg p}{w_2} \rightarrow \overset{p}{w_1} \rightarrow \ w_0 \ \rightarrow \overset{\neg p}{w_3}$$

(e) $\vdash [F](p \supset q) \supset (\langle F\rangle p \supset \langle F\rangle q)$

$$\begin{array}{c}
\neg([F](p \supset q) \supset (\langle F\rangle p \supset \langle F\rangle q)), 0\\
[F](p \supset q), 0\\
\neg(\langle F\rangle p \supset \langle F\rangle q), 0\\
\langle F\rangle p, 0\\
\neg\langle F\rangle q, 0\\
[F]\neg q, 0\\
0\mathrm{r}1\\
p, 1\\
\neg q, 1\\
p \supset q, 1\\
\begin{array}{cc}
\neg p, 1 & q, 1\\
\otimes & \otimes
\end{array}
\end{array}$$

(f) $\langle F\rangle\langle F\rangle p \supset \langle F\rangle p \nvdash \langle P\rangle\langle P\rangle p \supset \langle P\rangle p$

$$\begin{array}{c}
\langle F\rangle\langle F\rangle p \supset \langle F\rangle p, 0\\
\neg(\langle P\rangle\langle P\rangle p \supset \langle P\rangle p), 0\\
\langle P\rangle\langle P\rangle p, 0\\
\neg\langle P\rangle p, 0\\
[P]\neg p, 0\\
1\mathrm{r}0\\
\langle P\rangle p, 1\\
\neg p, 1\\
2\mathrm{r}1\\
p, 2\\
\begin{array}{cc}
\neg\langle F\rangle\langle F\rangle p, 0 & \langle F\rangle p, 0\\
[F]\neg\langle F\rangle p & 0\mathrm{r}3\\
 & p
\end{array}
\end{array}$$

The following interpretation, taken from the right branch, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, w_3\}$$

$$w_1 R w_0, w_2 R w_1, w_0 R w_3$$

$$v_{w_1}(p) = 0, v_{w_2}(p) = 1, v_{w_3}(p) = 1$$

This can be represented in the following diagram:

$$\overset{p}{w_2} \rightarrow \overset{\neg p}{w_1} \rightarrow \; w_0 \; \rightarrow \overset{p}{w_3}$$

(g) $\vdash ([P]p \vee [P]q) \supset [P](p \vee q)$

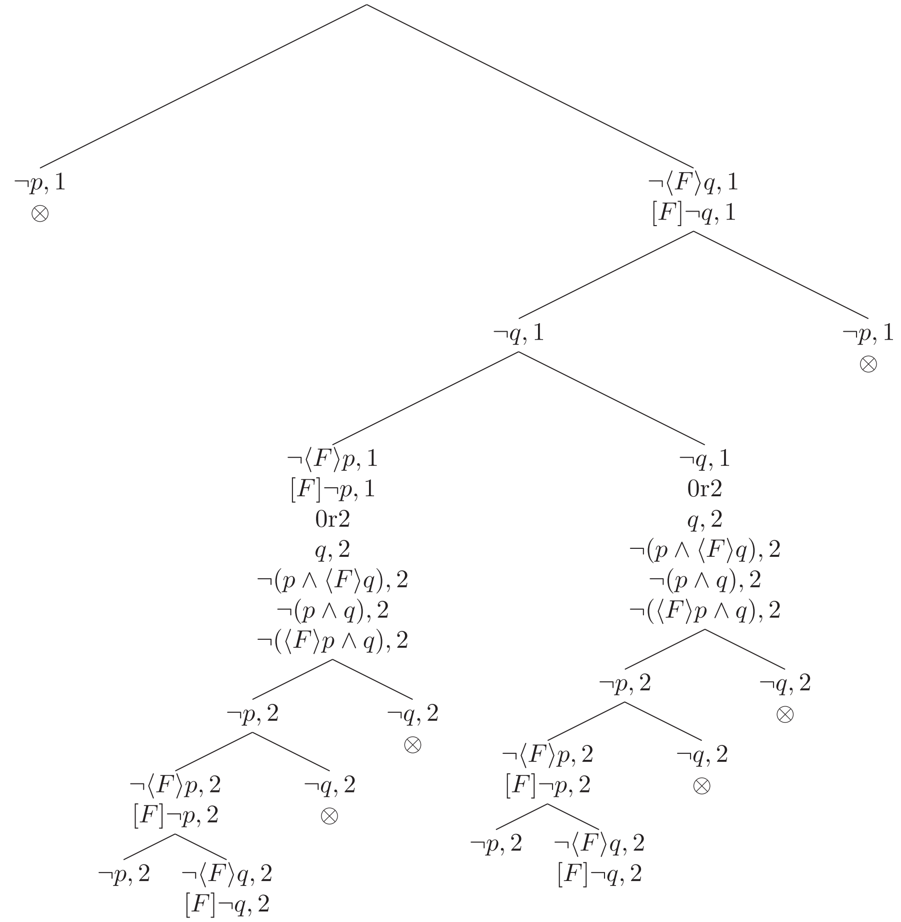
$$\begin{array}{c}
\neg(([P]p \vee [P]q) \supset [P](p \vee q)), 0 \\
[P]p \vee [P]q, 0 \\
\neg[P](p \vee q), 0 \\
\langle P\rangle\neg(p \vee q), 0 \\
1r0 \\
\neg(p \vee q), 1 \\
\neg p, 1 \\
\neg q, 1 \\
\begin{array}{cc}
[P]p, 0 & [P]q, 0 \\
p, 1 & q, 1 \\
\otimes & \otimes
\end{array}
\end{array}$$

(h) $\vdash \langle P\rangle(p \wedge q) \supset (((\langle P\rangle p \wedge \langle P\rangle q)$

$$\begin{array}{c}
\neg(\langle P\rangle(p \wedge q) \supset (((\langle P\rangle p \wedge \langle P\rangle q)), 0 \\
\langle P\rangle(p \wedge q), 0 \\
\neg(\langle P\rangle p \wedge \langle P\rangle q), 0 \\
1r0 \\
p \wedge q, 1 \\
p, 1 \\
q, 1 \\
\begin{array}{cc}
\neg\langle P\rangle p, 0 & \neg\langle P\rangle q, 0 \\
[P]\neg p, 0 & [P]\neg q, 0 \\
\neg p, 1 & \neg q, 1 \\
\otimes & \otimes
\end{array}
\end{array}$$

(i) $\nvdash (\langle F\rangle p \wedge \langle F\rangle q) \supset (((\langle F\rangle(p \wedge \langle F\rangle q)) \vee \langle F\rangle(p \wedge q) \vee ((\langle F\rangle((\langle F\rangle p \wedge q)))$

$$\neg(((\langle F\rangle p \wedge \langle F\rangle q) \supset (((\langle F\rangle(p \wedge \langle F\rangle q)) \vee \langle F\rangle(p \wedge q) \vee ((\langle F\rangle((\langle F\rangle p \wedge q)))), 0$$
$$\langle F\rangle p \wedge \langle F\rangle q, 0$$
$$\neg(((\langle F\rangle(p \wedge \langle F\rangle q)) \vee \langle F\rangle(p \wedge q) \vee ((\langle F\rangle((\langle F\rangle p \wedge q))), 0$$
$$\neg\langle F\rangle(p \wedge \langle F\rangle q), 0$$
$$\neg\langle F\rangle(p \wedge q), 0$$
$$\neg\langle F\rangle((\langle F\rangle p \wedge q), 0$$
$$[F]\neg(p \wedge \langle F\rangle q), 0$$
$$[F]\neg(p \wedge q), 0$$
$$[F]\neg((\langle F\rangle p \wedge q), 0$$
$$\langle F\rangle p, 0$$
$$\langle F\rangle q, 0$$
$$0\mathrm{r}1$$
$$p, 1$$
$$\neg(p \wedge \langle F\rangle q), 1$$
$$\neg(p \wedge q), 1$$
$$\neg((\langle F\rangle p \wedge q), 1$$



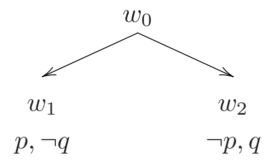
The following interpretation, taken from the left-most open branch, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, \}$$

$$w_0 R w_1, w_0 R w_2$$

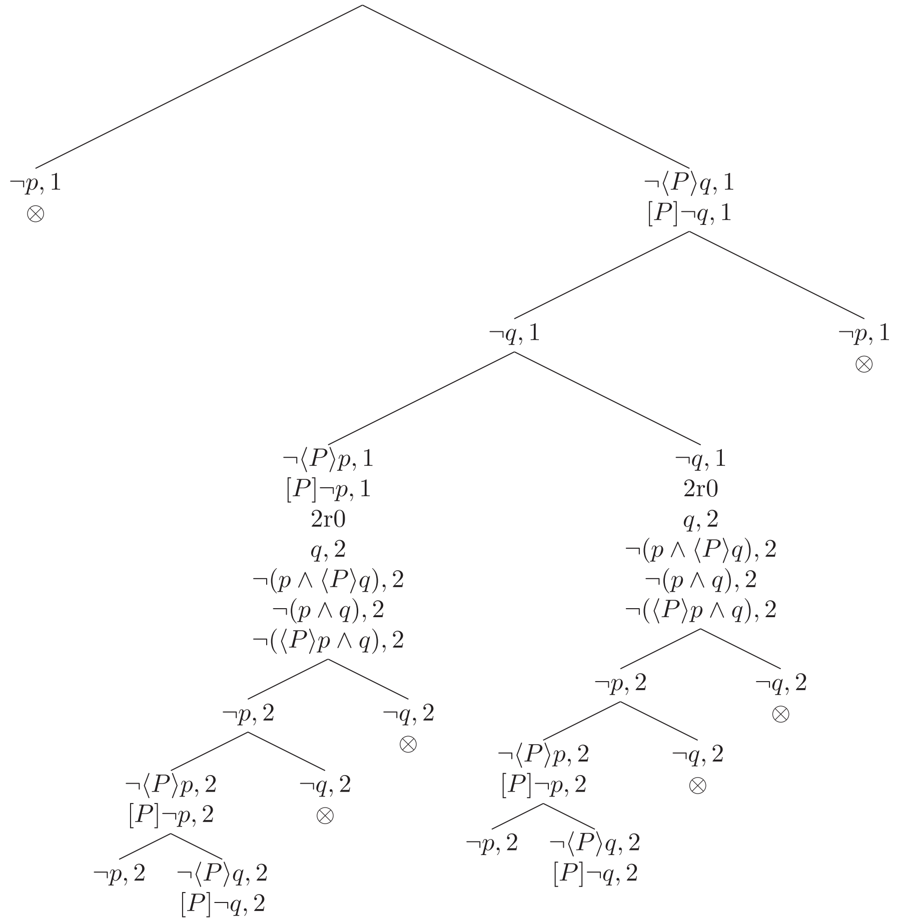
$$v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_2}(p) = 0, v_{w_2}(q) = 1,$$

This can be represented in the following diagram:



(j) $\nvdash (\langle P\rangle p \wedge \langle P\rangle q) \supset (((\langle P\rangle(p \wedge \langle P\rangle q)) \vee \langle P\rangle(p \wedge q) \vee ((\langle P\rangle(\langle P\rangle p \wedge q)))$

$$\neg(((\langle P\rangle p \wedge \langle P\rangle q) \supset (((\langle P\rangle(p \wedge \langle P\rangle q)) \vee \langle P\rangle(p \wedge q) \vee ((\langle P\rangle(\langle P\rangle p \wedge q)))), 0$$
$$\langle P\rangle p \wedge \langle P\rangle q, 0$$
$$\neg(((\langle P\rangle(p \wedge \langle P\rangle q)) \vee \langle P\rangle(p \wedge q) \vee ((\langle P\rangle(\langle P\rangle p \wedge q))), 0$$
$$\neg\langle P\rangle(p \wedge \langle P\rangle q), 0$$
$$\neg\langle P\rangle(p \wedge q), 0$$
$$\neg\langle P\rangle(\langle P\rangle p \wedge q), 0$$
$$[P]\neg(p \wedge \langle P\rangle q), 0$$
$$[P]\neg(p \wedge q), 0$$
$$[P]\neg(\langle P\rangle p \wedge q), 0$$
$$\langle P\rangle p, 0$$
$$\langle P\rangle q, 0$$
$$1\text{r}0$$
$$p, 1$$
$$\neg(p \wedge \langle P\rangle q), 1$$
$$\neg(p \wedge q), 1$$
$$\neg(\langle P\rangle p \wedge q), 1$$



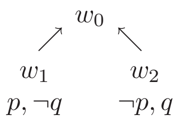
The following interpretation, taken from the left-most open branch, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, \}$$

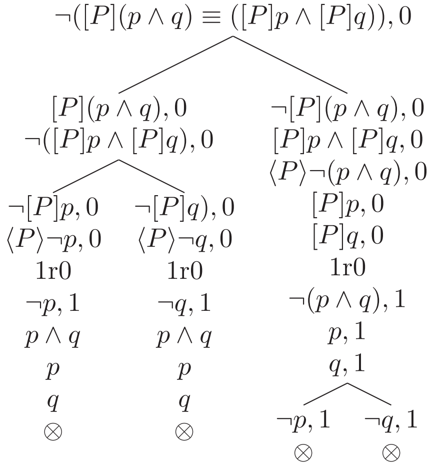
$$w_1 R w_0, w_2 R w_0$$

$$v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_2}(p) = 0, v_{w_2}(q) = 1,$$

