1. This exercise concerns combinations of relations.

(a) For each of ρ, σ, τ and η, produce a relation which satisfies one of these but none of the others (except that ρ implies η, so this case is impossible).

$\boxed{\rho}$ (η)

$w_0 \mathrm{R} w_1$, $w_1 \mathrm{R} w_2$, $w_0 \mathrm{R} w_0$, $w_1 \mathrm{R} w_1$, $w_2 \mathrm{R} w_2$

$$\overset{\curvearrowright}{w_0} \rightarrow \overset{\curvearrowright}{w_1} \rightarrow \overset{\curvearrowright}{w_2}$$

$\boxed{\sigma}$

$w_0 \mathrm{R} w_1$, $w_1 \mathrm{R} w_2$, $w_2 \mathrm{R} w_1$, $w_1 \mathrm{R} w_0$

$$w_0 \rightleftarrows w_1 \rightleftarrows w_2$$

$\boxed{\tau}$

$w_0 \mathrm{R} w_1$, $w_1 \mathrm{R} w_2$, $w_0 \mathrm{R} w_2$

$$\begin{tikzcd} w_0 \arrow[r] \arrow[rr, bend right] & w_1 \arrow[r] & w_2 \end{tikzcd}$$

$\boxed{\eta}$

$w_0 \mathrm{R} w_1$, $w_1 \mathrm{R} w_2$, $w_2 \mathrm{R} w_3$ ...

$$w_0 \rightarrow w_1 \rightarrow w_2 \rightarrow w_3 \rightarrow ...$$

(b) There are six pairs of these conditions: $\rho\sigma$, $\rho\tau$, $(\rho\eta)$, $\sigma\tau$, $\sigma\eta$, $\tau\eta$. Since ρ entails η, the third of these is simply ρ. For each of the five genuine compound pairs, produce a relation that satisfies this condition, but none of the others (except that any relation that is ρ must also be η.

$\boxed{\rho\sigma}$

$w_0 \mathrm{R} w_1$, $w_1 \mathrm{R} w_2$, $w_0 \mathrm{R} w_0$, $w_1 \mathrm{R} w_1$, $w_2 \mathrm{R} w_2$, $w_2 \mathrm{R} w_1$, $w_1 \mathrm{R} w_0$

$$\overset{\curvearrowright}{w_0} \rightleftarrows \overset{\curvearrowright}{w_1} \rightleftarrows \overset{\curvearrowright}{w_2}$$

$\boxed{\rho\tau}$

$w_0\mathrm{R}w_1$, $w_1\mathrm{R}w_2$, $w_0\mathrm{R}w_0$, $w_1\mathrm{R}w_1$, $w_2\mathrm{R}w_2$, $w_0\mathrm{R}w_2$

$$\begin{tikzcd}
w_0 \arrow[loop above] \arrow[r] \arrow[rr, bend right] & w_1 \arrow[loop above] \arrow[r] & w_2 \arrow[loop above]
\end{tikzcd}$$

$\boxed{\sigma\tau}$

$w_0Rw_1, w_1Rw_2, w_2Rw_1, w_1Rw_0, w_0Rw_2, w_2Rw_0$

$$\begin{tikzcd}
w_0 \arrow[r, shift left] \arrow[rr, bend right, leftrightarrow] & w_1 \arrow[l, shift left] \arrow[r, shift left] & w_2 \arrow[l, shift left]
\end{tikzcd}$$

$\boxed{\sigma\eta}$

$w_0Rw_1, w_1Rw_2, w_2Rw_1, w_1Rw_0 ...$

$$w_0 \rightleftarrows w_1 \rightleftarrows w_2 \rightleftarrows ...$$

$\boxed{\tau\eta}$

w_0Rw_1, w_1Rw_2, w_2Rw_3, w_3Rw_2, w_2Rw_1, w_1Rw_0, w_0Rw_2, w_0Rw_3, $w_1Rw_3 ...$

$$\begin{tikzcd}
w_0 \arrow[r] \arrow[rr, bend right] \arrow[rrr, bend right=50] & w_1 \arrow[r] \arrow[rr, bend right] & w_2 \arrow[r] & w_3 \arrow[r] & ...
\end{tikzcd}$$

(c) Check the following. There are four triples of these conditions: $\rho\sigma\tau$, $(\rho\sigma\eta)$, $(\rho\tau\eta)$, $(\sigma\tau\eta)$. Because ρ entails η, the middle two are simply $\rho\sigma$ and $\rho\tau$. Moreover, for the same reason, and because $\sigma\tau\eta$ entails ρ (as we noted in 3.2.6), the first and last are identical. (And for good measure, $\rho\sigma\tau\eta$ is simply $\rho\sigma\tau$ as well.) Hence, there is only one genuine triple.

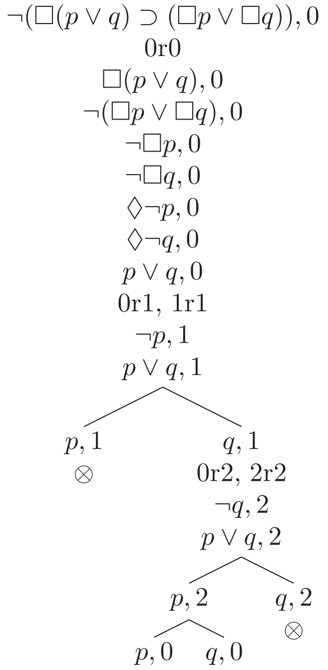
ρ specifies that all worlds are related to themselves. η specifies that all worlds are related to some world. If ρ, all worlds are related to themselves, then obviously η, all worlds are related to some world. See 3.2.6 "If a world accesses itself, it certainly accesses something."(Note the reverse is not true; It is not the case that if a world accesses something, it accesses itself.)

For the second part, the footnote to 3.2.6 explains: "Consider any world, w. By η, wRw' for some w'. So, by σ, $w'Rw$, and, by τ, wRw." The combination of the three conditions produces the ρ condition, and thus is equivalent to $\rho\sigma\tau\eta$. Additionally, because ρ entails η, both are equivalent to $\rho\sigma\tau$ - the first triple. Hence there is only one genuine triple.

2. Which of the inferences of 2.12, problems 2(l)-(v) hold in $K\rho$, $K\sigma, K\tau$ and $K\eta$? Check with appropriate tableaux. If a tableau does not close, define and draw a counter-model.

$\boxed{K\rho}$

(l) $\nvdash \Box(p \vee q) \supset (\Box p \vee \Box q)$



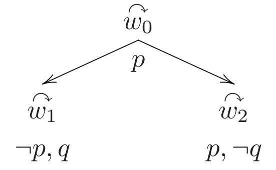
The following interpretation, taken from the left-most open branch, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_1, w_0 R w_2, w_0 R w_0, w_1 R w_1, w_2 R w_2$$

$$v_{w_0}(p) = 1, v_{w_1}(p) = 0, v_{w_1}(q) = 1, v_{w_2}(p) = 1, v_{w_2}(q) = 0$$

This can be represented in the following diagram:



(m) $\Box p, \Box\neg q \nvdash \Box(p \supset q)$

$$\Box p, 0$$
$$\Box\neg q, 0$$
$$\neg\Box(p \supset q), 0$$
$$0\text{r}0$$
$$p, 0$$
$$\neg q, 0$$
$$\Diamond\neg(p \supset q), 0$$
$$0\text{r}1,\ 1\text{r}1$$
$$\neg(p \supset q), 1$$
$$p, 1$$
$$\neg q, 1$$
$$p, 1$$
$$\neg q, 1$$

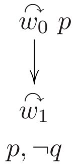
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

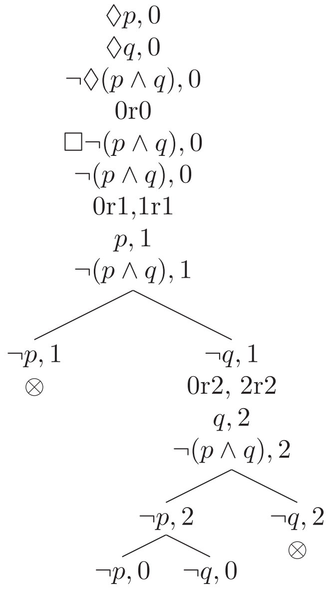
$$w_0 R w_1, w_0 R w_0, w_1 R w_1$$

$$v_{w_0}(p) = 1, v_{w_0}(q) = 0, v_{w_1}(p) = 1, v_{w_1}(q) = 0$$

This can be represented in the following diagram:



(n) $\Diamond p, \Diamond q \nvdash \Diamond(p \wedge q)$



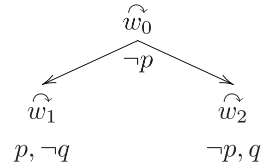
The following interpretation, taken from the left-most open branch, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0Rw_1, w_0Rw_2, w_0Rw_0, w_1Rw_1, w_2Rw_2$$

$$v_{w_0}(p) = 0, v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_2}(p) = 0, v_{w_2}(q) = 1$$

This can be represented in the following diagram:



(o) $\vdash \Box p \supset p$

$$\begin{array}{c}
\neg(\Box p \supset p), 0 \\
0\mathrm{r}0 \\
\Box p, 0 \\
\neg p, 0 \\
p, 0 \\
\otimes
\end{array}$$

(p) $\vdash \Box p \supset \Diamond p$

$$\begin{array}{c}
\neg(\Box p \supset \Diamond p), 0 \\
0\mathrm{r}0 \\
\Box p, 0 \\
\neg\Diamond p, 0 \\
\Box\neg p, 0 \\
p, 0 \\
\neg p, 0 \\
\otimes
\end{array}$$