This can be represented in the following diagram:



(k) $\vdash [P](p \wedge q) \equiv ([P]p \wedge [P]q)$



(l) $\nvdash [P]p \supset \langle P\rangle p$

$$\neg([P]p \supset \langle P\rangle p), 0$$
$$[P]p, 0$$
$$\neg\langle P\rangle p, 0$$
$$[P]\neg p, 0$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0\}$$

This can be represented in the following diagram:

$$w_0$$

(m) $\nvdash (p \wedge [P]p) \supset \langle F\rangle[P]p$

$$\begin{gathered}
\neg((p \wedge [P]p) \supset \langle F\rangle[P]p), 0 \\
p \wedge [P]p, 0 \\
\neg\langle F\rangle[P]p), 0 \\
p, 0 \\
[P]p, 0 \\
[F]\neg[P]p, 0
\end{gathered}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0\}$$

$$v_{w_0}(p) = 1$$

This can be represented in the following diagram:

$$\begin{gathered}
w_0 \\
p
\end{gathered}$$

(n) $\vdash \langle P\rangle[F]p \supset p$

$$\begin{gathered}
\neg(\langle P\rangle[F]p \supset p), 0 \\
\langle P\rangle[F]p, 0 \\
\neg p, 0 \\
1r0 \\
[F]p, 1 \\
p, 0 \\
\otimes
\end{gathered}$$

(o) $\nvdash [P]([P]p \supset p) \supset [P]p$

$$\begin{array}{c}
\neg([P]([P]p \supset p) \supset [P]p), 0 \\
[P]([P]p \supset p), 0 \\
\neg[P]p, 0 \\
\langle P\rangle\neg p, 0 \\
1r0 \\
\neg p, 1 \\
[P]p \supset p, 1 \\
\begin{array}{cc}
\neg[P]p, 1 & p, 1 \\
\langle P\rangle\neg p, 1 & \otimes \\
2r1 & \\
\neg p, 2 &
\end{array}
\end{array}$$

The following interpretation, not taken from the tree, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_1 R w_0, w_2 R w_1$$

$$v_{w_1}(p) = 0, v_{w_2}(p) = 0$$

This can be represented in the following diagram:

$$\overset{\neg p}{w_2} \rightarrow \overset{\neg p}{w_1} \rightarrow \ w_0$$

(p) $\nvdash \langle P\rangle[P]p \supset [P]\langle P\rangle p$

$$\begin{array}{c}
\neg(\langle P\rangle[P]p \supset [P]\langle P\rangle p), 0 \\
\langle P\rangle[P]p, 0 \\
\neg[P]\langle P\rangle p, 0 \\
\langle P\rangle\neg\langle P\rangle p \\
1r0 \\
[P]p, 1 \\
2r0 \\
\neg\langle P\rangle p, 2 \\
[P]\neg p, 2
\end{array}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_1 R w_0, w_2 R w_0$$

This can be represented in the following diagram:

$$\begin{tikzcd}
 & w_0 & \\
w_1 \arrow[ur] & & w_2 \arrow[ul]
\end{tikzcd}$$

(q) $\nvdash \langle F\rangle[F]p \supset p$

$$\neg(\langle F\rangle[F]p \supset p), 0$$
$$\langle F\rangle[F]p, 0$$
$$\neg p, 0$$
$$0\text{r}1$$
$$[F]p, 1$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

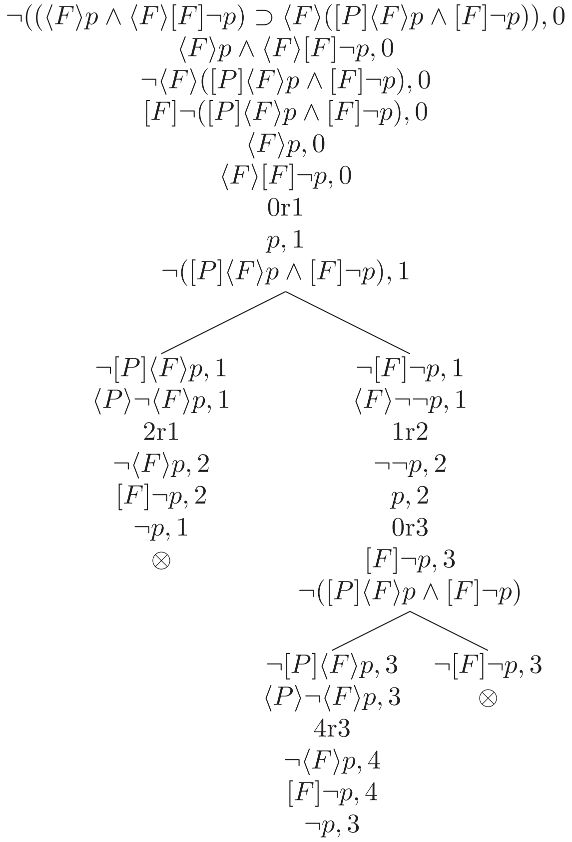
$$w_0 R w_1$$

$$v_{w_0}(p) = 0$$

This can be represented in the following diagram:

$$\begin{tikzcd}
\overset{\neg p}{w_0} \arrow[r] & w_1
\end{tikzcd}$$

(r) $\nvdash (\langle F\rangle p \wedge \langle F\rangle[F]\neg p) \supset \langle F\rangle([P]\langle F\rangle p \wedge [F]\neg p)$



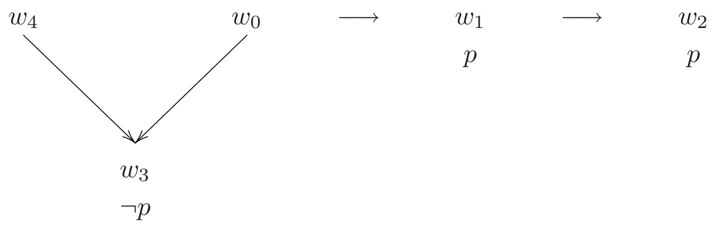
The interpretation taken from the open middle branch shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, w_3, w_4\}$$

$$w_0Rw_1, w_1Rw_2, w_0Rw_3, w_4Rw_3$$

$$v_{w_1}(p) = 1, v_{w_2}(p) = 1, v_{w_3}(p) = 0$$

This can be represented in the following diagram:



11. In the previous question, if the inference is invalid, repeat the question in K^t_τ, K^t_δ and K^t_ϕ

(d) $[F]p \supset [F][F]p \vdash_{K^t_\tau} [P]p \supset [P][P]p$

$[F]p \supset [F][F]p, 0$
$\neg([P]p \supset [P][P]p), 0$
$[P]p, 0$
$\neg[P][P]p, 0$
$\langle P\rangle\neg[P]p, 0$
1r0
$\neg[P]p, 1$
$\langle P\rangle\neg p, 1$
$p, 1$
2r1, 2r0
$\neg p, 2$
$p, 2$
$\otimes$

(d) $[F]p \supset [F][F]p \nvdash_{K^t_\delta} [P]p \supset [P][P]p$

$$\begin{array}{c}
[F]p \supset [F][F]p, 0\\
\neg([P]p \supset [P][P]p), 0\\
[P]p, 0\\
\neg[P][P]p, 0\\
\langle P\rangle\neg[P]p, 0\\
1\text{r}0\\
\neg[P]p, 1\\
\langle P\rangle\neg p, 1\\
p, 1\\
2\text{r}0,\ 1\text{r}2\\
p, 2\\
3\text{r}1\\
\neg p, 3\\
4\text{r}1,\ 3\text{r}4\\
\begin{array}{cc}
\neg[F]p, 0 & [F][F]p, 0\\
\langle F\rangle\neg p, 0 & 3\text{r}0,\ 2\text{r}3\\
0\text{r}5 & \vdots\\
\neg p, 5 & \\
3\text{r}0,\ 2\text{r}3 & \\
\vdots &
\end{array}
\end{array}$$

The following finite interpretation, not taken from the tree, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_1Rw_1, w_2Rw_2, w_2Rw_1, w_1Rw_0$$

$$v_{w_1}(p) = 1, v_{w_2}(p) = 0$$

This can be represented in the following diagram:

$$\begin{array}{ccccc}
\overset{\frown}{w}_2 & \rightarrow & \overset{\frown}{w}_1 & \rightarrow & w_0\\
\neg p & & p & &
\end{array}$$

Note that the idea of a world being related to itself is slightly counterintuitive in tense logic — if a world is related to itself, the point in time it represents is both before and after itself.

Using this feature of reflexive worlds I have constructed the above finite interpretation that satisfies the ‘denseness’ condition of K_δ. The K_δ condition specifies, if w_i is related to w_j then w_i is related to some w_k, and w_k is related to w_j. This is satisfied because i, j, and k do not have to be distinct. In the case above, as w_2 is related to w_1, there must be a world that w_2 is related to, which is related to w_1. This world is w_2 once more: w_2Rw_2 (w_iRw_k) and w_2Rw_1 (w_kRw_j). If we think of the denseness condition as specifying that all related worlds must have a world between them, w_2 is the world between w_2 and w_1, and w_1 is the world between w_1 and w_0.

Let us check that the interpretation works:

The premise $[F]p \supset [F][F]p$ is true at w_0, because w_0 is not related to any world.

$[P]p$ is true at w_0 because at all worlds related to w_0, p is true.

$[P][P]p$ is false at w_0 because there is a world related to a world related to w_0, that is, w_2, and p is false there.

Therefore the conclusion, $[P]p \supset [P][P]p$, is false at w_0, as required.

(d) $[F]p \supset [F][F]p \nvdash_{K^t_\phi} [P]p \supset [P][P]p$

$$\begin{array}{c}
[F]p \supset [F][F]p, 0 \\
\neg([P]p \supset [P][P]p), 0 \\
[P]p, 0 \\
\neg[P][P]p, 0 \\
\langle P\rangle\neg[P]p, 0 \\
1\text{r}0 \\
\neg[P]p, 1 \\
\langle P\rangle\neg p, 1 \\
p, 1 \\
2\text{r}1 \\
\neg p, 2 \\
\swarrow \qquad \searrow \\
\begin{array}{cc}
\begin{array}{c} \neg[F]p, 0 \\ \langle F\rangle\neg p, 0 \\ 0\text{r}3 \\ \neg p, 3 \end{array} &
\begin{array}{c} [F][F]p, 0 \\ \\ \\ \\ \end{array}
\end{array}
\end{array}$$

The following interpretation, taken from the left branch, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, w_3\}$$

$$w_1Rw_0, w_2Rw_1, w_0Rw_3$$

$$v_{w_1}(p) = 1, v_{w_2}(p) = 0, v_{w_3}(p) = 0$$

This can be represented in the following diagram:

$$\begin{array}{ccccccc} w_2 & \rightarrow & w_1 & \rightarrow & w_0 & \rightarrow & w_3 \\ \neg p & & p & & & & \neg p \end{array}$$

(f) $\langle F\rangle\langle F\rangle p \supset \langle F\rangle p \vdash_{K^t_\tau} \langle P\rangle\langle P\rangle p \supset \langle P\rangle p$

$$\begin{array}{c}
\langle F\rangle\langle F\rangle p \supset \langle F\rangle p, 0 \\
\neg(\langle P\rangle\langle P\rangle p \supset \langle P\rangle p), 0 \\
\langle P\rangle\langle P\rangle p, 0 \\
\neg\langle P\rangle p, 0 \\
[P]\neg p, 0 \\
1\text{r}0 \\
\langle P\rangle p, 1 \\
\neg p, 1 \\
2\text{r}1,\ 2\text{r}0 \\
p, 2 \\
\neg p, 2 \\
\otimes
\end{array}$$

(f) $\langle F\rangle\langle F\rangle p \supset \langle F\rangle p \vdash_{K^t_\delta} \langle P\rangle\langle P\rangle p \supset \langle P\rangle p$

$$\begin{array}{c}
\langle F\rangle\langle F\rangle p \supset \langle F\rangle p, 0 \\
\neg(\langle P\rangle\langle P\rangle p \supset \langle P\rangle p), 0 \\
\langle P\rangle\langle P\rangle p, 0 \\
\neg\langle P\rangle p, 0 \\
[P]\neg p, 0 \\
1\text{r}0 \\
\langle P\rangle p, 1 \\
\neg p, 1 \\
2\text{r}1 \\
p, 2
\end{array}$$

$$\begin{array}{cc}
\swarrow & \searrow \\
\neg\langle F\rangle\langle F\rangle p, 0 & \langle F\rangle p, 0 \\
[F]\neg\langle F\rangle p, 0 & 0\text{r}3 \\
1\text{r}3,\ 3\text{r}0,\ 2\text{r}4,\ 4\text{r}1 & p, 3 \\
\vdots & 1\text{r}4,\ 4\text{r}0,\ 2\text{r}5,\ 5\text{r}1,\ 3\text{r}6,\ 6\text{r}0 \\
 & \vdots
\end{array}$$

The following interpretation, not taken from the tree, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_1Rw_1, w_2Rw_2, w_2Rw_1, w_1Rw_0$$

$$v_{w_1}(p) = 0, v_{w_2}(p) = 1$$

This can be represented in the following diagram:

$$\begin{array}{ccccc} \overset{\curvearrowright}{w_2} & \rightarrow & \overset{\curvearrowright}{w_1} & \rightarrow & w_0 \\ p & & \neg p & & \end{array}$$

For an explanation of why this kind of interpretation works, see the solution to (d) for K^t_δ.

Let us check that it works:

The premise $\langle F\rangle\langle F\rangle p \supset \langle F\rangle p$ is true at w_0, because the antecedent is false at w_0; w_0 is related to no future worlds.

$\langle P\rangle\langle P\rangle p$ is true at w_0, because there is a world, related to a world, related to w_0 where p is true, that is, w_2.

$\langle P\rangle p$ is false at w_0, because in all worlds related to w_0, p is false.

Therefore, the conclusion, $\langle P\rangle\langle P\rangle p \supset \langle P\rangle p$, is false at w_0, as required.

(f) $\langle F\rangle\langle F\rangle p \supset \langle F\rangle p \nvdash_{K^t_\phi} \langle P\rangle\langle P\rangle p \supset \langle P\rangle p$

$$\begin{array}{c} \langle F\rangle\langle F\rangle p \supset \langle F\rangle p, 0 \\ \neg(\langle P\rangle\langle P\rangle p \supset \langle P\rangle p), 0 \\ \langle P\rangle\langle P\rangle p, 0 \\ \neg\langle P\rangle p, 0 \\ [P]\neg p, 0 \\ 1r0 \\ \langle P\rangle p, 1 \\ \neg p, 1 \\ 2r1 \\ p, 2 \end{array}$$

$$\begin{array}{cc} \neg\langle F\rangle\langle F\rangle p, 0 & \langle F\rangle p, 0 \\ [F]\neg\langle F\rangle p & 0r3 \\ & p \end{array}$$

The following interpretation from the right-hand branch shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, w_3\}$$

$$w_1 R w_0, w_2 R w_1, w_0 R w_3$$

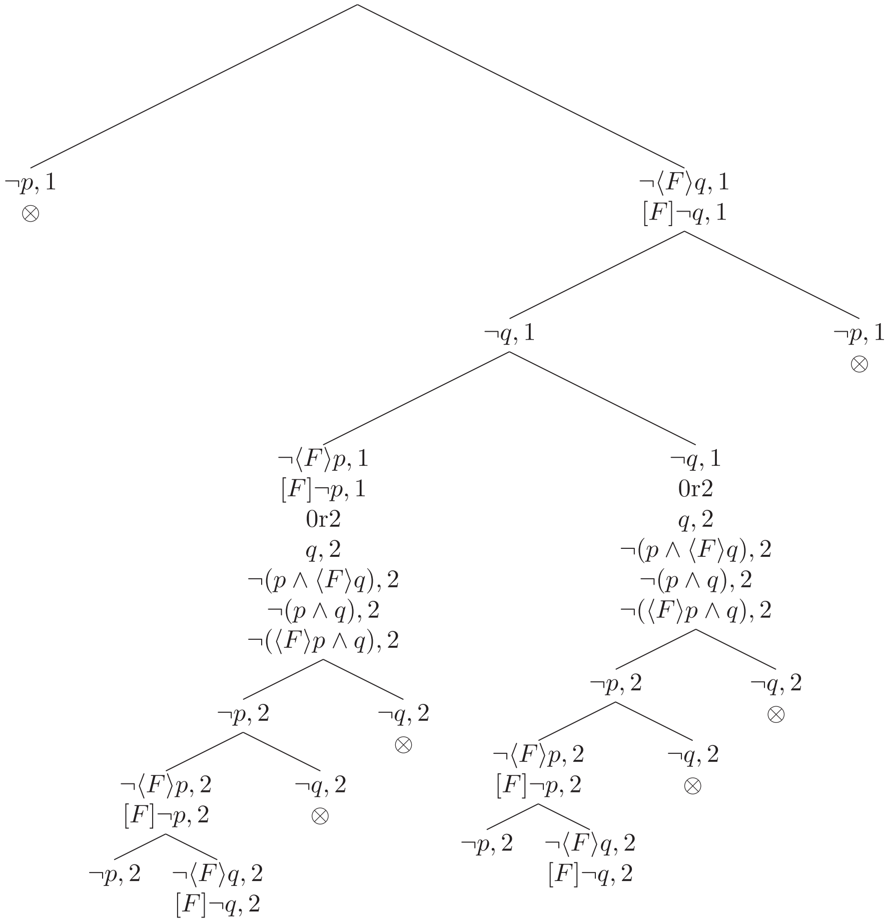
$$v_{w_1}(p) = 0, v_{w_2}(p) = 1, v_{w_3}(p) = 1$$

This can be represented in the following diagram:

$$\begin{tikzcd}
w_2 \arrow[r] & w_1 \arrow[r] & w_0 \arrow[r] & w_3 \\
p & \neg p & & p
\end{tikzcd}$$

(i) $\nvDash_{K^t_\tau} (\langle F\rangle p \wedge \langle F\rangle q) \supset (((\langle F\rangle (p \wedge \langle F\rangle q)) \vee \langle F\rangle (p \wedge q) \vee ((\langle F\rangle (\langle F\rangle p \wedge q)))$

$$\neg(((\langle F\rangle p \wedge \langle F\rangle q) \supset (((\langle F\rangle (p \wedge \langle F\rangle q)) \vee \langle F\rangle (p \wedge q) \vee ((\langle F\rangle (\langle F\rangle p \wedge q)))), 0$$
$$\langle F\rangle p \wedge \langle F\rangle q, 0$$
$$\neg(((\langle F\rangle (p \wedge \langle F\rangle q)) \vee \langle F\rangle (p \wedge q) \vee ((\langle F\rangle (\langle F\rangle p \wedge q))), 0$$
$$\neg\langle F\rangle (p \wedge \langle F\rangle q), 0$$
$$\neg\langle F\rangle (p \wedge q), 0$$
$$\neg\langle F\rangle (\langle F\rangle p \wedge q), 0$$
$$[F]\neg(p \wedge \langle F\rangle q), 0$$
$$[F]\neg(p \wedge q), 0$$
$$[F]\neg(\langle F\rangle p \wedge q), 0$$
$$\langle F\rangle p, 0$$
$$\langle F\rangle q, 0$$
$$0\mathrm{r}1$$
$$p, 1$$
$$\neg(p \wedge \langle F\rangle q), 1$$
$$\neg(p \wedge q), 1$$
$$\neg(\langle F\rangle p \wedge q), 1$$



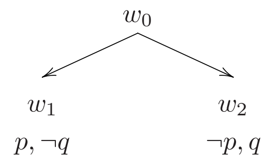
The following interpretation, taken from the left-most open branch, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, \}$$

$$w_0 R w_1, w_0 R w_2$$

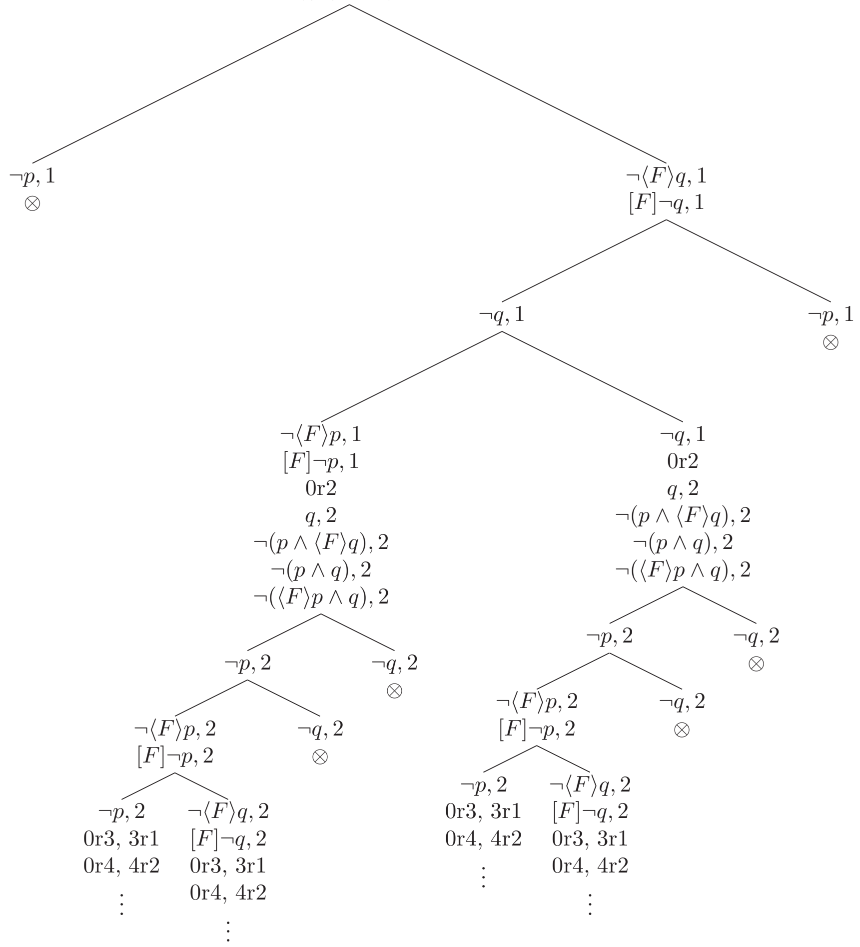
$$v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_2}(p) = 0, v_{w_2}(q) = 1,$$

This can be represented in the following diagram:



(i) $\nvdash_{K^t_\delta} (\langle F\rangle p \wedge \langle F\rangle q) \supset (((\langle F\rangle(p \wedge \langle F\rangle q)) \vee \langle F\rangle(p \wedge q) \vee ((\langle F\rangle(\langle F\rangle p \wedge q)))$

$$\neg((\langle F\rangle p \wedge \langle F\rangle q) \supset (((\langle F\rangle(p \wedge \langle F\rangle q)) \vee \langle F\rangle(p \wedge q) \vee ((\langle F\rangle(\langle F\rangle p \wedge q)))), 0$$
$$\langle F\rangle p \wedge \langle F\rangle q, 0$$
$$\neg(((\langle F\rangle(p \wedge \langle F\rangle q)) \vee \langle F\rangle(p \wedge q) \vee ((\langle F\rangle(\langle F\rangle p \wedge q))), 0$$
$$\neg\langle F\rangle(p \wedge \langle F\rangle q), 0$$
$$\neg\langle F\rangle(p \wedge q), 0$$
$$\neg\langle F\rangle(\langle F\rangle p \wedge q), 0$$
$$[F]\neg(p \wedge \langle F\rangle q), 0$$
$$[F]\neg(p \wedge q), 0$$
$$[F]\neg(\langle F\rangle p \wedge q), 0$$
$$\langle F\rangle p, 0$$
$$\langle F\rangle q, 0$$
$$0\mathrm{r}1$$
$$p, 1$$
$$\neg(p \wedge \langle F\rangle q), 1$$
$$\neg(p \wedge q), 1$$
$$\neg(\langle F\rangle p \wedge q), 1$$



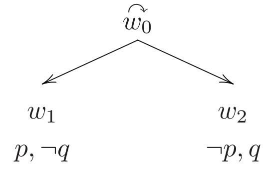
The following interpretation, not taken from the tree shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_0, w_0 R w_1, w_0 R w_2$$

$$v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_2}(p) = 0, v_{w_2}(q) = 1,$$

This can be represented in the following diagram:



For more on why this kind of counter-model satisfies the denseness condition of K^t_δ, check the solution to (d) for K^t_δ.

Let us check that this counter-model works.

$\langle F \rangle p$ is true at w_0 because w_0 is related to w_1, and p is true there.

$\langle F \rangle q$ is true at w_0 because w_0 is related to w_2, and q is true there.

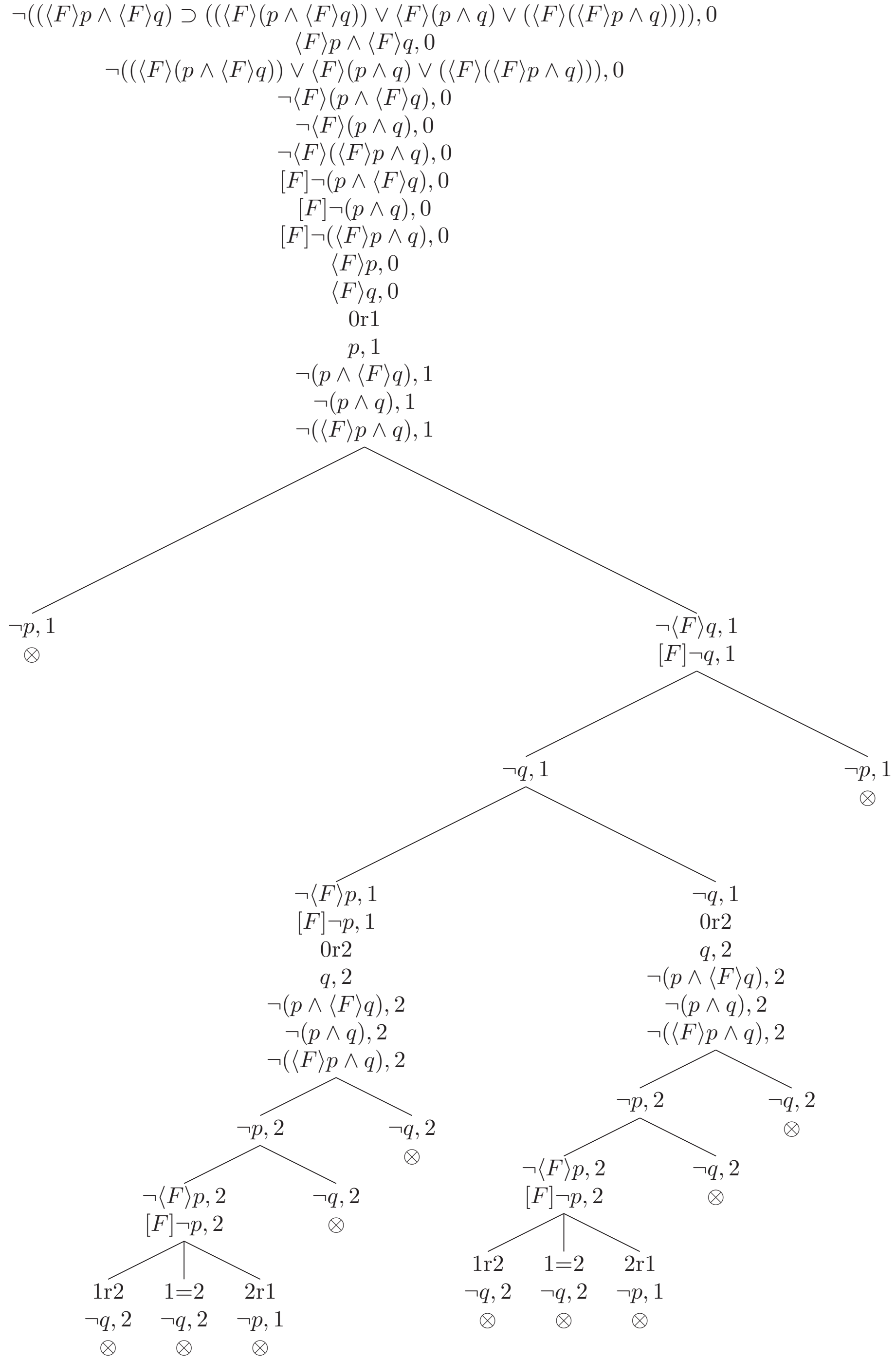
Thus, the antecedent $(\langle F \rangle p \wedge \langle F \rangle q)$, is true at w_0.