(q) $p \nvdash \Box p$

$$\begin{array}{c}
p, 0 \\
\neg\Box p, 0 \\
0\mathrm{r}0 \\
\Diamond\neg p, 0 \\
0\mathrm{r}1,\ 1\mathrm{r}1 \\
\neg p, 1
\end{array}$$

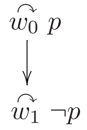
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

$$w_0 R w_1, w_0 R w_0, w_1 R w_1$$

$$v_{w_0}(p) = 1, v_{w_1}(p) = 0$$

This can be represented in the following diagram:



(r) $\nvdash \Box p \supset \Box\Box p$

$$\begin{array}{c}
\neg(\Box p \supset \Box\Box p), 0 \\
0\text{r}0 \\
\Box p, 0 \\
\neg\Box\Box p, 0 \\
\Diamond\neg\Box p, 0 \\
p, 0 \\
0\text{r}1, 1\text{r}1 \\
\neg\Box p, 1 \\
\Diamond\neg p, 1 \\
p, 1 \\
1\text{r}2,\ 2\text{r}2 \\
\neg p, 2
\end{array}$$

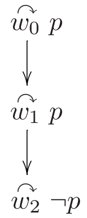
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0Rw_1, w_0Rw_0, w_1Rw_1, w_1Rw_2, w_2Rw_2$$

$$v_{w_0}(p) = 1, v_{w_1}(p) = 1, v_{w_2}(p) = 0$$

This can be represented in the following diagram:



(s) $\vdash \Diamond p \supset \Diamond\Diamond p$

$$\begin{gathered}
\neg(\Diamond p \supset \Diamond\Diamond p), 0 \\
0\mathrm{r}0 \\
\Diamond p, 0 \\
\neg\Diamond\Diamond p, 0 \\
\Box\neg\Diamond p, 0 \\
\neg\Diamond p, 0 \\
\Box\neg p, 0 \\
\neg p, 0 \\
0\mathrm{r}1,\ 1\mathrm{r}1 \\
p, 1 \\
\neg\Diamond p, 1 \\
\Box\neg p, 1 \\
\neg p, 1 \\
\otimes
\end{gathered}$$

(t) $\nvdash p \supset \Box\Diamond p$

$$\begin{gathered}
\neg(p \supset \Box\Diamond p), 0 \\
0\mathrm{r}0 \\
p, 0 \\
\neg\Box\Diamond p, 0 \\
\Diamond\neg\Diamond p, 0 \\
0\mathrm{r}1,\ 1\mathrm{r}1 \\
\neg\Diamond p, 1 \\
\Box\neg p, 1 \\
\neg p, 1
\end{gathered}$$

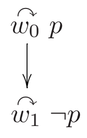
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

$$w_0 R w_1, w_0 R w_0, w_1 R w_1$$

$$v_{w_0}(p) = 1, v_{w_1}(p) = 0$$

This can be represented in the following diagram:



(u) $\nvdash \Diamond p \supset \Box\Diamond p$

$$\begin{array}{c}
\neg(\Diamond p \supset \Box\Diamond p), 0 \\
0\mathrm{r}0 \\
\Diamond p, 0 \\
\neg\Box\Diamond p, 0 \\
\Diamond\neg\Diamond p, 0 \\
0\mathrm{r}1,\ 1\mathrm{r}1 \\
p, 1 \\
0\mathrm{r}2,\ 2\mathrm{r}2 \\
\neg\Diamond p, 2 \\
\Box\neg p, 2 \\
\neg p, 2
\end{array}$$

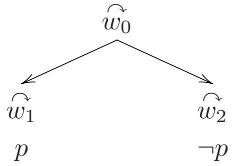
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_1, w_0 R w_0, w_1 R w_1, w_0 R w_2, w_2 R w_2$$

$$v_{w_1}(p) = 1, v_{w_2}(p) = 0$$

This can be represented in the following diagram:



(v) $\vdash \Diamond(p \vee \neg p)$

$$\begin{array}{c}
\neg\Diamond(p \vee \neg p), 0 \\
0\mathrm{r}0 \\
\Box\neg(p \vee \neg p), 0 \\
\neg(p \vee \neg p), 0 \\
\neg p, 0 \\
\neg\neg p, 0 \\
\otimes
\end{array}$$

$\boxed{K_\sigma}$

(l) $\nvdash \Box(p \lor q) \supset (\Box p \lor \Box q)$

$$\begin{array}{c}
\neg(\Box(p \lor q) \supset (\Box p \lor \Box q)), 0\\
\Box(p \lor q), 0\\
\neg(\Box p \lor \Box q), 0\\
\neg\Box p, 0\\
\neg\Box q, 0\\
\Diamond\neg p, 0\\
\Diamond\neg q, 0\\
\text{0r1, 1r0}\\
\neg p, 1\\
p \lor q, 1\\
\begin{array}{cc}
p, 1 & q, 1\\
\otimes & \text{0r2, 2r0}\\
 & \neg q, 2\\
 & p \lor q, 2\\
 & \begin{array}{cc} p, 2 & q, 2\\ & \otimes \end{array}
\end{array}
\end{array}$$

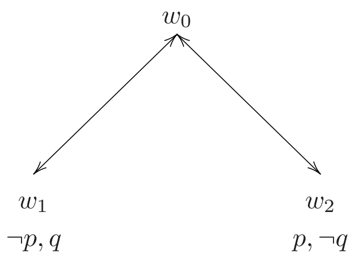
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_1, w_1 R w_0, w_0 R w_2, w_2 R w_0$$

$$v_{w_1}(p) = 0, v_{w_1}(q) = 1, v_{w_2}(p) = 1, v_{w_2}(q) = 0$$

This can be represented in the following diagram:



(m) $\Box p, \Box\neg q \nvdash \Box(p \supset q)$

$$\begin{gathered}
\Box p, 0 \\
\Box\neg q, 0 \\
\neg\Box(p \supset q), 0 \\
\Diamond\neg(p \supset q), 0 \\
0\mathrm{r}1,\ 1\mathrm{r}0 \\
\neg(p \supset q), 1 \\
p, 1 \\
\neg q, 1 \\
p, 1 \\
\neg q, 1
\end{gathered}$$

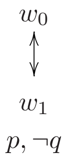
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

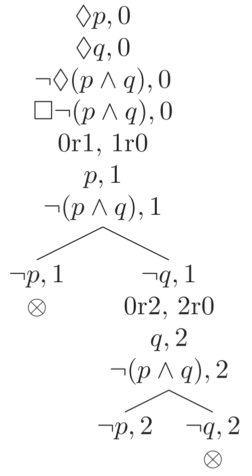
$$w_0 R w_1, w_1 R w_0$$

$$v_{w_1}(p) = 1, v_{w_1}(q) = 0$$

This can be represented in the following diagram:



(n) $\Diamond p, \Diamond q \nvdash \Diamond(p \wedge q)$



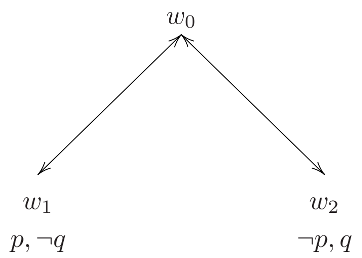
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0Rw_1, w_1Rw_0, w_0Rw_2, w_2Rw_0$$

$$v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_2}(p) = 0, v_{w_2}(q) = 1$$

This can be represented in the following diagram:



(o) $\nvdash \Box p \supset p$

$$\begin{gathered} \neg(\Box p \supset p), 0 \\ \Box p, 0 \\ \neg p, 0 \end{gathered}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0\}$$

$$v_{w_0}(p) = 0$$

This can be represented in the following diagram:

$$\begin{gathered} w_0 \\ \neg p \end{gathered}$$

(p) $\nvdash \Box p \supset \Diamond p$

$$\begin{gathered} \neg(\Box p \supset \Diamond p), 0 \\ \Box p, 0 \\ \neg \Diamond p, 0 \\ \Box \neg p, 0 \end{gathered}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0\}$$

This can be represented in the following diagram:

$$w_0$$

(q) $p \nvdash \Box p$

$p, 0$
$\neg\Box p, 0$
$\Diamond\neg p, 0$
0r1, 1r0
$\neg p, 1$

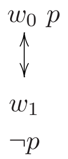
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

$$w_0 R w_1, w_1 R w_0$$

$$v_{w_0}(p) = 1, v_{w_1}(p) = 0$$

This can be represented in the following diagram:



(r) $\nvdash \Box p \supset \Box\Box p$

$\neg(\Box p \supset \Box\Box p), 0$
$\Box p, 0$
$\neg\Box\Box p, 0$
$\Diamond\neg\Box p, 0$
0r1, 1r0
$\neg\Box p, 1$
$\Diamond\neg p, 1$
$p, 1$
1r2, 2r1
$\neg p, 2$

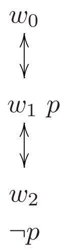
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_1, w_1 R w_0, w_1 R w_2, w_2 R w_1$$

$$v_{w_1}(p) = 1, v_{w_2}(p) = 0$$

This can be represented in the following diagram:



(s) $\nvdash \Diamond p \supset \Diamond\Diamond p$

$$\begin{array}{c}
\neg(\Diamond p \supset \Diamond\Diamond p), 0 \\
\Diamond p, 0 \\
\neg\Diamond\Diamond p, 0 \\
\Box\neg\Diamond p, 0 \\
0\text{r}1,\ 1\text{r}0 \\
p, 1 \\
\neg\Diamond p, 1 \\
\Box\neg p, 1 \\
\neg p, 0
\end{array}$$