$\langle F \rangle (p \wedge \langle F \rangle q)$, and $\langle F \rangle (\langle F \rangle p \wedge q)$ are false at w_0, because the worlds w_0 is related to are not related to other worlds; $\langle F \rangle q$ and $\langle F \rangle p$ are false at both w_1 and w_2.

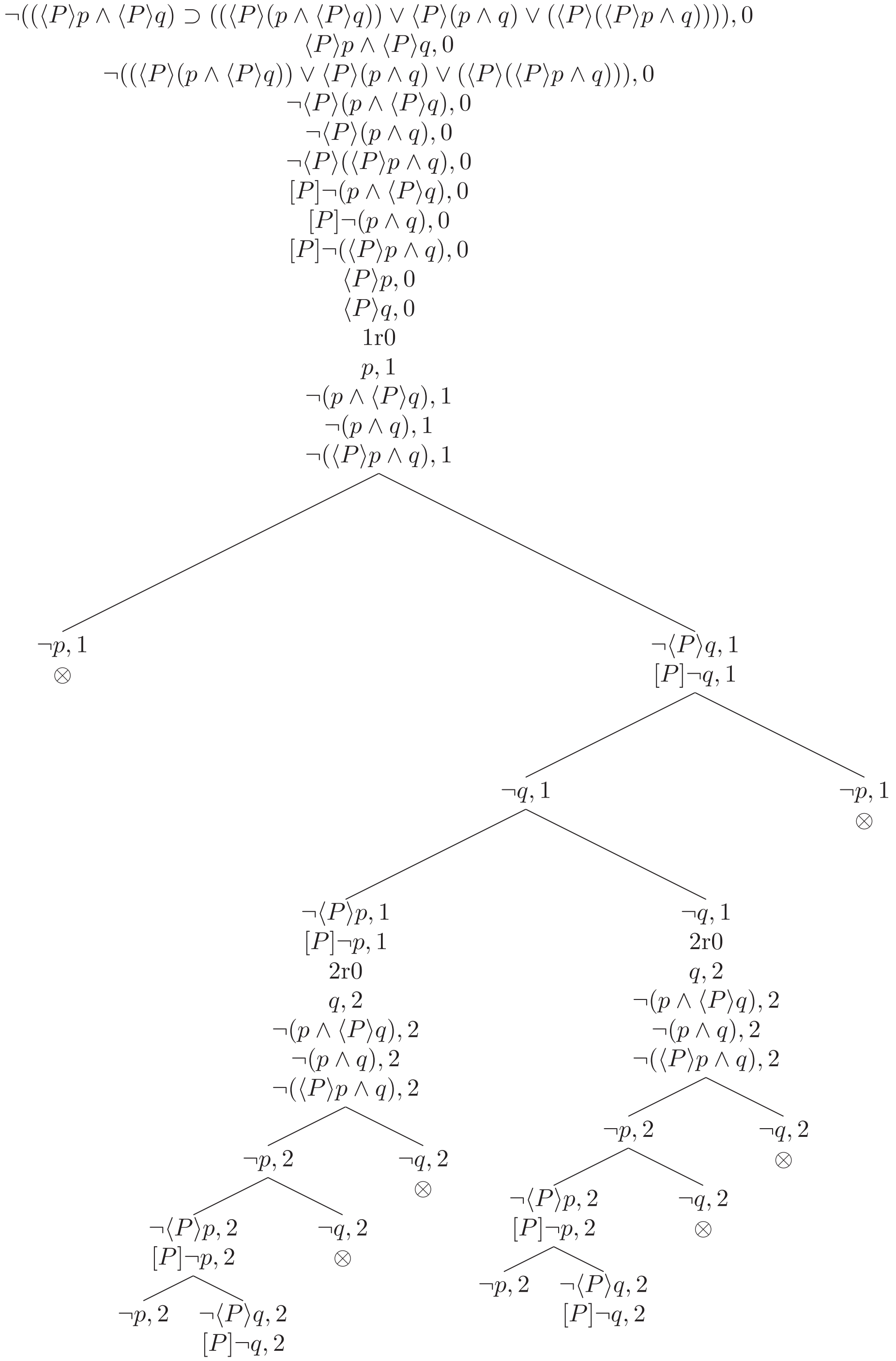
$\langle F \rangle (p \wedge q)$ is false at w_0 because w_0 is related to no world where $p \wedge q$ is true.

Therefore, the consequent, $(((\langle F \rangle (p \wedge \langle F \rangle q)) \vee \langle F \rangle (p \wedge q) \vee (\langle F \rangle (\langle F \rangle p \wedge q)))$ is false at w_0, as required.

(i) $\vdash_{K^t_\phi} (\langle F\rangle p \wedge \langle F\rangle q) \supset (((\langle F\rangle(p \wedge \langle F\rangle q)) \vee \langle F\rangle(p \wedge q) \vee ((\langle F\rangle((\langle F\rangle p \wedge q)))$



(j) $\nvdash_{K^t_\tau} (\langle P\rangle p \wedge \langle P\rangle q) \supset (((\langle P\rangle(p \wedge \langle P\rangle q)) \vee \langle P\rangle(p \wedge q) \vee ((\langle P\rangle((\langle P\rangle p \wedge q)))$



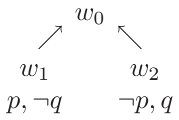
The following interpretation, taken from the left-most open branch, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

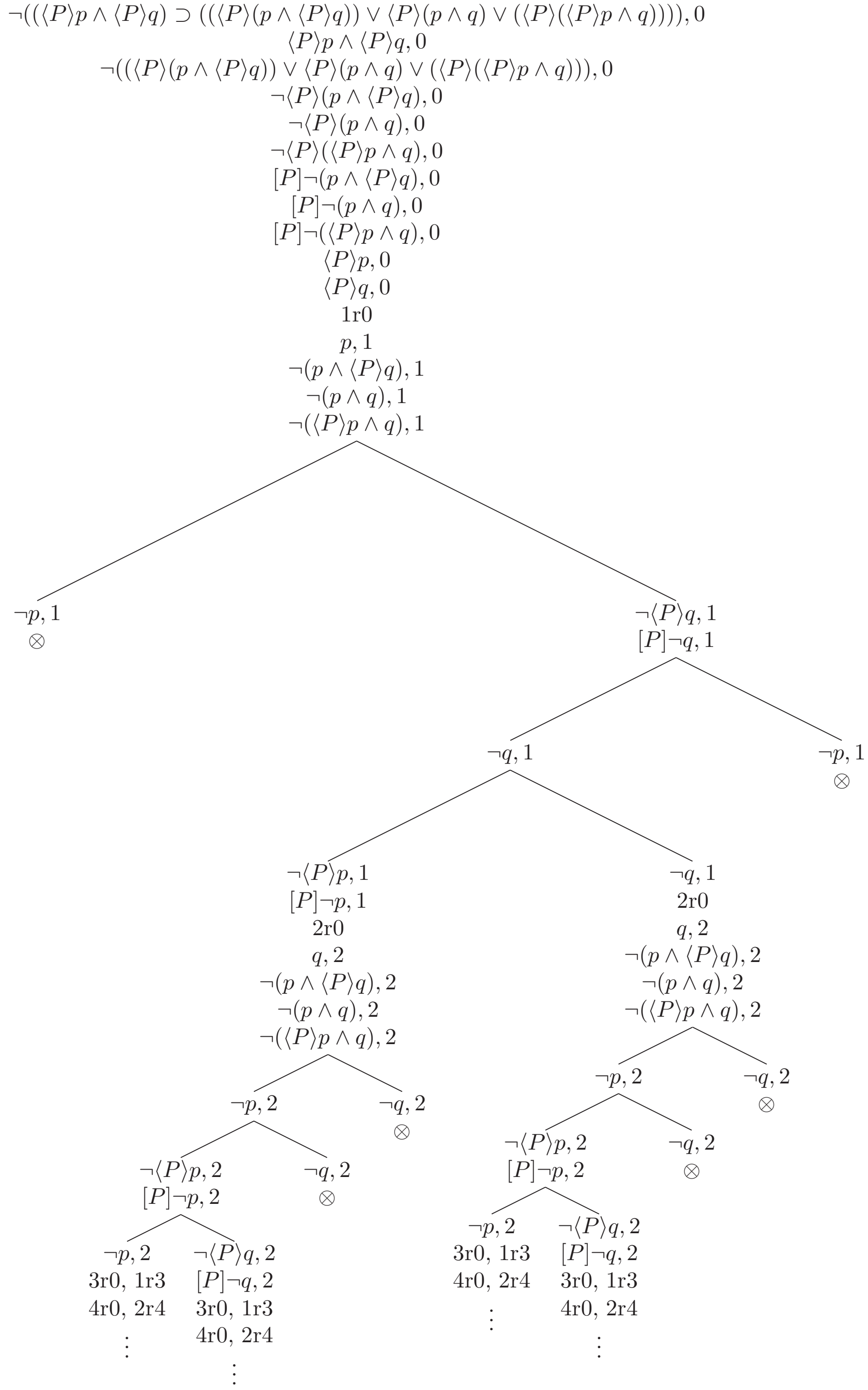
$$w_1 R w_0, w_2 R w_0$$

$$v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_2}(p) = 0, v_{w_2}(q) = 1$$

This can be represented in the following diagram:



(j) $\nvdash_{K^t_\delta} (\langle P\rangle p \wedge \langle P\rangle q) \supset (((\langle P\rangle(p \wedge \langle P\rangle q)) \vee \langle P\rangle(p \wedge q) \vee ((\langle P\rangle((\langle P\rangle p \wedge q)))$



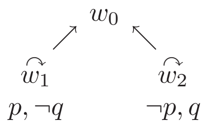
The following interpretation, not taken from the tree, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_1Rw_1, w_2Rw_2, w_1Rw_0, w_2Rw_0$$

$$v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_2}(p) = 0, v_{w_2}(q) = 1$$

This can be represented in the following diagram:



For more on why this kind of counter-model satisfies the denseness condition of K^t_δ, check the solution to (d) for K^t_δ.

Let us check that this counter-model works.

$\langle P\rangle p$ is true at w_0 because w_1 is related to w_0, and p is true at w_1.

$\langle P\rangle q$ is true at w_0 because w_2 is related to w_0, and q is true at w_2.

Thus, the antecedent $(\langle P\rangle p \wedge \langle P\rangle q)$, is true at w_0.

$\langle P\rangle(p \wedge \langle P\rangle q)$ is false at w_0 because p is false at w_2, and q is false at w_1.

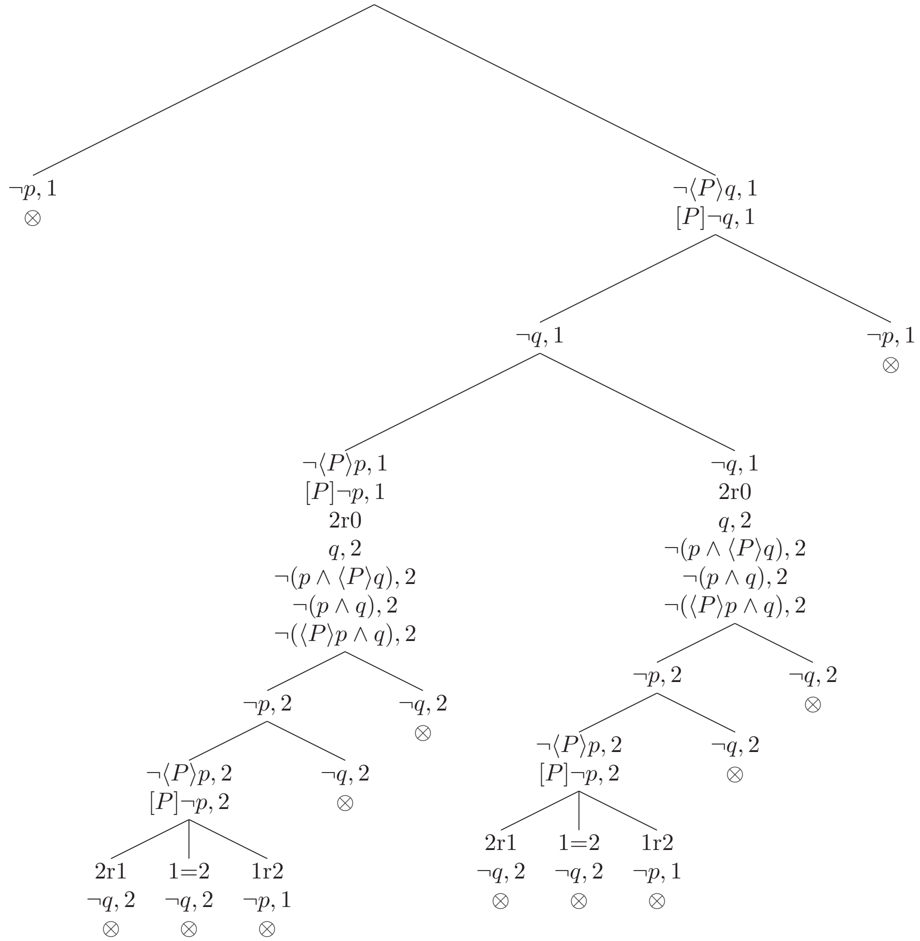
$\langle P\rangle(\langle P\rangle p \wedge q)$ is false at at w_0, because q is false at w_1 and p is false at w_2.

$\langle P\rangle(p \wedge q)$ is false at w_0 because $p \wedge q$ is false at w_1 and w_2.

Therefore, the consequent, $((\langle P\rangle(p \wedge \langle P\rangle q)) \vee \langle P\rangle(p \wedge q) \vee (\langle P\rangle(\langle P\rangle p \wedge q)))$ is false at w_0, as required.

(j) $\vdash_{K^t_\phi} (\langle P\rangle p \wedge \langle P\rangle q) \supset (((\langle P\rangle(p \wedge \langle P\rangle q)) \vee \langle P\rangle(p \wedge q) \vee (\langle P\rangle(\langle P\rangle p \wedge q)))$

$$\neg(((\langle P\rangle p \wedge \langle P\rangle q) \supset (((\langle P\rangle(p \wedge \langle P\rangle q)) \vee \langle P\rangle(p \wedge q) \vee (\langle P\rangle(\langle P\rangle p \wedge q)))), 0$$
$$\langle P\rangle p \wedge \langle P\rangle q, 0$$
$$\neg(((\langle P\rangle(p \wedge \langle P\rangle q)) \vee \langle P\rangle(p \wedge q) \vee (\langle P\rangle(\langle P\rangle p \wedge q))), 0$$
$$\neg\langle P\rangle(p \wedge \langle P\rangle q), 0$$
$$\neg\langle P\rangle(p \wedge q), 0$$
$$\neg\langle P\rangle(\langle P\rangle p \wedge q), 0$$
$$[P]\neg(p \wedge \langle P\rangle q), 0$$
$$[P]\neg(p \wedge q), 0$$
$$[P]\neg(\langle P\rangle p \wedge q), 0$$
$$\langle P\rangle p, 0$$
$$\langle P\rangle q, 0$$
$$1\text{r}0$$
$$p, 1$$
$$\neg(p \wedge \langle P\rangle q), 1$$
$$\neg(p \wedge q), 1$$
$$\neg(\langle P\rangle p \wedge q), 1$$



(l) $\nvdash_{K^t_\tau} [P]p \supset \langle P\rangle p$

$$\begin{array}{c} \neg([P]p \supset \langle P\rangle p), 0 \\ [P]p, 0 \\ \neg\langle P\rangle p, 0 \\ [P]\neg p, 0 \end{array}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0\}$$

This can be represented in the following diagram:

$$w_0$$

(l) $\nvdash_{K^t_\delta} [P]p \supset \langle P\rangle p$

$$\begin{array}{c} \neg([P]p \supset \langle P\rangle p), 0 \\ [P]p, 0 \\ \neg\langle P\rangle p, 0 \\ [P]\neg p, 0 \end{array}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0\}$$

This can be represented in the following diagram:

$$w_0$$

(l) $\nvdash_{K^t_\phi} [P]p \supset \langle P\rangle p$

$$\begin{array}{c} \neg([P]p \supset \langle P\rangle p), 0 \\ [P]p, 0 \\ \neg\langle P\rangle p, 0 \\ [P]\neg p, 0 \end{array}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0\}$$

This can be represented in the following diagram:

$$w_0$$

(m) $\nvdash (p \wedge [P]p) \supset_{K^t_\tau} \langle F\rangle[P]p$

$$\begin{gathered}\neg((p \wedge [P]p) \supset \langle F\rangle[P]p), 0\\ p \wedge [P]p, 0\\ \neg\langle F\rangle[P]p), 0\\ p, 0\\ [P]p, 0\\ [F]\neg[P]p, 0\end{gathered}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0\}$$

$$v_{w_0}(p) = 1$$

This can be represented in the following diagram:

$$\begin{gathered}w_0\\ p\end{gathered}$$

(m) $\nvdash (p \wedge [P]p) \supset_{K^t_\delta} \langle F\rangle[P]p$

$$\begin{gathered}\neg((p \wedge [P]p) \supset \langle F\rangle[P]p), 0\\ p \wedge [P]p, 0\\ \neg\langle F\rangle[P]p), 0\\ p, 0\\ [P]p, 0\\ [F]\neg[P]p, 0\end{gathered}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0\}$$

$$v_{w_0}(p) = 1$$

This can be represented in the following diagram:

$$\begin{gathered}w_0\\ p\end{gathered}$$

(m) $\nvdash (p \wedge [P]p) \supset_{K^t_\phi} \langle F\rangle[P]p$

$$\begin{array}{c}
\neg((p \wedge [P]p) \supset \langle F\rangle[P]p), 0 \\
p \wedge [P]p, 0 \\
\neg\langle F\rangle[P]p), 0 \\
p, 0 \\
[P]p, 0 \\
[F]\neg[P]p, 0
\end{array}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0\}$$

$$v_{w_0}(p) = 1$$

This can be represented in the following diagram:

$$\begin{array}{c} w_0 \\ p \end{array}$$

(o) $\nvdash_{K^t_\tau} [P]([P]p \supset p) \supset [P]p$

$$\begin{array}{c}
\neg([P]([P]p \supset p) \supset [P]p), 0 \\
[P]([P]p \supset p), 0 \\
\neg[P]p, 0 \\
\langle P\rangle\neg p, 0 \\
1\text{r}0 \\
\neg p, 1 \\
[P]p \supset p, 1 \\
\begin{array}{cc}
\begin{array}{c}
\neg[P]p, 1 \\
\langle P\rangle\neg p, 1 \\
2\text{r}1,\ 2\text{r}0 \\
\neg p, 2 \\
[P]p \supset p, 2 \\
\begin{array}{cc}
\begin{array}{c}
\neg[P]p, 2 \\
\langle P\rangle\neg p, 2 \\
3\text{r}2,\ 3\text{r}1,\ 3\text{r}0 \\
\vdots
\end{array} &
\begin{array}{c}
p, 2 \\
\otimes \\
\\
\\
\end{array}
\end{array}
\end{array} &
\begin{array}{c}
p, 1 \\
\otimes \\
\\
\\
\\
\\
\\
\\
\\
\end{array}
\end{array}
\end{array}$$

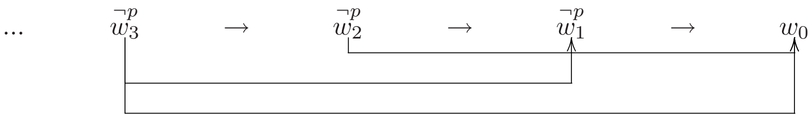
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, w_3 ...\}$$

$$w_1Rw_0, w_2Rw_1, w_2Rw_0, w_3Rw_2, w_3Rw_1, w_3Rw_0...$$

$$v_{w_1}(p) = 0, v_{w_2}(p) = 0, v_{w_3}(p) = 0...$$

This can be represented in the following diagram:



(o) $\nvdash_{K^t_\delta} [P]([P]p \supset p) \supset [P]p$

$$\neg([P]([P]p \supset p) \supset [P]p), 0$$
$$[P]([P]p \supset p), 0$$
$$\neg[P]p, 0$$
$$\langle P\rangle\neg p, 0$$
$$\text{1r0}$$
$$\neg p, 1$$
$$[P]p \supset p, 1$$

$$\begin{array}{cc} \neg[P]p, 1 & p, 1 \\ \langle P\rangle\neg p, 1 & \otimes \\ \text{2r1} & \\ \neg p, 2 & \\ \text{3r0, 1r3, 4r1, 2r4} & \\ \vdots & \end{array}$$

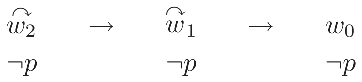
The following interpretation, not taken from the tree, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_1Rw_1, w_2Rw_2, w_1Rw_0, w_2Rw_1$$

$$v_{w_0}(p) = 0, v_{w_1}(p) = 0, v_{w_2}(p) = 0$$

This can be represented in the following diagram:



For more on why these kind of interpretations satisfy the denseness requirement of K^t_δ, see the solution to (d) for K^t_δ.

Let us check that the counter-model works.

The antecedent, $[P]([P]p \supset p)$ is true at w_0, because at all worlds related to w_0, i.e., w_1, $[P]p$ is false, and hence $[P]p \supset p$ is true.

The consequent $[P]p$ is false at w_0, because p is false at w_1, as required.

(o) $\nvdash_{K^t_\phi}$ $[P]([P]p \supset p) \supset [P]p$

$$\begin{array}{c}
\neg([P]([P]p \supset p) \supset [P]p), 0 \\
[P]([P]p \supset p), 0 \\
\neg[P]p, 0 \\
\langle P\rangle\neg p, 0 \\
1r0 \\
\neg p, 1 \\
[P]p \supset p, 1 \\
\begin{array}{cc}
\neg[P]p, 1 & p, 1 \\
\langle P\rangle\neg p, 1 & \otimes \\
2r1 & \\
\neg p, 2 &
\end{array}
\end{array}$$

The following interpretation, not taken from the tree, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, \}$$

$$w_1 R w_0, w_2 R w_1$$

$$v_{w_1}(p) = 0, v_{w_2}(p) = 0$$

This can be represented in the following diagram:

$$\begin{tikzcd}
w_2 \arrow[r] & w_1 \arrow[r] & w_0 \\
\neg p & \neg p &
\end{tikzcd}$$

Let us check that the counter-model works.

The antecedent, $[P]([P]p \supset p)$ is true at w_0, because at all worlds related to w_0, i.e., w_1, $[P]p$ is false, and hence $[P]p \supset p$ is true.

The consequent $[P]p$ is false at w_0, because p is false at w_1, as required.

(p) $\nvdash_{K^t_\tau} \langle P\rangle[P]p \supset [P]\langle P\rangle p$

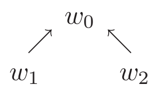
$$
\begin{gathered}
\neg(\langle P\rangle[P]p \supset [P]\langle P\rangle p), 0 \\
\langle P\rangle[P]p, 0 \\
\neg[P]\langle P\rangle p, 0 \\
\langle P\rangle\neg\langle P\rangle p \\
1r0 \\
[P]p, 1 \\
2r0 \\
\neg\langle P\rangle p, 2 \\
[P]\neg p, 2
\end{gathered}
$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_1Rw_0, w_2Rw_0$$

This can be represented in the following diagram:



(p) $\nvdash_{K^t_\delta} \langle P\rangle[P]p \supset [P]\langle P\rangle p$

$$
\begin{gathered}
\neg(\langle P\rangle[P]p \supset [P]\langle P\rangle p), 0 \\
\langle P\rangle[P]p, 0 \\
\neg[P]\langle P\rangle p, 0 \\
\langle P\rangle\neg\langle P\rangle p \\
1r0 \\
[P]p, 1 \\
2r0 \\
\neg\langle P\rangle p, 2 \\
[P]\neg p, 2 \\
1r3,\ 3r0,\ 2r4,\ 4r1 \\
\vdots
\end{gathered}
$$

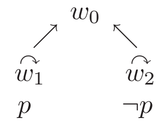
The following interpretation, not taken from the tree, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_1 R w_1, w_2 R w_2, w_1 R w_0, w_2 R w_0$$

$$v_{w_1}(p) = 1, v_{w_2}(p) = 0$$

This can be represented in the following diagram:



For more on why these kind of interpretations satisfy the denseness requirement of K^t_δ, see the solution to (d) for K^t_δ.

Let us check that the counter-model works.

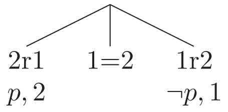
The antecedent, $\langle P\rangle[P]p$ is true at w_0, because at a world related to w_0, i.e. w_1, $[P]p$ is true.

The consequent $[P]\langle P\rangle p$ is false at w_0, because at w_2 related to w_0, $\langle P\rangle p$ is false.

Thus the inference is invalid, as required.