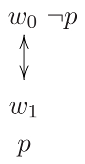
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

$$w_0 R w_1, w_1 R w_0$$

$$v_{w_0}(p) = 0, v_{w_1}(p) = 1$$

This can be represented in the following diagram:



(t) $\vdash p \supset \Box\Diamond p$

$$\begin{gathered}
\neg(p \supset \Box\Diamond p), 0 \\
p, 0 \\
\neg\Box\Diamond p, 0 \\
\Diamond\neg\Diamond p, 0 \\
0\mathrm{r}1,\ 1\mathrm{r}0 \\
\neg\Diamond p, 1 \\
\Box\neg p, 1 \\
\neg p, 0 \\
\otimes
\end{gathered}$$

(u) $\nvdash \Diamond p \supset \Box\Diamond p$

$$\begin{gathered}
\neg(\Diamond p \supset \Box\Diamond p), 0 \\
\Diamond p, 0 \\
\neg\Box\Diamond p, 0 \\
\Diamond\neg\Diamond p, 0 \\
0\mathrm{r}1,\ 1\mathrm{r}0 \\
p, 1 \\
0\mathrm{r}2,\ 2\mathrm{r}0 \\
\neg\Diamond p, 2 \\
\Box\neg p, 2 \\
\neg p, 0
\end{gathered}$$

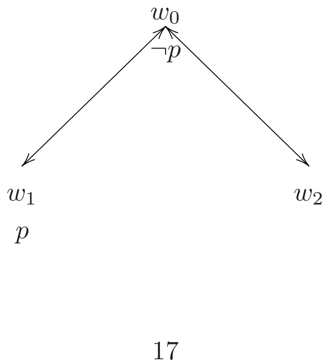
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_1, w_1 R 2_0, w_0 R w_2, w_2 R w_0$$

$$v_{w_0}(p) = 0, v_{w_1}(p) = 1$$

This can be represented in the following diagram:



(v) $\nvdash \Diamond(p \vee \neg p)$

$$\begin{array}{c} \neg\Diamond(p \vee \neg p), 0 \\ \Box\neg(p \vee \neg p), 0 \end{array}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0\}$$

This can be represented in the following diagram:

$$w_0$$

$\boxed{K_\tau}$

(l) $\nvdash \Box(p \vee q) \supset (\Box p \vee \Box q)$

$$\begin{array}{c} \neg(\Box(p \vee q) \supset (\Box p \vee \Box q)), 0 \\ \Box(p \vee q), 0 \\ \neg(\Box p \vee \Box q), 0 \\ \neg\Box p, 0 \\ \neg\Box q, 0 \\ \Diamond\neg p, 0 \\ \Diamond\neg q, 0 \\ 0r1 \\ \neg p, 1 \\ p \vee q, 1 \end{array}$$

$$\begin{array}{cc} p, 1 & q, 1 \\ \otimes & 0r2 \\ & \neg q, 2 \\ & p \vee q, 2 \\ & \begin{array}{cc} p, 2 & q, 2 \\ & \otimes \end{array} \end{array}$$

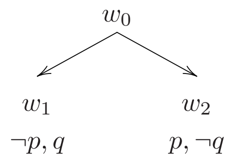
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_1, w_0 R w_2$$

$$v_{w_1}(p) = 0, v_{w_1}(q) = 1, v_{w_2}(p) = 1, v_{w_2}(q) = 0$$

This can be represented in the following diagram:



(m) $\Box p, \Box\neg q \nvdash \Box(p \supset q)$

$$\begin{gathered}
\Box p, 0 \\
\Box\neg q, 0 \\
\neg\Box(p \supset q), 0 \\
\Diamond\neg(p \supset q), 0 \\
0r1 \\
\neg(p \supset q), 1 \\
p, 1 \\
\neg q, 1 \\
p, 1 \\
\neg q, 1
\end{gathered}$$

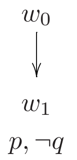
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

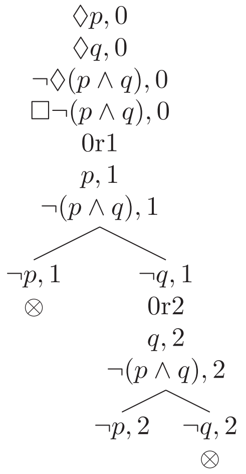
$$w_0 R w_1$$

$$v_{w_1}(p) = 1, v_{w_1}(q) = 0$$

This can be represented in the following diagram:



(n) $\Diamond p, \Diamond q \nvdash \Diamond(p \wedge q)$



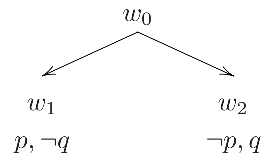
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_1, w_0 R w_2$$

$$v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_2}(p) = 0, v_{w_2}(q) = 1$$

This can be represented in the following diagram:



(o) $\nvdash \Box p \supset p$

$$\begin{gathered}\neg(\Box p \supset p), 0\\ \Box p, 0\\ \neg p, 0\end{gathered}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0\}$$

$$v_{w_0}(p) = 0$$

This can be represented in the following diagram:

$$\begin{gathered}w_0\\ \neg p\end{gathered}$$

(p) $\nvdash \Box p \supset \Diamond p$

$$\begin{gathered}\neg(\Box p \supset \Diamond p), 0\\ \Box p, 0\\ \neg\Diamond p, 0\\ \Box\neg p, 0\end{gathered}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0\}$$

This can be represented in the following diagram:

$$w_0$$

(q) $p \nvdash \Box p$

$$\begin{gathered}p, 0\\ \neg\Box p, 0\\ \Diamond\neg p, 0\\ 0\text{r}1\\ \neg p, 1\end{gathered}$$

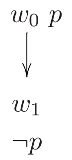
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

$$w_0 R w_1$$

$$v_{w_0}(p) = 1, v_{w_1}(p) = 0$$

This can be represented in the following diagram:



(r) $\vdash \Box p \supset \Box\Box p$

$$\begin{gathered}
\neg(\Box p \supset \Box\Box p), 0 \\
\Box p, 0 \\
\neg\Box\Box p, 0 \\
\Diamond\neg\Box p, 0 \\
0r1 \\
\neg\Box p, 1 \\
\Diamond\neg p, 1 \\
p, 1 \\
1r2,\ 0r2 \\
\neg p, 2 \\
p, 2 \\
\otimes
\end{gathered}$$

(s) $\nvdash \Diamond p \supset \Diamond\Diamond p$

$$\begin{gathered}
\neg(\Diamond p \supset \Diamond\Diamond p), 0 \\
\Diamond p, 0 \\
\neg\Diamond\Diamond p, 0 \\
\Box\neg\Diamond p, 0 \\
0r1 \\
p, 1 \\
\neg\Diamond p, 1 \\
\Box\neg p, 1
\end{gathered}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

$$w_0 R w_1$$

$$v_{w_1}(p) = 1$$

This can be represented in the following diagram:

(t) $\nvdash p \supset \Box\Diamond p$

$$\begin{gathered}
\neg(p \supset \Box\Diamond p), 0 \\
p, 0 \\
\neg\Box\Diamond p, 0 \\
\Diamond\neg\Diamond p, 0 \\
0\mathrm{r}1 \\
\neg\Diamond p, 1 \\
\Box\neg p, 1
\end{gathered}$$

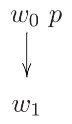
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

$$w_0 R w_1$$

$$v_{w_0}(p) = 1$$

This can be represented in the following diagram:



(u) $\nvdash \Diamond p \supset \Box\Diamond p$

$$\begin{gathered}\neg(\Diamond p \supset \Box\Diamond p), 0\\ \Diamond p, 0\\ \neg\Box\Diamond p, 0\\ \Diamond\neg\Diamond p, 0\\ 0\text{r}1\\ p, 1\\ 0\text{r}2\\ \neg\Diamond p, 2\\ \Box\neg p, 2\end{gathered}$$

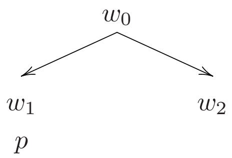
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_1, w_0 R w_2$$

$$v_{w_1}(p) = 1$$

This can be represented in the following diagram:



(v) $\nvdash \Diamond(p \vee \neg p)$

$$\begin{gathered}\neg\Diamond(p \vee \neg p), 0\\ \Box\neg(p \vee \neg p), 0\end{gathered}$$

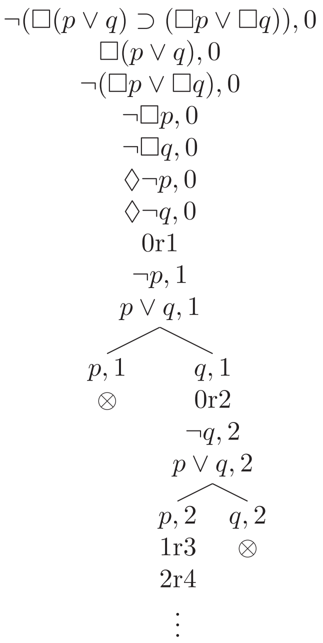
The following interpretation shows this inference to be invalid:

$$W = \{w_0\}$$

This can be represented in the following diagram:

$\boxed{K_\eta}$

(l) $\nvdash \Box(p \vee q) \supset (\Box p \vee \Box q)$



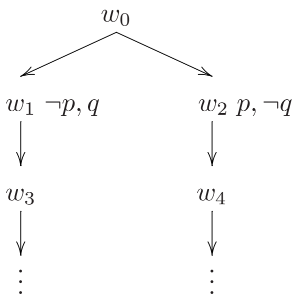
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, w_3, w_4 ...\}$$

$$w_0 R w_1, w_1 R w_3, w_0 R w_2, w_2 R w_4 ...$$

$$v_{w_1}(p) = 0, v_{w_1}(q) = 1, v_{w_2}(p) = 1, v_{w_2}(q) = 0$$

This can be represented in the following diagram:



(m) $\Box p, \Box\neg q \nvdash \Box(p \supset q)$

$$\begin{gathered}
\Box p, 0\\
\Box\neg q, 0\\
\neg\Box(p \supset q), 0\\
\Diamond\neg(p \supset q), 0\\
0\text{r}1\\
\neg(p \supset q), 1\\
p, 1\\
\neg q, 1\\
p, 1\\
\neg q, 1\\
1\text{r}2\\
\vdots
\end{gathered}$$

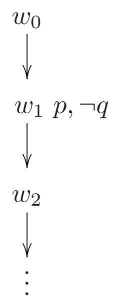
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2...\}$$

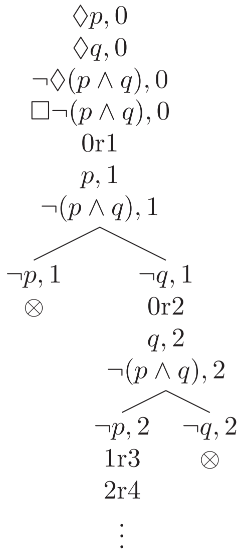
$$w_0 R w_1, w_1 R w_2...$$

$$v_{w_1}(p) = 1, v_{w_1}(q) = 0$$

This can be represented in the following diagram:



(n) $\Diamond p, \Diamond q \nvdash \Diamond(p \wedge q)$



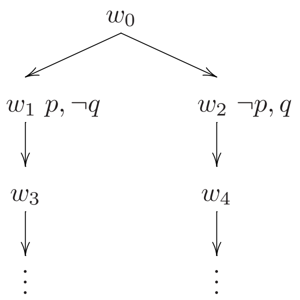
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, w_3, w_4...\}$$

$$w_0Rw_1, w_0Rw_2, w_1Rw_3, w_2Rw_4...$$

$$v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_2}(p) = 0, v_{w_2}(q) = 1$$

This can be represented in the following diagram:



(o) $\nvdash \Box p \supset p$

$$\begin{gathered}
\neg(\Box p \supset p), 0 \\
\Box p, 0 \\
\neg p, 0 \\
0\text{r}1 \\
p, 1 \\
1\text{r}2 \\
\vdots
\end{gathered}$$

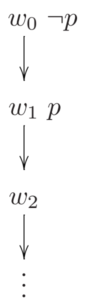
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2 ...\}$$

$$w_0 R w_1, w_1 R w_2 ...$$

$$v_{w_0}(p) = 0, v_{w_1}(p) = 1$$

This can be represented in the following diagram:



(p) $\vdash \Box p \supset \Diamond p$

$$\begin{gathered}
\neg(\Box p \supset \Diamond p), 0 \\
\Box p, 0 \\
\neg \Diamond p, 0 \\
\Box \neg p, 0 \\
0\text{r}1 \\
p, 1 \\
\neg p, 1 \\
\otimes
\end{gathered}$$

(q) $p \nvdash \Box p$

$$
\begin{gathered}
p, 0 \\
\neg\Box p, 0 \\
\Diamond\neg p, 0 \\
0\mathrm{r}1 \\
\neg p, 1 \\
1\mathrm{r}2 \\
\vdots
\end{gathered}
$$

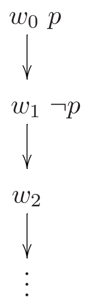
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2 ...\}$$

$$w_0 R w_1, w_1 R w_2 ...$$

$$v_{w_0}(p) = 1, v_{w_1}(p) = 0$$

This can be represented in the following diagram:



(r) $\nvdash \Box p \supset \Box\Box p$

$$\begin{array}{c}
\neg(\Box p \supset \Box\Box p), 0 \\
\Box p, 0 \\
\neg\Box\Box p, 0 \\
\Diamond\neg\Box p, 0 \\
0r1 \\
\neg\Box p, 1 \\
\Diamond\neg p, 1 \\
p, 1 \\
1r2 \\
\neg p, 2 \\
2r3 \\
\vdots
\end{array}$$

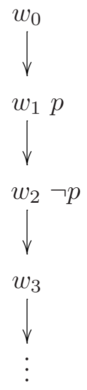
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, w_3...\}$$

$$w_0 R w_1, w_1 R w_2, w_2 R w_3...$$

$$v_{w_1}(p) = 1, v_{w_2}(p) = 0$$

This can be represented in the following diagram:



(s) $\nvdash \Diamond p \supset \Diamond\Diamond p$

$$\begin{gathered}
\neg(\Diamond p \supset \Diamond\Diamond p), 0 \\
\Diamond p, 0 \\
\neg\Diamond\Diamond p, 0 \\
\Box\neg\Diamond p, 0 \\
0\mathrm{r}1 \\
p, 1 \\
\neg\Diamond p, 1 \\
\Box\neg p, 1 \\
1\mathrm{r}2 \\
\neg p, 2 \\
2\mathrm{r}3 \\
\vdots
\end{gathered}$$

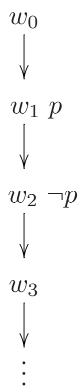
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, w_3...\}$$

$$w_0Rw_1, w_1Rw_2, w_2Rw_3...$$

$$v_{w_1}(p) = 1, v_{w_2}(p) = 0$$

This can be represented in the following diagram:



(t) $\nvdash p \supset \Box\Diamond p$

$$\begin{array}{c}
\neg(p \supset \Box\Diamond p), 0 \\
p, 0 \\
\neg\Box\Diamond p, 0 \\
\Diamond\neg\Diamond p, 0 \\
0\text{r}1 \\
\neg\Diamond p, 1 \\
\Box\neg p, 1 \\
1\text{r}2 \\
\neg p, 2 \\
2\text{r}3 \\
\vdots
\end{array}$$

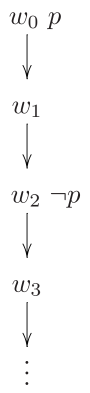
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, w_3 ...\}$$

$$w_0 R w_1, w_1 R w_2, w_2 R w_3 ...$$

$$v_{w_0}(p) = 1, v_{w_2}(p) = 0$$

This can be represented in the following diagram:



(u) $\nvdash \Diamond p \supset \Box\Diamond p$

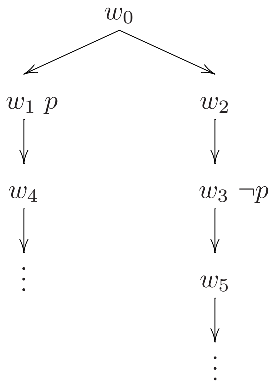
$$\begin{gathered}
\neg(\Diamond p \supset \Box\Diamond p), 0 \\
\Diamond p, 0 \\
\neg\Box\Diamond p, 0 \\
\Diamond\neg\Diamond p, 0 \\
0\mathrm{r}1 \\
p, 1 \\
0\mathrm{r}2 \\
\neg\Diamond p, 2 \\
\Box\neg p, 2 \\
2\mathrm{r}3 \\
\neg p, 3 \\
1\mathrm{r}4 \\
3\mathrm{r}5 \\
\vdots
\end{gathered}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2, w_3, w_4, w_5...\}$$

$$w_0Rw_1, w_1Rw_4, w_0Rw_2, w_2Rw_3, w_3Rw_5...$$
$$v_{w_1}(p) = 1, v_{w_3}(p) = 0$$

This can be represented in the following diagram:



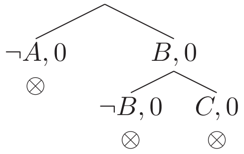
(v) $\vdash \Diamond(p \vee \neg p)$

$$\neg\Diamond(p \vee \neg p), 0$$
$$\Box\neg(p \vee \neg p), 0$$
$$0r1$$
$$\neg(p \vee \neg p), 1$$
$$\neg p, 1$$
$$\neg\neg p, 1$$
$$\otimes$$

3. Show the following in K_ρ:

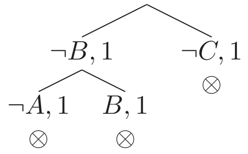
(a) $\vdash (\Box(A \supset B) \wedge \Box(B \supset C)) \supset (A \supset C)$

$$\neg((\Box(A \supset B) \wedge \Box(B \supset C)) \supset (A \supset C)), 0$$
$$0r0$$
$$(\Box(A \supset B) \wedge \Box(B \supset C), 0$$
$$\neg(A \supset C), 0$$
$$A, 0$$
$$\neg C, 0$$
$$\Box(A \supset B), 0$$
$$\Box(B \supset C), 0$$
$$A \supset B, 0$$
$$B \supset C, 0$$



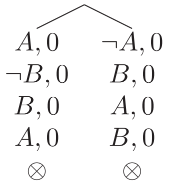
(b) $\vdash (\Box(A \supset B) \wedge \Diamond(A \wedge C)) \supset \Diamond(B \wedge C)$

$$\neg((\Box(A \supset B) \wedge \Diamond(A \wedge C)) \supset \Diamond(B \wedge C)), 0$$
$$0r0$$
$$\Box(A \supset B) \wedge \Diamond(A \wedge C), 0$$
$$\neg\Diamond(B \wedge C), 0$$
$$\Box\neg(B \wedge C), 0$$
$$\Box(A \supset B), 0$$
$$\Diamond(A \wedge C), 0$$
$$0r1,\ 1r1$$
$$A \wedge C, 1$$
$$\neg(B \wedge C), 1$$
$$A \supset B, 1$$
$$A, 1$$
$$C, 1$$