(p) $\nvdash_{K^t_\phi} \langle P\rangle[P]p \supset [P]\langle P\rangle p$

$$\neg(\langle P\rangle[P]p \supset [P]\langle P\rangle p), 0$$
$$\langle P\rangle[P]p, 0$$
$$\neg[P]\langle P\rangle p, 0$$
$$\langle P\rangle\neg\langle P\rangle p$$
$$1r0$$
$$[P]p, 1$$
$$2r0$$
$$\neg\langle P\rangle p, 2$$
$$[P]\neg p, 2$$



The following interpretation, from the middle branch, shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

$$w_1 R w_0$$

This can be represented in the following diagram:

$$\begin{tikzcd}
w_0 \\
w_1 \arrow[u]
\end{tikzcd}$$

(q) $\nvdash_{K^t_\tau} \langle F\rangle[F]p \supset p$

$$\begin{gathered}
\neg(\langle F\rangle[F]p \supset p), 0 \\
\langle F\rangle[F]p, 0 \\
\neg p, 0 \\
0\mathrm{r}1 \\
[F]p, 1
\end{gathered}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

$$w_0 R w_1$$

$$v_{w_0}(p) = 0$$

This can be represented in the following diagram:

$$\begin{tikzcd}
w_0 \arrow[r] & w_1 \\
\neg p &
\end{tikzcd}$$

(q) $\nvdash_{K^t_\delta} \langle F\rangle[F]p \supset p$

$$\begin{gathered}
\neg(\langle F\rangle[F]p \supset p), 0 \\
\langle F\rangle[F]p, 0 \\
\neg p, 0 \\
0\mathrm{r}1 \\
[F]p, 1 \\
0\mathrm{r}2,\ 2\mathrm{r}1 \\
\vdots
\end{gathered}$$

The following interpretation, not taken from the tree, shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

$$w_0 R w_0, w_0 R w_1$$

$$v_{w_0}(p) = 0$$

This can be represented in the following diagram:

$$\begin{array}{ccc} \overset{\frown}{w_0} & \longrightarrow & w_1 \\ \neg p & & \end{array}$$

For more on why these kind of interpretations satisfy the denseness requirement of K^t_δ, see the solution to (d) for K^t_δ.

Let us check that the counter-model works.

$[F]p$ is true at w_1, because w_1 is not related to any worlds. So, the antecedent, $\langle F\rangle[F]p$ is true at w_0, because w_0 is related to w_1.

The consequent p is false at w_0, because p is false at w_0.

Thus the inference is invalid, as required.

(q) $\nvdash_{K^t_\phi} \langle F\rangle[F]p \supset p$

$$\begin{array}{c} \neg(\langle F\rangle[F]p \supset p), 0 \\ \langle F\rangle[F]p, 0 \\ \neg p, 0 \\ 0r1 \\ [F]p, 1 \end{array}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

$$w_0 R w_1$$

$$v_{w_0}(p) = 0$$

This can be represented in the following diagram:

$$\overset{\neg p}{w_0} \rightarrow \; w_1$$

(r) $\nvdash_{K^t_\tau} (\langle F\rangle p \wedge \langle F\rangle[F]\neg p) \supset \langle F\rangle([P]\langle F\rangle p \wedge [F]\neg p)$

$$\begin{array}{c}
\neg((\langle F\rangle p \wedge \langle F\rangle[F]\neg p) \supset \langle F\rangle([P]\langle F\rangle p \wedge [F]\neg p)), 0 \\
\langle F\rangle p \wedge \langle F\rangle[F]\neg p, 0 \\
\neg\langle F\rangle([P]\langle F\rangle p \wedge [F]\neg p), 0 \\
[F]\neg([P]\langle F\rangle p \wedge [F]\neg p), 0 \\
\langle F\rangle p, 0 \\
\langle F\rangle[F]\neg p, 0 \\
0r1 \\
p, 1 \\
\neg([P]\langle F\rangle p \wedge [F]\neg p), 1
\end{array}$$

$$\begin{array}{cc}
\neg[P]\langle F\rangle p & \neg[F]\neg p, 1 \\
\langle P\rangle\neg\langle F\rangle p & \langle F\rangle\neg\neg p, 1 \\
2r1 & 1r2,\ 0r2 \\
\neg\langle F\rangle p, 2 & \neg\neg p, 2 \\
[F]\neg p, 2 & p, 2 \\
\neg p, 1 & 0r3 \\
\otimes & [F]\neg p, 3 \\
 & [P]\langle F\rangle p \wedge [F]\neg p, 2 \\
 & \vdots
\end{array}$$

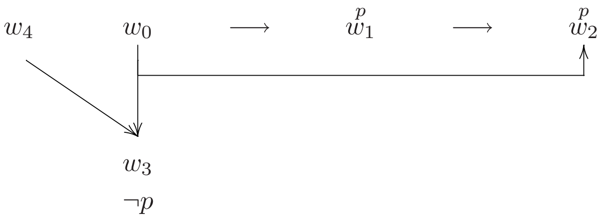
This interpretation, not taken from the tree, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, w_3, w_4\}$$

$$w_0Rw_1, w_1Rw_2, w_0Rw_2, w_0Rw_3, w_4Rw_3$$

$$v_{w_1}(p) = 1, v_{w_2}(p) = 1, v_{w_3}(p) = 0$$

This can be represented in the following diagram:



Let us check that this counter-model works:

$\langle F\rangle p$ is made true at at w_0 by p being true at the related w_1.

$\langle F\rangle[F]\neg p$ is made true at w_0 by w_3.

Therefore the antecedent, $\langle F\rangle p \wedge \langle F\rangle[F]\neg p$, is true at at w_0.

However, there is no world making $\langle F\rangle([P]\langle F\rangle p \wedge [F]\neg p)$ true at w_0. For this, a world w_0 is related to would have to both make $[P]\langle F\rangle p$, and $[F]\neg p$ true. w_1 makes the former true, but not the latter; w_3 makes the latter true, but not the former, because w_4 is related to w_3, and hence a past world in relation to w_3.

Therefore, the consequent $\langle F\rangle([P]\langle F\rangle p \wedge [F]\neg p)$ is false at w_0, as required.

(r) $\nvdash_{K^t_\delta} (\langle F\rangle p \wedge \langle F\rangle[F]\neg p) \supset \langle F\rangle([P]\langle F\rangle p \wedge [F]\neg p)$

$$\begin{gathered}
\neg((\langle F\rangle p \wedge \langle F\rangle[F]\neg p) \supset \langle F\rangle([P]\langle F\rangle p \wedge [F]\neg p)), 0\\
\langle F\rangle p \wedge \langle F\rangle[F]\neg p, 0\\
\neg\langle F\rangle([P]\langle F\rangle p \wedge [F]\neg p), 0\\
[F]\neg([P]\langle F\rangle p \wedge [F]\neg p), 0\\
\langle F\rangle p, 0\\
\langle F\rangle[F]\neg p, 0\\
0r1\\
p, 1\\
\neg([P]\langle F\rangle p \wedge [F]\neg p), 1
\end{gathered}$$

$\neg[P]\langle F\rangle p$	$\neg[F]\neg p, 1$
$\langle P\rangle\neg\langle F\rangle p$	$\langle F\rangle\neg\neg p, 1$
2r1	1r2
$\neg\langle F\rangle p, 2$	$\neg\neg p, 2$
$[F]\neg p, 2$	$p, 2$
$\neg p, 1$	0r3
$\otimes$	$[F]\neg p, 3$
	0r4, 4r1, 1r5, 5r2, 0r6, 6r3
	$\vdots$

This interpretation, not taken from the tree, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, w_3\}$$

$$w_0Rw_0, w_1Rw_1, w_3Rw_3, w_0Rw_1, w_1Rw_2, w_0Rw_3$$

$$v_{w_1}(p) = 1, v_{w_2}(p) = 1, v_{w_3}(p) = 0$$

This can be represented in the following diagram:

$$\begin{tikzcd}
\overset{\frown}{w_0} \arrow[r] \arrow[dd] & \overset{\frown}{w_1} \arrow[r] & w_2 \\
 & p & p \\
\overset{\frown}{w_3} & & \\
\neg p & &
\end{tikzcd}$$

Let us check that this counter-model works:

$\langle F\rangle p$ is made true at w_0 by p being true at the related w_1.

$\langle F\rangle[F]\neg p$ is made true at w_0 by $\neg p$ being true at the related w_3.

Therefore the antecedent, $\langle F\rangle p \wedge \langle F\rangle[F]\neg p$, is true at w_0.

However, there is no world making $\langle F\rangle([P]\langle F\rangle p \wedge [F]\neg p)$ true at w_0. For this, a world w_0 is related to would have to both make $[P]\langle F\rangle p$, and $[F]\neg p$ true. w_1 makes the former true, but not the latter; w_3 makes the latter true, but not the former, because it is related to itself, and hence a past world in relation to itself. (See the solution to (d) for δ for a little more on this.)

Therefore, the consequent $\langle F\rangle([P]\langle F\rangle p \wedge [F]\neg p)$ is false at w_0, as required.

(r) $\nvdash_{K^t_\phi} ((\langle F\rangle p \wedge \langle F\rangle[F]\neg p) \supset \langle F\rangle([P]\langle F\rangle p \wedge [F]\neg p)$

$$\begin{array}{c}
\neg(((\langle F\rangle p \wedge \langle F\rangle[F]\neg p) \supset \langle F\rangle([P]\langle F\rangle p \wedge [F]\neg p)), 0] \\
\langle F\rangle p \wedge \langle F\rangle[F]\neg p, 0 \\
\neg\langle F\rangle([P]\langle F\rangle p \wedge [F]\neg p), 0 \\
[F]\neg([P]\langle F\rangle p \wedge [F]\neg p), 0 \\
\langle F\rangle p, 0 \\
\langle F\rangle[F]\neg p, 0 \\
0r1 \\
p, 1 \\
\neg([P]\langle F\rangle p \wedge [F]\neg p), 1
\end{array}$$

$$\begin{array}{cc}
\neg[P]\langle F\rangle p & \neg[F]\neg p, 1 \\
\langle P\rangle\neg\langle F\rangle p & \langle F\rangle\neg\neg p, 1 \\
2r1 & 1r2 \\
\neg\langle F\rangle p, 2 & \neg\neg p, 2 \\
[F]\neg p, 2 & p, 2 \\
\neg p, 1 & 0r3 \\
\otimes & [F]\neg p, 3 \\
 & \neg([P]\langle F\rangle p \wedge [F]\neg p), 3 \\
 & \begin{array}{ccc} 1r3 & 1{=}3 & 3r1 \\ \vdots & \vdots & \vdots \end{array}
\end{array}$$

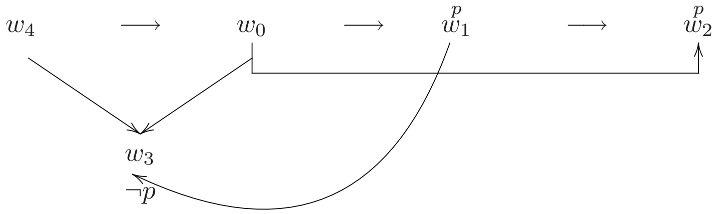
This interpretation, not taken from the tree, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, w_3, w_4\}$$

$$w_0Rw_1, w_1Rw_2, w_0Rw_3, w_4Rw_3, w_1Rw_3, w_4Rw_0$$

$$v_{w_1}(p) = 1, v_{w_2}(p) = 1, v_{w_3}(p) = 0$$

This can be represented in the following diagram:



Let us check that this counter-model works:

$\langle F\rangle p$ is made true at w_0 by p being true at w_1.

$\langle F\rangle[F]\neg p$ is made true at w_0 by $\neg p$ being true at w_3.

Therefore the antecedent, $\langle F\rangle p \wedge \langle F\rangle[F]\neg p$, is true at w_0.

However, there is no world making $\langle F\rangle([P]\langle F\rangle p \wedge [F]\neg p)$ true at w_0. For this, a world w_0 is related to would have to both make $[P]\langle F\rangle p$, and $[F]\neg p$ true. w_1 makes the former true, but not the latter; w_3 makes the latter true, but not the former, because w_4 is related to w_3 (a past world in relation to w_3).

Therefore, the consequent $\langle F\rangle([P]\langle F\rangle p \wedge [F]\neg p)$ is false at w_0, as required.

12. Consider a tense logic in which the relation R is constrained by the following condition. There is an x such that: (i) for no y, xRy; and (ii) for all y distinct from x, yRx. Show that $[F](A\wedge\neg A)\vee\langle F\rangle[F](A\wedge\neg A)$ is a logical truth.

Take any interpretation. Let w be the the world which is not related to any world. Then $[F](A\wedge\neg A)$ is true at w. Therefore $[F](A\wedge\neg A)\vee\langle F\rangle[F](A\wedge\neg A)$ is true at w.

Let w' be any other world. w' is related to w. So, $\langle F\rangle[F](A\wedge\neg A)$ is true at w'. Therefore $[F](A\wedge\neg A)\vee\langle F\rangle[F](A\wedge\neg A)$ is true at w'.

In any interpretation, $[F](A\wedge\neg A)\vee\langle F\rangle[F](A\wedge\neg A)$ is true at w, and at all other worlds, therefore it is a logical truth.

13. If an inference is valid in K^t_τ, does it follow that its mirror image is? What about K^t_δ and K^t_ϕ?

Consider any interpretation. Let its mirror image be the same, except that we replace R with its converse, Rc , where xRy iff yR_cx.

$\boxed{K^t_\tau}$

Suppose that an inference is invalid in K^t_τ, and that I is a counter-model. The mirror image of I is a counter-model for the mirror image of the inference, and it is still an interpretation for K^t_τ, because the τ restriction is symmetric. That is, suppose that xR_cy and yR_cz. Then yRx and zRy. So zRx, and xR_cz.

$\boxed{K^t_\delta}$

The same can be said for K^t_δ. Suppose that an inference is invalid in K^t_δ, and that I is a counter-model. The mirror image of I is a counter-model for the mirror image of the inference, and it is still an interpretation for K^t_δ, because the δ restriction is symmetric. That is, suppose that xR_cy, and so yRx. Then yRz and zRx. So xR_cz and zR_cy.

$\boxed{K^t_\phi}$

In fact the result fails for K^t_ϕ: We saw in 3.6v.7 that $\langle F\rangle p \wedge \langle F\rangle q \vdash_{K^t\phi} \langle F\rangle(p \wedge q) \vee \langle F\rangle(p \wedge \langle F\rangle q) \vee \langle F\rangle(\langle F\rangle p \wedge q)$ is valid; its mirror image is not, as we saw in 11(j):

$\langle P\rangle p \wedge \langle P\rangle q \nvdash_{K^t\phi} \langle P\rangle(p \wedge q) \vee \langle P\rangle(p \wedge \langle P\rangle q) \vee \langle P\rangle(\langle P\rangle p \wedge q)$

Tree as in 11(j).

15. *Fill in the details omitted in 3.7

3.7.5 Check the case for $\Diamond$ in the proof of RTL $\Sigma \vDash_{K_{\rho\sigma\tau}}$ iff $\Sigma \vDash_{K_v A}$

$$\begin{aligned} v'_w(\Diamond A) = 1 \;\; & \text{iff} \;\; \text{for some } x \,\epsilon\, W' \text{ such that } wR'x, v'_x(A) = 1 \\ & \text{iff} \;\; \text{for some } x \,\epsilon\, W' \text{ such that } wR'x, v_x(A) = 1 \text{ (by IH)} \\ & \text{iff} \;\; \text{for some } x \,\epsilon\, W \text{ such that } wRx, v_x(A) = 1 \text{ (*)} \\ & \text{iff} \;\; v_w(\Diamond A) = 1 \end{aligned}$$

The line (*) holds since wRx iff $x \,\epsilon\, W'$ iff $wR'x$ ■

3.7.6 Check the new cases for $[P]$ and $\langle P\rangle$ in the Soundness and Completeness Lemmas.

Soundness Lemma

Let f be a function which shows interpretation I to be faithful to branch of a tableau b. Suppose $[P]A, i$ is on b, and that we apply the rule. Since I is faithful to b, $[P]A$ is true at $f(i)$. Moreover, for any i and j, such that jri is on b, $f(j)Rf(i)$. Hence, by the truth conditions for for $[P]$, A is true at $f(j)$ and so I is faithful to the extension of the branch.

Let f be a function which shows interpretation I to be faithful to branch of a tableau b. Suppose $\neg[P]A, i$ is on b, and that we apply the rule. Since I is faithful to b, $\neg[P]A$ is true at $f(i)$. Moreover, $\neg[P]A$ iff $\langle P\rangle\neg A$:

$$\begin{aligned}
v_w(\neg[P]) = 1 \quad & \text{iff } v_w([P]) = 0 \\
& \text{iff for some } w' \text{ such that } w'Rw,\ v'_w(A) = 0 \\
& \text{iff for some } w' \text{ such that } w'Rw,\ v'_w(\neg A) = 1 \\
& \text{iff } v_w(\langle F\rangle\neg A) = 1
\end{aligned}$$

Therefore, f shows I to be faithful to the extension of the branch.

Suppose $\langle P\rangle A, i$ is on b, and that we apply the rule to get nodes of the form jri and A, j. Since I is faithful to b, $\langle P\rangle A$ is true at $f(i)$. Moreover, for some $w\epsilon W$, $wRf(i)$, and A is true at w. Let f' be the same as f except that $f'(j) = w$. f' shows I to be faithful to b, since f and f' differ only at j. Further, by definition, $f'(j)Rf'(i)$, and A is true at $f'(j)$. Hence f' shows I to be faithful to the extended branch.

Let f be a function which shows interpretation I to be faithful to branch of a tableau b. Suppose $\neg\langle P\rangle A, i$ is on b, and that we apply the rule. Since I is faithful to b, $\neg\langle P\rangle A$ is true at $f(i)$. Moreover, $\neg\langle P\rangle A$ iff $[P]\neg A$:

$$\begin{aligned}
v_w(\neg\langle P\rangle) = 1 \quad & \text{iff } v_w(\langle P\rangle) = 0 \\
& \text{iff for all } w' \text{ such that } w'Rw,\ v'_w(A) = 0 \\
& \text{iff for all } w' \text{ such that } w'Rw,\ v'_w(\neg A) = 1 \\
& \text{iff } v_w([F]\neg A) = 1
\end{aligned}$$

Therefore, f shows I to be faithful to the extension of the branch.

■

Completeness Lemma

If formula A occurs on open complete branch of a tableau b, and is of the form $\langle P\rangle B$, then the rule has been applied to $\langle P\rangle B, i$. Thus, for some j such that jRi is on b, B, j is on b. By construction and the induction hypothesis, for some w_j such that $w_j R w_i$, B is true at w_j. Hence $\langle P\rangle B$ is true at w_i, as required.

If A occurs on b, and is of the form $\neg\langle P\rangle B$, then the rule for negated possibility has been applied to $\neg\langle P\rangle B, i$. Thus, $[P]\neg B, i$ is on b. So, for all j such that jRi, $\neg B, j$ is on b. By induction hypothesis, for all j such that $w_j R w_i$, B is false at w_j. Hence $\Diamond B$ is false at w_i, as required.

If formula A occurs on an open complete branch of a tableau b, and is of the form $[P]B$, then the rule has been applied to $[P]B, i$. Thus, for all j such that jRi is on b, B, j is on b. By construction and the induction hypothesis, for all w_j such that $w_j R w_i$, B is true at w_j. Hence $[P]B$ is true at w_i, as required.

If A occurs on b, and is of the form $\neg[P]B$, then the rule for negated necessity has been applied to $\neg[P]B, i$. Thus, $\langle P\rangle\neg B, i$ is on b. So, for some j such that jRi, $\neg B, j$ is on b. By induction hypothesis, for some j such that $w_j R w_i$, B is false at w_j. Hence $[P]B$ is false at w_i, as required.

■

3.7.7 Check the Completeness Lemma case for η', and the last two possibilities for the $=$ case.

η'

If $w_i \epsilon W$ then for some j, jri is on b. Hence, for some j, $w_j R w_i$ as required.

$=$

Suppose that $\alpha(i)$ and $i = j$ are on b, and that we apply the rule to get $\alpha(j)$. Since f shows I to be faithful to b, $f(i) = f(j)$. If $\alpha(i)$ is A, i then A is true at $f(i)$. Hence A is true at $f(j)$ as required. If $\alpha(i)$ is kri, then $f(k)Rf(i)$ and so $f(k)Rf(j)$, as required. If $\alpha(i)$ is $i = k$, then, because $f(j) = f(k)$, $j = k$ as required.

3.7.8 Part 1. Check that the revised completeness lemma still applies to $\langle F\rangle, [P], and\langle P\rangle$. and the ρ, τ, σ, η, η' and β cases.

$\boxed{\langle F\rangle}$

Suppose that $\langle F\rangle B, i$ is on b. Then there is some $j \epsilon I$, such that irj is on b, and B, j is on b. Hence by construction and induction hypothesis, for some $w_{[j]}$ such that $w_{[i]}Rw_{[j]}$, B is true at $w_{[j]}$, as required. Suppose that $\neg\langle F\rangle B, i$ is on b. Then $[F]\neg B, i$ is on b. Therefore, for all $j \epsilon I$, such that irj is on b, $\neg B, j$ is on b. Hence by construction and induction hypothesis, for all $w_{[j]}$ such that $w_{[i]}Rw[j]$, $\neg B$ is true at $w_{[j]}$, and hence B is false at $w_{[j]}$, as required.

$\boxed{[P]}$

Suppose that $[P]B, i$ is on b. Then for all $j \epsilon I$ such that jri is on b, B, j is on b. Hence, by construction and induction hypothesis, for all $w_{[j]}$ such that $w_{[j]}Rw_{[i]}$, B is true at $w_{[j]}$, as required. Suppose that $\neg[P]B, i$ is on b. Then $\langle P\rangle\neg B, i$ is on b, as, therefore are irj and $\neg B, j$, for some j. By construction and induction hypothesis, $w_{[j]}Rw_{[i]}$ and B is false at $w_{[j]}$. Hence, $[F]B$ is false at $w_{[i]}$ as required.

$\boxed{\langle P\rangle}$

Suppose that $\langle P\rangle B, i$ is on b. Then there is some $j \epsilon I$, such that jri is on b, and B, j is on b. Hence by construction and induction hypothesis, for some $w_{[j]}$ such that $w_{[j]}Rw_{[i]}$, B is true at $w_{[j]}$, as required. Suppose that $\neg\langle P\rangle B, i$ is on b. Then $[P]\neg B, i$ is on b. Therefore, for all $j \epsilon I$, such that jri is on b, $\neg B, j$ is on b. Hence by construction and induction hypothesis, for all $w_{[j]}$ such that $w_{[j]}Rw[i]$, $\neg B$ is true at $w_{[j]}$, and hence B is false at $w_{[j]}$, as required.

3.7.8 Part 2. Check that the induced interpretation is of the appropriate kind if the corresponding tableau rule is used, for the ρ, τ, σ, η, η' and β cases.

$\boxed{\rho}$

Suppose $w_{[i]} \epsilon W$. Then i occurs on b and by the ρ rule iri is on b. Thus $w_{[i]}Rw_{[i]}$ as required.

$\boxed{\tau}$

Suppose that $w_{[i]}Rw_{[j]}$ and $w_{[j]}Rw_{[k]}$. Then irj, and jrk are on b. So by the τ rule, irk is on b. Hence $w_{[i]}Rw_{[k]}$, as required.

$\boxed{\sigma}$

Suppose that $w_{[i]}Rw_{[j]}$. Then irj is on b. So by the σ rule, jri is on b. Hence $w_{[j]}Rw_{[i]}$, as required.

$\boxed{\eta}$

Suppose that $w_{[i]} \epsilon W$. Then i occurs on b and by the η rule, irj is on b, for some new j. Hence $w_{[i]}Rw_{[j]}$, as required.

$\boxed{\eta'}$

Suppose that $w_{[i]} \epsilon W$. Then i occurs on b and by the η' rule jri is on b, for some new j. Hence $w_{[j]}Rw_{[i]}$, as required.

$\boxed{\beta}$

Suppose that $w_{[j]}Rw_{[i]}$ and $w_{[k]}Rw_{[i]}$. (Where $[i]$, $[j]$ and $[k]$ are distinct). Then jri and kri are on b. So, by the β rule , either jrk, krj, or $j = k$ are on b; so either $w_{[j]}Rw_{[i]}$, $w_{[k]}Rw_{[j]}$ or $j \sim k$. If $\alpha_{(}i)$ is $i = k$ then $f(i) = f(k)$. But $f(i) = f(j)$, so $f(j) = f(k)$, as required. In all three cases therefore, we have what we need.

■

16. *Work out the details of the semantics and tableaux for a language with both modal and tense operators.

We can create a joint system by postulating two accessibility relations. Let the accessibility relation for modal logic be R and that for temporal logic be S. Interpretations for this joint language are now of the form $I = \langle W, R, S, v \rangle$.

Intuitively, worlds are now related in two ways - via time and necessity.

We must distinguish between the two relations in tableaux as well. Let us keep the standard modal relation as irj, and change the temporal relation to isj. Thus, s is substituted for r in all the temporal tableau rules:

$$\begin{array}{c} [F]A, i \\ isj \\ \downarrow \\ A, j \end{array}$$

$$< F > A, i$$
$$\downarrow$$
$$isj$$
$$A, j$$

$$[P]A, i$$
$$jsi$$
$$\downarrow$$
$$A, j$$

$$< P > A, i$$
$$\downarrow$$
$$jsi$$
$$A, j$$

The Soundness and Completeness proofs are exactly the same as for standard temporal logic, with the addition of duplicate cases for 's'

■

This is a very basic model, where there are no inter-relations between R and S. However, it could be expanded upon to produce stronger logics by adding constraints on the relations. A natural one might be, anything that is happening, has happened, or will happen is possible. This could be fomulated by the addition of the following constraints: for all w, wRw; if wSw' then wRw', and if $w'Sw$ then wRw'. The details are left to the interested reader.

17. *Show that the tableaux for K_v, as described in 3.5.3, are sound and complete with respect to the semantics, as described in 3.5.2.

The proof proceeds in exactly the same fashion as the proofs for K in section 2, because K_v is not treated as a restriction - rather as new rules for $\Box$ and $\Diamond$, and their negations. We shall simply show that the Soundness and Completeness Lemmas hold with the new rules.

Soundness:

Suppose that f shows I to be faithful to branch section b, that $\Box A, i$ appears on b, and that we apply the relevant rule to it. Since I is faithful to b, $\Box A$ is true at $f(i)$. Moreover, for all i and j, $f(i)Rf(j)$. Hence by the truth conditions for $\Box$, A is true at $f(j)$, and so I is faithful to the extension of the branch.