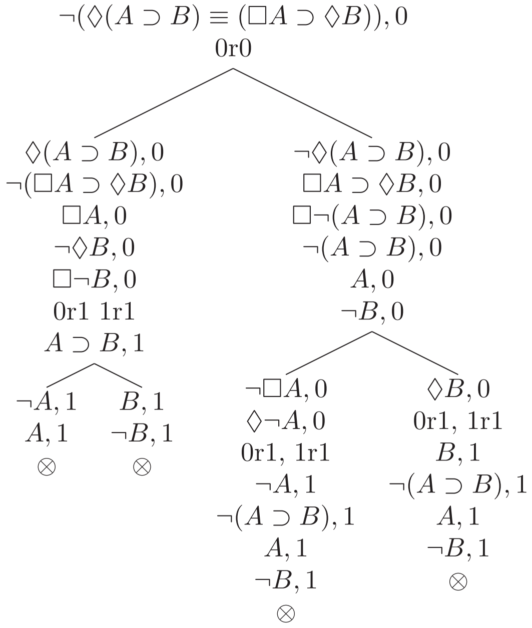


(c) $\vdash (\Box A \wedge \Box B) \supset (A \equiv B)$

$$\neg((\Box A \wedge \Box B) \supset (A \equiv B)), 0$$
$$0r0$$
$$\Box A \wedge \Box B, 0$$
$$\neg(A \equiv B), 0$$
$$\Box A, 0$$
$$\Box B, 0$$



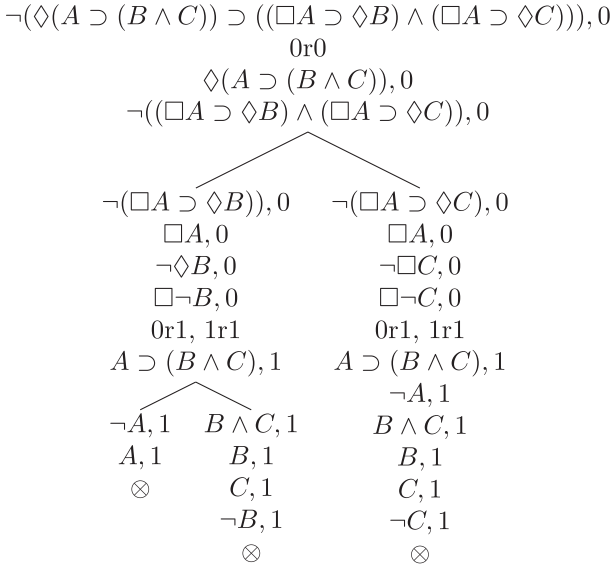
(d) $\vdash \Diamond(A \supset B) \equiv (\Box A \supset \Diamond B)$



(e) $\vdash (\Diamond\neg A \vee \Diamond\neg B) \vee \Diamond(A \vee B)$

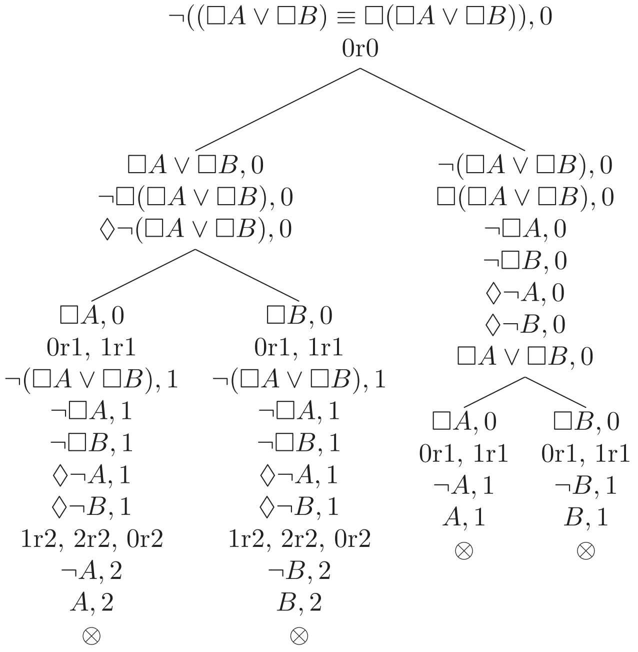
$\neg((\Diamond\neg A \vee \Diamond\neg B) \vee \Diamond(A \vee B)), 0$
0r0
$\neg(\Diamond\neg A \vee \Diamond\neg B), 0$
$\neg\Diamond(A \vee B), 0$
$\Box\neg(A \vee B), 0$
$\neg\Diamond\neg A, 0$
$\neg\Diamond\neg B, 0$
$\Box\neg\neg A, 0$
$\Box\neg\neg B, 0$
$\neg(A \vee B), 0$
$\neg\neg A, 0$
$\neg\neg B, 0$
$\neg A, 0$
$\neg B, 0$
$A, 0$
$B, 0$
⊗

(f) $\vdash \Diamond(A \supset (B \wedge C)) \supset ((\Box A \supset \Diamond B) \wedge (\Box A \supset \Diamond C))$



4. Show the following in $K_{\rho\tau}$:

(a) $\vdash (\Box A \vee \Box B) \equiv \Box(\Box A \vee \Box B)$



(b) $\vdash \Box(\Box(A \equiv B) \supset C) \supset (\Box(A \equiv B) \supset \Box C)$

$$\begin{array}{c}
\neg(\Box(\Box(A \equiv B) \supset C) \supset (\Box(A \equiv B) \supset \Box C)), 0 \\
0\text{r}0 \\
\Box(\Box(A \equiv B) \supset C), 0 \\
\neg(\Box(A \equiv B) \supset \Box C), 0 \\
\Box(A \equiv B), 0 \\
\neg\Box C, 0 \\
\Diamond\neg C, 0 \\
0\text{r}1,\ 1\text{r}1 \\
\neg C, 1 \\
A \equiv B, 1 \\
\Box(A \equiv B) \supset C, 1 \\
\begin{array}{cc}
\neg\Box(A \equiv B), 1 & C, 1 \\
\Diamond\neg(A \equiv B), 1 & \otimes \\
1\text{r}2,\ 2\text{r}2,\ 0\text{r}2 & \\
\neg(A \equiv B), 2 & \\
A \equiv B, 2 & \\
\otimes &
\end{array}
\end{array}$$

5. Show the following in K_υ:

(a) $\vdash \Diamond A \supset \Diamond\Diamond A$

$$\begin{array}{c}
\neg(\Diamond A \supset \Diamond\Diamond A), 0 \\
\Diamond A, 0 \\
\neg\Diamond\Diamond A, 0 \\
\Box\neg\Diamond A, 0 \\
\neg\Diamond A, 0 \\
\otimes
\end{array}$$

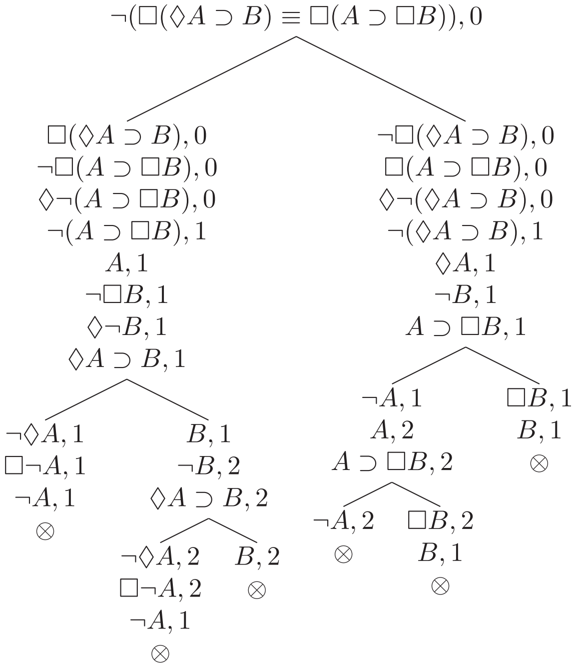
(b) $\vdash \Diamond A \supset \Box\Diamond A$

$$\begin{array}{c}
\neg(\Diamond A \supset \Box\Diamond A), 0 \\
\Diamond A, 0 \\
\neg\Box\Diamond A, 0 \\
\Diamond\neg\Diamond A, 0 \\
\neg\Diamond A, 1 \\
\Box\neg A, 1 \\
A, 2 \\
\neg A, 2 \\
\otimes
\end{array}$$

(c) $\vdash \Box(\Box A \supset \Box B) \vee \Box(\Box B \supset \Box A)$

$$\begin{gathered}
\neg(\Box(\Box A \supset \Box B) \vee \Box(\Box B \supset \Box A)), 0 \\
\neg\Box(\Box A \supset \Box B), 0 \\
\neg\Box(\Box B \supset \Box A), 0 \\
\Diamond\neg(\Box A \supset \Box B), 0 \\
\Diamond\neg(\Box B \supset \Box A), 0 \\
\neg(\Box A \supset \Box B), 1 \\
\Box A, 1 \\
\neg\Box B, 1 \\
\Diamond\neg B, 1 \\
\neg B, 2 \\
\neg(\Box B \supset \Box A), 3 \\
\Box B, 3 \\
\neg\Box A, 3 \\
B, 2 \\
\otimes
\end{gathered}$$

(d) $\vdash \Box(\Diamond A \supset B) \equiv \Box(A \supset \Box B)$



6. Which of the following hold in $K_{\rho\tau}$?

(a) $\nvdash \Diamond\Box p \supset \Box\Diamond p$

$$\begin{array}{c}
\neg(\Diamond\Box p \supset \Box\Diamond p), 0 \\
0r0 \\
\Diamond\Box p, 0 \\
\neg\Box\Diamond p, 0 \\
\Diamond\neg\Diamond p, 0 \\
0r1,\ 1r1 \\
\Box p, 1 \\
p, 1 \\
0r2,\ 2r2 \\
\neg\Diamond p, 2 \\
\Box\neg p, 2 \\
\neg p, 2
\end{array}$$

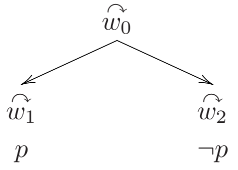
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0Rw_0, w_0Rw_1, w_1Rw_1, w_0Rw_2, w_2Rw_2$$

$$v_{w_1}(p) = 1, v_{w_2}(p) = 0$$

This can be represented in the following diagram:



(b) $\nvdash \Box(\Box p \supset q) \vee \Box(\Box q \supset p)$

$$\neg(\Box(\Box p \supset q) \vee \Box(\Box q \supset p)), 0$$
$$0\mathrm{r}0$$
$$\neg\Box(\Box p \supset q), 0$$
$$\neg\Box(\Box q \supset p), 0$$
$$\Diamond\neg(\Box p \supset q), 0$$
$$\Diamond\neg(\Box q \supset p), 0$$
$$0\mathrm{r}1,\ 1\mathrm{r}1$$
$$\neg(\Box p \supset q), 1$$
$$\Box p, 1$$
$$\neg q, 1$$
$$p, 1$$
$$0\mathrm{r}2,\ 2\mathrm{r}2$$
$$\neg(\Box q \supset p), 2$$
$$\Box q, 2$$
$$\neg p, 2$$
$$q, 2$$

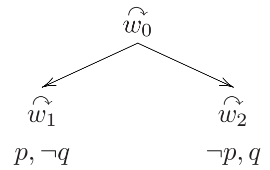
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

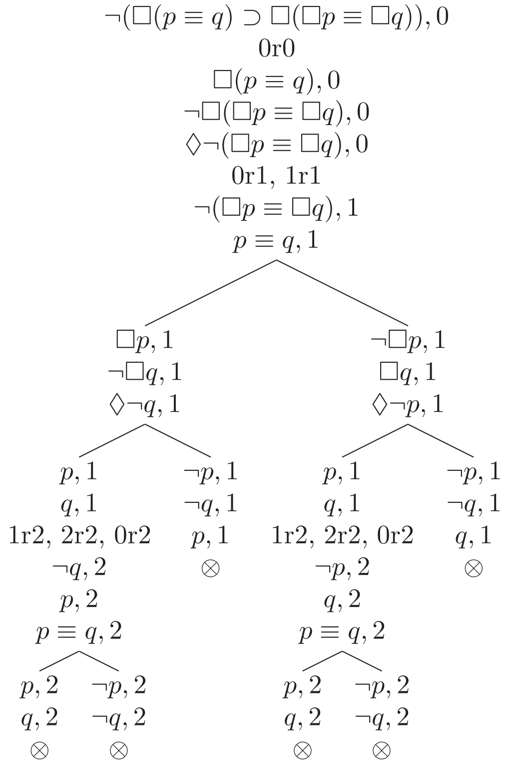
$$w_0Rw_0, w_0Rw_1, w_1Rw_1, w_0Rw_2, w_2Rw_2$$

$$v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_2}(p) = 0, v_{w_2}(q) = 1$$

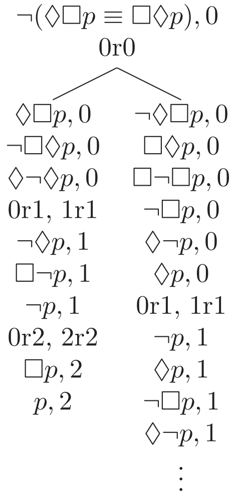
This can be represented in the following diagram:



(c) $\vdash \Box(p \equiv q) \supset \Box(\Box p \equiv \Box q)$



(d) $\nvdash \Diamond\Box p \equiv \Box\Diamond p$



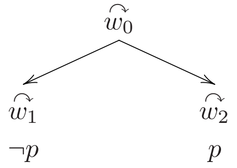
The following interpretation, taken from the finite left-hand branch, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0Rw_0, w_0Rw_1, w_1Rw_1, w_0Rw_2, w_2Rw_2$$

$$v_{w_1}(p) = 0, v_{w_2}(p) = 1$$

This can be represented in the following diagram:



7. The following exercises concern the relationships between various normal modal logics.

(a) If R is reflexive (ρ), it is extendable (η). Hence, if truth is preserved at all worlds of all η-interpretations, it is preserved at all worlds of all ρ-interpretations. Consequently, the system K_ρ is an extension of the system K_η. Find an inference demonstrating that it is a proper extension.

$\boxed{\Box A \supset A}$

$\vdash_{K_\rho} (\Box A \supset A)$

$$\begin{gathered} \neg(\Box A \supset A), 0 \\ 0\text{r}0 \\ \Box A, 0 \\ \neg A, 0 \\ A, 0 \\ \otimes \end{gathered}$$

$\nvdash_{K_\eta} (\Box A \supset A)$

$$\begin{gathered} \neg(\Box A \supset A), 0 \\ \Box A, 0 \\ \neg A, 0 \\ 0\text{r}1 \\ A, 1 \\ 1\text{r}2 \\ \vdots \end{gathered}$$

$$W = \{w_0, w_1, w_2 ...\}$$

$$w_0 R w_1, w_1 R w_2 ...$$

$$v_{w_0}(p) = 0, v_{w_1}(p) = 1$$

$$\begin{array}{llll} w_0 \rightarrow & w_1 \rightarrow & w_2 \rightarrow & ... \\ \neg p & p & & \end{array}$$

■

This interpretation shows that it is not the case that if truth is preserved at all worlds of all ρ-interpretations, it is preserved at all worlds of all η-interpretations: i.e. that ρ is a proper extension of η.

(b) Show that none of the systems K_ρ, K_σ and K_τ is an extension of any of the others (i.e., for each pair, find an inference that is valid in one but not the other, and then vice versa).(Hint: see 3.5.10.)

$$\boxed{\text{There is at least one inference valid in } K_\rho \text{ that is not valid in } K_\sigma \text{ or } K_\tau}$$

$\vdash_{K_\rho} \Box p \supset p$

$$\begin{array}{c} \neg(\Box p \supset p), 0 \\ 0r0 \\ \Box p, 0 \\ \neg p, 0 \\ p, 0 \\ \otimes \end{array}$$

The same tableau shows this inference to be invalid in both K_σ and K_τ:

$\nvdash_{K_\sigma} \Box p \supset p$

$\nvdash_{K_\tau} \Box p \supset p$

$$\neg(\Box p \supset p), 0$$
$$\Box p, 0$$
$$\neg p, 0$$

$$W = \{w_0\}$$

$$v_{w_0}(p) = 0$$

$$w_0$$
$$\neg p$$

$\blacksquare$

$\boxed{\text{There is at least one inference valid in } K_\sigma \text{ that is not valid in } K_\rho \text{ or } K_\tau}$

$\vdash_{K_\sigma} p \supset \Box\Diamond p$

$$\neg(p \supset \Box\Diamond p), 0$$
$$p, 0$$
$$\neg\Box\Diamond p, 0$$
$$\Diamond\neg\Diamond p, 0$$
$$0r1,\ 1r0$$
$$\neg\Diamond p, 1$$
$$\Box\neg p, 1$$
$$\neg p, 0$$
$$\otimes$$

$\nvdash_{K_\rho} p \supset \Box\Diamond p$

$$\neg(p \supset \Box\Diamond p), 0$$
$$0r0$$
$$p, 0$$
$$\neg\Box\Diamond p, 0$$
$$\Diamond\neg\Diamond p, 0$$
$$0r1,\ 1r1$$
$$\neg\Diamond p, 1$$
$$\Box\neg p, 1$$
$$\neg p, 1$$

$$W = \{w_0, w_1\}$$

$$w_0Rw_1, w_0Rw_0, w_1Rw_1$$

$$v_{w_0}(p) = 1, v_{w_1}(p) = 0$$

$$\begin{array}{ccc} \overset{\curvearrowright}{w_0} & \to & \overset{\curvearrowright}{w_1} \\ p & & \neg p \end{array}$$

$\nvdash_{K_\tau} p \supset \Box\Diamond p$

$$\begin{array}{c} \neg(p \supset \Box\Diamond p), 0 \\ p, 0 \\ \neg\Box\Diamond p, 0 \\ \Diamond\neg\Diamond p, 0 \\ 0\text{r}1 \\ \neg\Diamond p, 1 \\ \Box\neg p, 1 \end{array}$$

$$W = \{w_0, w_1\}$$

$$w_0Rw_1$$

$$v_{w_0}(p) = 1$$

$$\begin{array}{ccc} w_0 & \to & w_1 \\ p & & \end{array}$$

■

$\boxed{\text{There is at least one inference valid in } K_\tau \text{ that is not valid in } K_\rho \text{ or } K_\sigma}$

$\vdash_{K_\tau} \Box p \supset \Box\Box p$

$$\begin{array}{c}
\neg(\Box p \supset \Box\Box p), 0 \\
\Box p, 0 \\
\neg\Box\Box p, 0 \\
\Diamond\neg\Box p, 0 \\
0r1 \\
\neg\Box p, 1 \\
p, 1 \\
\Diamond\neg p, 1 \\
1r2,\ 0r2 \\
\neg p, 2 \\
p, 2 \\
\otimes
\end{array}$$