Suppose that f shows I to be faithful to branch section b, that $\neg\Box A, i$ appears on b, and that we apply the relevant rule to it. Since I is faithful to b, $\Box A$ is false at $f(i)$. So there is some $w \epsilon W$ such that $f(i)Rw$, and A is false at w. Let f' be exactly the same as f except that $f(j) = w$. Since all worlds are related to one another, $f'(i)Rf'(j)$, and A is false at $f'(j)$, hence f' shows I to be faithful to the extended branch.

Suppose that f shows I to be faithful to branch section b, that $\Diamond A, i$ appears on b, and that we apply the relevant rule to it, generating an extension of the branch with A, j where j is new. Since I is faithful to b, $\Diamond A$ is true at $f(i)$. So there is some $w \epsilon W$, such that $f(i)Rw$ and A is true at w. Let f' be exactly the same as f except that $f'(j) = w$. Since all worlds are related to one another $f'(i)Rf'(j)$, and A is true at $f'(j)$, hence f' shows I to be faithful to the extended branch.

Suppose that f shows I to be faithful to branch section b, that $\neg\Diamond A, i$ appears on b, and that we apply the relevant rule to it. Since I is faithful to b, $\Diamond A$ is false at $f(i)$. So there is no $w \epsilon W$, such that $f(i)Rw$ and A is true at w. Hence in all $w' \epsilon W$, such that $f(i)Rw'$, by v, all $w \epsilon W$, $\neg A$ is true at w as required.

Completeness:

Suppose A occurs on an open complete branch b and is of the form $\Box B$. If $\Box B, i$ is on b, then the rule has been applied, and for all j such that j is on b, B, j is on b. By construction and the induction hypothesis, for all w_j, B is true at w_j. Hence $\Box B$ is true at w_i.

Suppose A occurs an on open complete branch b and is of the form $\neg\Box B$. If $\neg\Box B, i$ is on b, then the rule has been applied, and $\Diamond\neg B, i$ is on b. So for some new j, such that j is on b, $\neg B, j$ is on b. By construction and the induction hypothesis, for some $w_i R w_j$, B is false at w_j. Hence $\Box B$ is false at w_i, as required.

If A occurs on an open complete branch b, and is of the form $\Diamond B$, then the rule for possibility has been applied to $\Diamond B, i$. Thus, for some new j, B, j is on b. By construction and the induction hypothesis, for some w_j, B is true at w_j. Hence $\Diamond B$ is true at w_i, as required.

If A occurs on an open complete branch b and is of the form $\neg\Diamond B$ then the rule for negated possibility has already been applide to $\neg\Diamond B, i$. Thus $\Box\neg B, i$ is on the branch. So for all j such that j is on b, $\neg B, j$ is on b. Therefore there is no $w \epsilon W$ such that $w_i R w$ and B is true at w. Thus $\Diamond B$ is false at w_i as required.

■

18. *Let α (anti-reflexivity) be the condition: for all w, it is not the case that wRw. Show that the logic K_α is the same as the logic K. (Hint: think about the interpretations produced by K-tableaux.)

If an inference is valid in K, it is valid in $K\alpha$, because $K\alpha$ is an extension of K. If it is invalid in $K\alpha$, the tableau for it does not close. Consider an interpretation induced by an open branch. Examining the rules, it is clear that there is no line of the form iri on the tableau, so the interpretation is a K interpretation. Thus $K\alpha$ and K are the same.

■

19. *A relation, R, is *Euclidean* iff, if wRu and wRv then uRv (and also, of course, vRu). An ϵ-interpretation is one in which R is Euclidean. What tableau rules are sound and complete for K_ϵ? Show that K_ϵ is distinct from K, K_ρ, K_σ, K_τ and K_η. (Hint: consider the formula $\Diamond A \supset \Box\Diamond A$.)

There is a new rule which is sound and complete for K_ϵ:

$$\begin{array}{c} irj \\ irk \\ \downarrow \\ jrk \\ krj \end{array}$$

Soundness:

The proof is as for K(2.9.6-2.9.7). All we must do is check that the Soundness Lemma still works given the new rules:

Suppose that f shows I to be faithful to b, that irj and irk are on branch segment b, and that we apply the ϵ rule. Since $f(i)Rf(j)$, $f(i)Rf(k)$ and R is Euclidean, $f(j)Rf(k)$, and $f(k)Rf(j)$, as required.

Completeness:

Proof as for K, all we must do in addition is check that the interpretation induced by the open branch b is of the required kind:

Suppose that w_iRw_j, and w_iRw_k; then irj and irk occur on an open complete branch b. Then the ϵ rule above has been applied, so jrk and krj also occur on b. Hence by construction and the induction hypothesis, w_jRw_k, and w_kRw_j, as required.

■

Show that K_ϵ is distinct from K, K_ρ, K_σ, K_τ and K_η.

We will find a characteristic formula for K_ϵ, and then show that it is invalid in the other logics.

Characteristic formula:
$\vdash_{K_\epsilon} \Diamond A \supset \Box\Diamond A$

$$\begin{array}{c}
\neg(\Diamond A \supset \Box\Diamond A), 0 \\
\Diamond A, 0 \\
\neg\Box\Diamond A, 0 \\
\Diamond\neg\Diamond A, 0 \\
0\text{r}1 \\
A, 1 \\
0\text{r}2,\ 1\text{r}2,\ 2\text{r}1 \\
\neg\Diamond A, 2 \\
\Box\neg A, 2 \\
\neg A, 1 \\
\otimes
\end{array}$$

This formula is not valid in any of the logics above:

$\nvdash_K \Diamond p \supset \Box\Diamond p$

$$\begin{array}{c}
\neg(\Diamond p \supset \Box\Diamond p), 0 \\
\Diamond p, 0 \\
\neg\Box\Diamond p, 0 \\
\Diamond\neg\Diamond p, 0 \\
0\text{r}1 \\
p, 1 \\
0\text{r}2 \\
\neg\Diamond p, 2 \\
\Box\neg p, 2
\end{array}$$

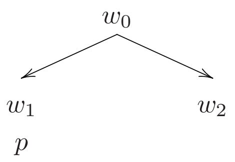
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_1, w_0 R w_2$$

$$v_{w_1}(p) = 1$$

This can be represented in the following diagram:



$\nvdash_{K_\rho} \Diamond p \supset \Box\Diamond p$

$$\begin{gathered}
\neg(\Diamond p \supset \Box\Diamond p), 0 \\
0\mathrm{r}0 \\
\Diamond p, 0 \\
\neg\Box\Diamond p, 0 \\
\Diamond\neg\Diamond p, 0 \\
0\mathrm{r}1,\ 1\mathrm{r}1 \\
p, 1 \\
0\mathrm{r}2,\ 2\mathrm{r}2 \\
\neg\Diamond p, 2 \\
\Box\neg p, 2 \\
\neg p, 2
\end{gathered}$$

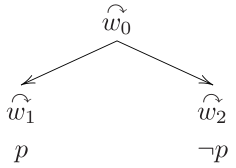
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0Rw_1, w_0Rw_2, w_0Rw_0, w_1Rw_1, w_2Rw_2$$

$$v_{w_1}(p) = 1, v_{w_2}(p) = 0$$

This can be represented in the following diagram:



$\nvdash_{K_\sigma} \Diamond p \supset \Box\Diamond p$

$$\begin{gathered}
\neg(\Diamond p \supset \Box\Diamond p), 0 \\
\Diamond p, 0 \\
\neg\Box\Diamond p, 0 \\
\Diamond\neg\Diamond p, 0 \\
0\mathrm{r}1,\ 1\mathrm{r}0 \\
p, 1 \\
0\mathrm{r}2,\ 2\mathrm{r}0 \\
\neg\Diamond p, 2 \\
\Box\neg p, 2 \\
\neg p, 0
\end{gathered}$$

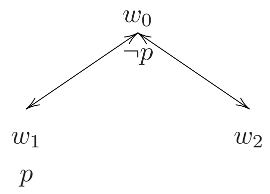
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0Rw_1, w_0Rw_2, w_1Rw_0, w_2Rw_0$$

$$v_{w_1}(p) = 1, v_{w_0}(p) = 0$$

This can be represented in the following diagram:



$\nvdash_{K_\tau} \Diamond p \supset \Box\Diamond p$

$$\neg(\Diamond p \supset \Box\Diamond p), 0$$
$$\Diamond p, 0$$
$$\neg\Box\Diamond p, 0$$
$$\Diamond\neg\Diamond p, 0$$
$$0r1$$
$$p, 1$$
$$0r2$$
$$\neg\Diamond p, 2$$
$$\Box\neg p, 2$$

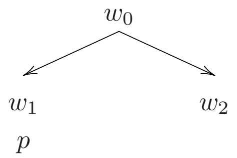
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_1, w_0 R w_2$$

$$v_{w_1}(p) = 1$$

This can be represented in the following diagram:



$\nvdash_{K_\eta} \Diamond p \supset \Box\Diamond p$

$$\begin{array}{c}
\neg(\Diamond p \supset \Box\Diamond p), 0 \\
\Diamond p, 0 \\
\neg\Box\Diamond p, 0 \\
\Diamond\neg\Diamond p, 0 \\
0\text{r}1 \\
p, 1 \\
0\text{r}2 \\
\neg\Diamond p, 2 \\
\Box\neg p, 2 \\
2\text{r}3 \\
\neg p, 3 \\
1\text{r}4,\ 3\text{r}5 \\
\vdots
\end{array}$$

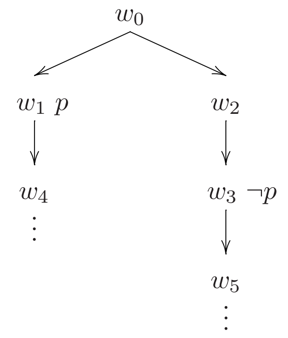
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, w_3, w_4, w_5 ...\}$$

$$w_0 R w_1, w_0 R w_2, w_2 R w_3, w_3 R w_5, w_1 R w_4 ...$$

$$v_{w_1}(p) = 1, v_{w_3}(p) = 0$$

This can be represented in the following diagram:



The characteristic formula valid in K_ϵ, is not valid in the other logics under consideration. Therefore, K_ϵ is distinct from all the systems above.

■

20. *Show that if a relation is reflexive and Euclidean then it is (a) symmetric and (b) transitive. Infer that $K_{\rho\epsilon}$, $K_{\rho\sigma\epsilon}$, $K_{\rho\epsilon\tau}$, and $K_{\epsilon\rho\sigma\tau}$ are all the same. Infer also that $K_{\rho\tau}$ is a subsystem of $K_{\rho\epsilon}$. Show that the converse is false.

Suppose that R is reflexive and Euclidean:

R is symmetric: Suppose that xRy. Then xRx by reflexivity. So yRx by Euclidianness.

R is transitive: Suppose that xRy and yRz. Then yRx by symmetry. Because yRx and yRz, by Euclideanness, xRz.

Thus, if a relation is reflexive and Euclidean, it is both symmetric and transitive.

Part 1: Infer that $K_{\rho\epsilon}$, $K_{\rho\sigma\epsilon}$, $K_{\rho\epsilon\tau}$, and $K_{\epsilon\rho\sigma\tau}$ are all the same.

If an inference is valid in $K_{\rho\epsilon}$ it is valid in all the extensions of $K_{\rho\epsilon}$. $K_{\rho\epsilon\tau}$, $K_{\rho\sigma\epsilon}$, and $K_{\epsilon\rho\sigma\tau}$ are clearly extensions.

If an inference is invalid for $K_{\rho\epsilon}$ we have a countermodel for it. Because the R in the countermodel interpretation satisfies the ρ and ϵ constraints, as shown above, R also satisfies the σ and τ constraints. Therefore the interpretation is a countermodel in all of the extensions of $K_{\rho\epsilon}$ with σ and τ above.

■

Infer also that $K_{\rho\tau}$ is a subsystem of $K_{\rho\epsilon}$, and show the converse is false.

Suppose an inference is invalid in $K_{\rho\epsilon}$ Consider an open branch of the tableau. The interpretation induced by that branch is a $K_{\rho\tau}$ interpretation, by the reasoning above. So the inference is also invalid in $K_{\rho\tau}$

The converse is false — $K_{\rho\epsilon}$ is not a subsystem of $K_{\rho\tau}$. This can be shown by finding an inference that is valid in $K_{\rho\epsilon}$, and invalid in $K_{\rho\tau}$.

The characteristic inference for ϵ will do the trick:

$\vdash_{K_\epsilon} \Diamond A \supset \Box\Diamond A$ (shown above)

$\nvdash_{K_{\rho\tau}} \Diamond p \supset \Box\Diamond p$:

$$\begin{gathered}
\neg(\Diamond p \supset \Box\Diamond p), 0 \\
\text{0r0} \\
\Diamond p, 0 \\
\neg\Box\Diamond p, 0 \\
\Diamond\neg\Diamond p, 0 \\
\text{0r1, 1r1} \\
p, 1 \\
\text{0r2, 2r2} \\
\neg\Diamond p, 2 \\
\Box\neg p, 2 \\
p, 2
\end{gathered}$$

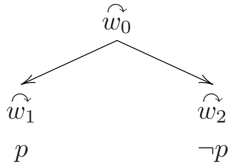
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_1, w_0 R w_2, w_0 R w_0, w_1 R w_1, w_2 R w_2$$

$$v_{w_1}(p) = 1, v_{w_2}(p) = 0$$

This can be represented in the following diagram:



Because $K_{\rho\epsilon}$ makes an inference valid that $K_{\rho\tau}$ does not, $K_{\rho\epsilon}$ is not a subsystem of $K_{\rho\tau}$.

■