$\nvdash_{K_\rho} \Box p \supset \Box\Box p$

$$\begin{array}{c}
\neg(\Box p \supset \Box\Box p), 0 \\
0r0 \\
\Box p, 0 \\
\neg\Box\Box p, 0 \\
\Diamond\neg\Box p, 0 \\
p, 0 \\
0r1,\ 1r1 \\
\neg\Box p, 1 \\
p, 1 \\
\Diamond\neg p, 1 \\
1r2,\ 2r2 \\
\neg p, 2
\end{array}$$

$$W = \{w_0, w_1, w_2\}$$

$$w_0Rw_1, w_0Rw_0, w_1Rw_1, w_1Rw_2, w_2Rw_2$$

$$v_{w_0}(p) = 1, v_{w_1}(p) = 1, v_{w_2}(p) = 0$$

$$\begin{array}{ccccc}
\overset{\curvearrowright}{w_0} & \rightarrow & \overset{\curvearrowright}{w_1} & \rightarrow & \overset{\curvearrowright}{w_2} \\
p & & p & & \neg p
\end{array}$$

$\nvdash_{K_\sigma} \Box p \supset \Box\Box p$

$$\neg(\Box p \supset \Box\Box p), 0$$
$$\Box p, 0$$
$$\neg\Box\Box p, 0$$
$$\Diamond\neg\Box p, 0$$
$$0\text{r}1,\ 1\text{r}0$$
$$\neg\Box p, 1$$
$$p, 1$$
$$\Diamond\neg p, 1$$
$$1\text{r}2,\ 2\text{r}1$$
$$\neg p, 2$$

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_1, w_1 R w_0, w_1 R w_2, w_2 R w_1$$

$$v_{w_1}(p) = 1, v_{w_2} = 0$$

$$\begin{array}{ccccc} w_0 & \rightleftarrows & w_1 & \rightleftarrows & w_2 \\ & & p & & \neg p \end{array}$$

■

In each of the three systems there is at least one inference that is valid under that system, but not under the other two. Therefore, none of the three are extensions of any of the others.

(c) By combining the individual conditions, we obtain the systems $K_{\rho\sigma}$, $K_{\rho\tau}$, $K_{\sigma\tau}$, $K_{\sigma\eta}$, and $K_{\tau\eta}$ (see problem 1(b)). $K_{\rho\sigma}$ is an extension of K_ρ and K_σ. Show that it is a proper extension of each of these. Do the same for the other four binary systems. Show that $K_{\rho\sigma}$is a proper extension of $K_{\eta\sigma}$, and that $K_{\rho\tau}$ is a proper extension of $K_{\eta\tau}$. Show that none of the other binary systems is an extension of any other.

In the previous exercise, we found inferences which were valid in only one of the unary systems K_ρ, K_σ and K_τ.

The corresponding inference for K_η is

$\vdash_{K_\eta} \Box p \supset \Diamond p$

$$\begin{array}{c}
\neg(\Box p \supset \Diamond p), 0 \\
\Box p, 0 \\
\neg\Diamond p, 0 \\
\Box\neg p, 0 \\
0\mathrm{r}1 \\
\neg p, 1 \\
p, 1 \\
\otimes
\end{array}$$

The same tableau shows this inference to be invalid in both K_σ and K_τ:

$\nvdash_{K_\sigma} \Box p \supset \Diamond p$

$\nvdash_{K_\tau} \Box p \supset \Diamond p$

$$\begin{array}{c}
\neg(\Box p \supset \Diamond p), 0 \\
\Box p, 0 \\
\neg\Diamond p, 0 \\
\Box\neg p, 0
\end{array}$$

$$W = \{w_0\}$$

$$w_0$$

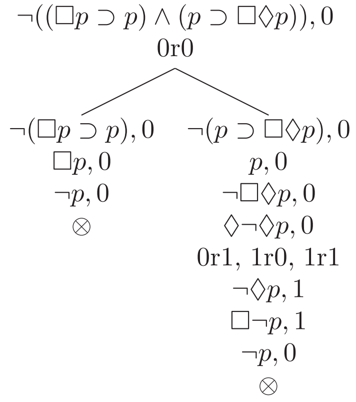
This inference is valid in K_ρ, because K_ρ is an extension of K_η.

We now have characteristic inferences for all four unary systems. Back to the question

> "$K_{\rho\sigma}$ is an extension of K_ρ and K_σ. Show that it is a proper extension of each of these. Do the same for the other four binary systems."

The conjunction of the sentences found in the previous exercise will do the trick:

$\vdash_{K_{\rho\sigma}} (\Box p \supset p) \wedge (p \supset \Box\Diamond p)$



This inference is a conjunction of two sentences, both of which have been seen in the last exercise to be true in only one unary system. Thus, all unary systems will make at least half of the conjunction false, and render the inference invalid.

A similar inference can be generated for all binary systems, from the conjunction of 'characteristic' sentences found in the last exercise (and above) for its constitutive unary systems. The inference will be valid in the relevant binary system, and invalid in the constitutive unary systems, showing that all the binary systems are proper extensions of their constitutive unary systems. (Binary systems which include η will not be extensions of the same systems with η substituted for ρ - but this is only relevent in the next part of the question.)

> "Show that $K_{\rho\sigma}$is a proper extension of $K_{\eta\sigma}$, and that $K_{\rho\tau}$ is a proper extension of $K_{\eta\tau}$."

$\Box p \supset p$ is, as we have seen, a tautology in K_ρ. It is also a tautology in $K_{\rho\sigma}$ because$K_{\rho\sigma}$ is an extension of K_ρ. It is invalid in $K_{\eta\sigma}$, as can be seen in the following tree diagram:

$\nvdash_{K_{\eta\sigma}} \Box p \supset p$

$$\begin{array}{c} \neg(\Box p \supset p), 0 \\ \Box p, 0 \\ \neg p, 0 \\ 0\text{r}1,\ 1\text{r}0 \\ p, 1 \\ 1\text{r}2,\ 2\text{r}1 \\ \vdots \end{array}$$

$$W = \{w_0, w_1, w_2 ...\}$$

$$w_0Rw_1, w_1Rw_0, w_1Rw_2, w_2Rw_1 ...$$

$$v_{w_0}(p) = 0, v_{w_1}(p) = 1$$

$$\begin{array}{ccccc} w_0 & \leftrightarrows & w_1 & \leftrightarrows & ... \\ \neg p & & p & & \end{array}$$

Since there is an inference which is valid in $K_{\rho\sigma}$ and invalid in $K_{\eta\sigma}$, $K_{\rho\sigma}$ is a proper extension of $K_{\eta\sigma}$.

■

The same is true of $K_{\rho\tau}$ and $K_{\eta\tau}$:

$\vdash_{K_{\rho\tau}} (\Box p \supset p)$, because $K_{\rho\tau}$ is an extension of K_ρ

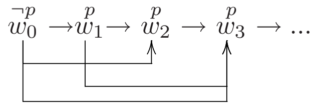
$K_{\eta\tau} \nvdash \Box p \supset p$

$$\begin{array}{c} \neg(\Box p \supset p), 0 \\ \Box p, 0 \\ \neg p, 0 \\ 0\text{r}1 \\ p, 1 \\ 1\text{r}2,\ 0\text{r}2 \\ p, 2 \\ 2\text{r}3,\ 0\text{r}3,\ 1\text{r}3 \\ \vdots \end{array}$$

$$W = \{w_0, w_1, w_2, w_3 ...\}$$

$$w_0Rw_1, w_1Rw_2, w_0Rw_2, w_2Rw_3, w_0R2_3, w_1Rw_3 ...$$

$$v_{w_0}(p) = 0, v_{w_1}(p) = 1, v_{w_2}(p) = 1, v_{w_3}(p) = 1 ...$$



■

Thus the conjunction of the two ‘distinctive’ sentences for the unary systems shows that $K_{\rho\sigma}$is a proper extension of $K_{\eta\sigma}$, and that $K_{\rho\tau}$ is a proper extension of $K_{\eta\tau}$.

Back to the question for the last time:

> “Show that none of the other binary systems is an extension of any other.”

There are 5 binary systems, $K_{\rho\sigma}$, $K_{\rho\tau}$, $K_{\sigma\tau}$, $K_{\sigma\eta}$, and $K_{\tau\eta}$. We showed that $K_{\rho\sigma}$is a proper extension of $K_{\eta\sigma}$, and that $K_{\rho\tau}$ is a proper extension of $K_{\eta\tau}$ in the last part of the question. Accordingly, if a system is not an extension of $K_{\rho\sigma}$, it is not an extension of $K_{\eta\sigma}$, and if a system is not an extension of $K_{\rho\tau}$, it is not an extension of $K_{\eta\tau}$. Therefore, there are only three systems that need to be shown to be mutually exclusive: $K_{\sigma\tau}$, $K_{\rho\sigma}$, and $K_{\rho\tau}$. To show this, it will suffice to find an inference that is valid in one, but not in the other, for each pair.

$\boxed{K_{\sigma\tau}}$

$\vdash_{K_{\sigma\tau}} \Box p \supset \Box\Box p$ (Because this inference is valid in K_τ, as we have seen, and $K_{\sigma\tau}$ is an extension of K_τ)

$\nvdash_{K_{\rho\sigma}} \Box p \supset \Box\Box p$

$$\neg(\Box p \supset \Box\Box p)$$
$$0r0$$
$$\Box p, 0$$
$$\neg\Box\Box p, 0$$
$$\Diamond\neg\Box p, 0$$
$$0r1,\ 1r1,\ 1r0$$
$$\neg\Box p, 1$$
$$p, 1$$
$$\Diamond\neg p, 1$$
$$1r2,\ 2r2,\ 2r1$$
$$\neg p, 2$$

■

$\vdash_{K_{\sigma\tau}} p \supset \Box\Diamond p$ (Because this inference is valid in K_σ, as we have seen, and $K_{\sigma\tau}$ is an extension of K_σ.)

$\nvdash_{K_{\rho\tau}} p \supset \Box\Diamond p$

$$\begin{array}{c} \neg(p \supset \Box\Diamond p), 0 \\ 0\text{r}0 \\ p, 0 \\ \neg\Box\Diamond p, 0 \\ \Diamond\neg\Diamond p, 0 \\ 0\text{r}1,\ 1\text{r}1 \\ \neg\Diamond p, 1 \\ \Box\neg p, 1 \\ \neg p, 1 \end{array}$$

■

Next,

$\boxed{K_{\rho\sigma}}$

$\vdash_{K_{\rho\sigma}} p \supset \Box\Diamond p$

$\nvdash_{K_{\rho\tau}} p \supset \Box\Diamond p$

$$\begin{array}{c} \neg(p \supset \Box\Diamond p), 0 \\ 0\text{r}0 \\ p, 0 \\ \neg\Box\Diamond p, 0 \\ \Diamond\neg\Diamond p, 0 \\ 0\text{r}1,\ 1\text{r}1 \\ \neg\Diamond p, 1 \\ \Box\neg p, 1 \\ \neg p, 1 \end{array}$$

■

$\vdash_{K_{\rho\sigma}} \Box p \supset p$

$\nvdash_{K_{\sigma\tau}} \Box p \supset p$

$$\begin{array}{c} \neg(\Box p \supset p) \\ \Box p, 0 \\ \neg p, 0 \end{array}$$

■

$\boxed{K_{\rho\tau}}$

$\vdash_{K_{\rho\tau}} \Box p \supset p$

$\nvdash_{K_{\sigma\tau}} \Box p \supset p$

$$\begin{array}{c} \neg(\Box p \supset p) \\ \Box p, 0 \\ \neg p, 0 \end{array}$$

■

$\vdash_{K_{\rho\tau}} \Box p \supset \Box\Box p$

$\nvdash_{K_{\rho\sigma}} \Box p \supset \Box\Box p$

$$\begin{array}{c} \neg(\Box p \supset \Box\Box p), 0 \\ 0\text{r}0 \\ \Box p, 0 \\ \neg\Box\Box p, 0 \\ \Diamond\neg\Box p, 0 \\ 0\text{r}1,\ 1\text{r}1,\ 1\text{r}0 \\ \neg\Box p, 1 \\ \Diamond\neg p, 1 \\ p, 1 \\ 1\text{r}2,\ 2\text{r}2,\ 2\text{r}1 \\ \neg p, 2 \end{array}$$

■

All three systems have been shown to make inferences valid which the other systems do not. Therefore, apart from those that were mentioned at the beginning of the solution, none of the binary systems is an extension of any another.

(d) Combining three (or four) of the conditions, we obtain only the system $K_{\rho\sigma\tau}$ (see problem 1(c)). Show that this is a proper extension of each of the binary systems of the last question.

The solution proceeds in an analogous way to those of 7(c).

$\vdash_{K_{\rho\sigma\tau}} \Box p \supset \Box\Box p$

(Because $K_{\rho\sigma\tau}$ is an extension of K_τ)

$\nvdash_{K_{\rho\sigma}} \Box p \supset \Box\Box p$ (Shown above)

■

$\vdash_{K_{\rho\sigma\tau}} \Box p \supset p$

(Because $K_{\rho\sigma\tau}$ is an extension of K_ρ)

$\nvdash_{K_{\sigma\tau}} \Box p \supset p$ (Shown above)

■

$\vdash_{K_{\rho\sigma\tau}} p \supset \Box\Diamond p$

(Because $K_{\rho\sigma\tau}$ is an extension of K_σ)

$\nvdash_{K_{\rho\tau}} p \supset \Box\Diamond p$ (Shown above)

■

$K_{\rho\sigma\tau}$ is an extension of each of the three binary systems. Further, it makes a theorem valid that the three true binary systems make invalid. Therefore it is a proper extension of each of them.

9. Check the details omitted in 3.6b.4, 3.6b.7.

3.6b.4: Check that $[F][F]A \nvdash [F]A$ and $[P][P]A \nvdash [P]A$ in K^t, and that $[F][F]A \vdash [F]A$ in K^t_δ

$\nvdash_{K^t} [F][F]A \nvdash [F]A$

$$\begin{gathered} [F][F]A, 0 \\ \neg[F]A, 0 \\ \langle F \rangle \neg A, 0 \\ 0r1 \\ \neg A, 1 \\ [F]A, 1 \end{gathered}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}; w_0 R w_1; v_{w_1}(p) = 0$$

This can be represented in the following diagram:

$$w_0 \ \rightarrow \overset{\neg p}{w_1}$$

$\nvdash_{K^t} [P][P]A \nvdash [P]A$

$$\begin{gathered} [P][P]A, 0 \\ \neg[P]A, 0 \\ \langle P \rangle \neg A, 0 \\ 1r0 \\ \neg A, 1 \\ [P]A, 1 \end{gathered}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}; w_1 R w_0; v_{w_1}(p) = 0$$

This can be represented in the following diagram:

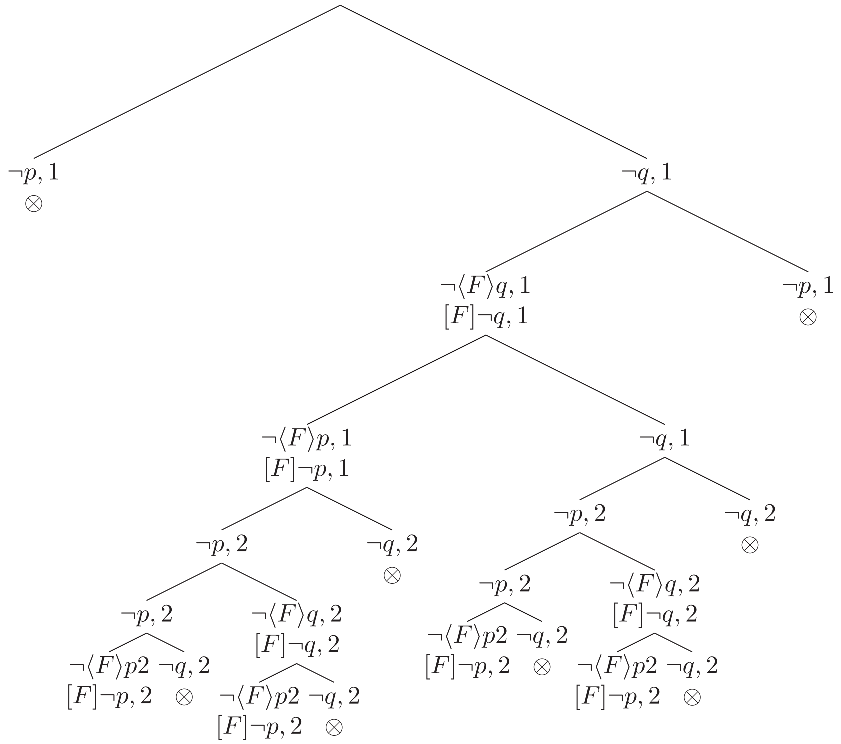
$$\overset{\neg p}{w_1} \rightarrow \ w_0$$

$[F][F]A \vdash_{K^t_\delta} [F]A$

$$\begin{array}{c} [F][F]A, 0 \\ \neg[F]A, 0 \\ \langle F\rangle\neg A, 0 \\ 0\mathrm{r}1 \\ \neg A, 1 \\ [F]A, 1 \\ 0\mathrm{r}2,\ 2\mathrm{r}1 \\ [F]A, 2 \\ A, 1 \\ \otimes \end{array}$$

3.6b.7 $\langle F\rangle p \wedge \langle F\rangle q \nvdash_{K^t} \langle F\rangle(p \wedge q) \vee \langle F\rangle(p \wedge \langle F\rangle q) \vee \langle F\rangle(\langle F\rangle p \wedge q)$

$$\langle F\rangle p \wedge \langle F\rangle q, 0$$
$$\neg(\langle F\rangle(p \wedge q) \vee \langle F\rangle(p \wedge \langle F\rangle q) \vee \langle F\rangle(\langle F\rangle p \wedge q)), 0$$
$$\langle F\rangle p, 0$$
$$\langle F\rangle q, 0$$
$$\neg\langle F\rangle(p \wedge q), 0$$
$$\neg\langle F\rangle(p \wedge \langle F\rangle q), 0$$
$$\neg\langle F\rangle((\langle F\rangle p \wedge q)), 0$$
$$[F]\neg(p \wedge q), 0$$
$$[F]\neg(p \wedge \langle F\rangle q), 0$$
$$[F]\neg((\langle F\rangle p \wedge q)), 0$$
$$0\mathrm{r}1$$
$$p, 1$$
$$0\mathrm{r}2$$
$$q, 2$$
$$\neg(p \wedge q), 1$$
$$\neg(p \wedge \langle F\rangle q), 1$$
$$\neg((\langle F\rangle p \wedge q)), 1$$
$$\neg(p \wedge q), 2$$
$$\neg(p \wedge \langle F\rangle q), 2$$
$$\neg((\langle F\rangle p \wedge q)), 2$$



The following interpretation, taken from the left-most open branch, shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_1, w_0 R w_2$$

$$v_{w_1}(q) = 0, v_{w_2}(p) = 0$$

This can be represented in the following diagram: