

Solutions

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1. Prove that the truth value of $\neg\Box A$ at a world is the same as that of $\Diamond\neg A$.

$$v_w(\neg\Box A) = 0$$

$$\text{iff } v_w(\Box A) = 1$$

$$\text{iff for all } w' \text{ such that } wRw', v_{w'}(A) = 1$$

$$\text{iff for all } w' \text{ such that } wRw', v_{w'}(\neg A) = 0$$

$$\text{iff } v_w(\Diamond\neg A) = 0$$

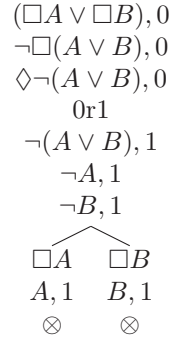
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2. Show the following. Where the tableau does not close, use it to define a countermodel, and draw this, as in 2.4.8

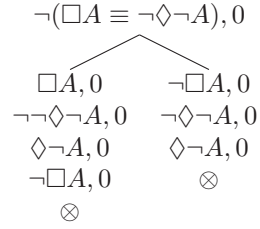
$$(a) \vdash (\Box A \wedge \Box B) \supset \Box(A \wedge B)$$

$$\begin{array}{c}
 (\Box A \wedge \Box B), 0 \\
 \neg\Box(A \wedge B), 0 \\
 \Box A, 0 \\
 \Box B, 0 \\
 \Diamond\neg(A \wedge B), 0 \\
 \text{or } 1 \\
 \neg(A \wedge B), 1 \\
 \swarrow \quad \searrow \\
 \neg A, 1 \quad \neg B, 1 \\
 A, 1 \quad B, 1 \\
 \otimes \quad \otimes
 \end{array}$$

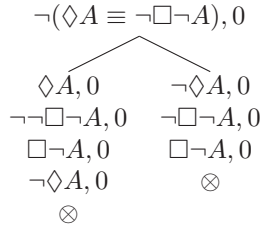
(b) $\vdash (\Box A \wedge \Box B) \supset \Box(A \vee B)$



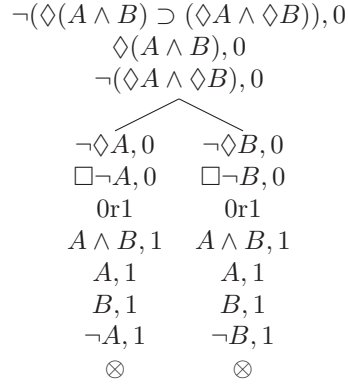
(c) $\vdash \Box A \equiv \neg \Diamond \neg A$



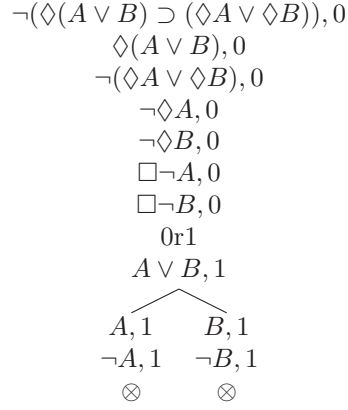
(d) $\vdash \Diamond A \equiv \neg \Box \neg A$



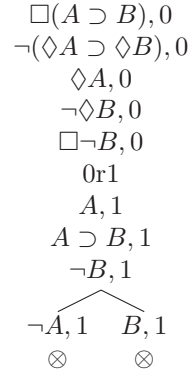
$$(e) \vdash \Diamond(A \wedge B) \supset (\Diamond A \wedge \Diamond B)$$



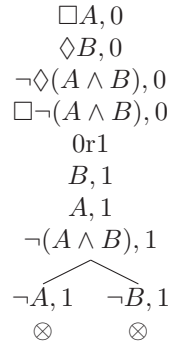
$$(f) \vdash \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$$



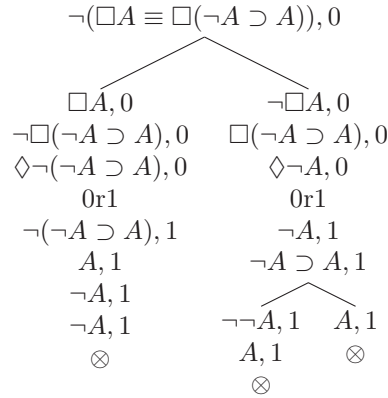
(g) $\Box(A \supset B) \vdash \Diamond A \supset \Diamond B$



(h) $\Box A, \Diamond B \vdash \Diamond(A \wedge B)$



(i) $\vdash \Box A \equiv \Box(\neg A \supset A)$



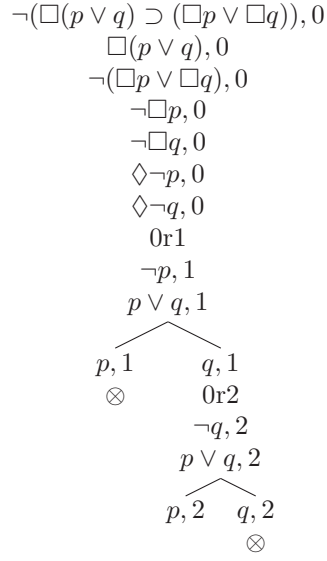
$$(j) \vdash \Box A \supset \Box(B \supset A)$$

$$\begin{array}{c}
\neg(\Box A \supset \Box(B \supset A)), 0 \\
\Box A, 0 \\
\neg\Box(B \supset A), 0 \\
\Diamond\neg(B \supset A), 0 \\
\text{or1} \\
\neg(B \supset A), 1 \\
B, 1 \\
\neg A, 1 \\
A, 1 \\
\otimes
\end{array}$$

$$(k) \vdash \neg\Diamond B \supset \Box(B \supset A)$$

$$\begin{array}{c}
\neg(\neg\Diamond B \supset \Box(B \supset A)), 0 \\
\neg\Diamond B, 0 \\
\neg\Box(B \supset A), 0 \\
\Diamond\neg(B \supset A), 0 \\
\Box\neg B, 0 \\
\text{or1} \\
\neg(B \supset A), 1 \\
B, 1 \\
\neg A, 1 \\
\neg B, 1 \\
\otimes
\end{array}$$

$$(1) \not\models \Box(p \vee q) \supset (\Box p \vee \Box q)$$



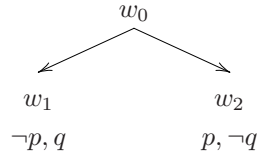
The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_1, w_0 R w_2$$

$$v_{w_1}(p) = 0, v_{w_1}(q) = 1, v_{w_2}(p) = 1, v_{w_2}(q) = 0$$

This can be represented in the following diagram:



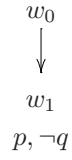
(m) $\Box p, \Box \neg q \not\vdash \Box(p \supset q)$

$$\begin{array}{c}
 \Box p, 0 \\
 \Box \neg q, 0 \\
 \neg \Box(p \supset q), 0 \\
 \Diamond \neg(p \supset q), 0 \\
 \text{Or1} \\
 \neg(p \supset q), 1 \\
 p, 1 \\
 \neg q, 1 \\
 p, 1 \\
 \neg q, 1
 \end{array}$$

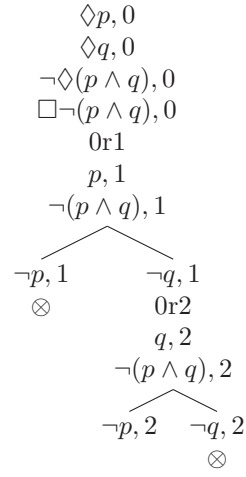
The following interpretation shows this inference to be invalid:

$$\begin{aligned}
 W &= \{w_0, w_1\} \\
 w_0 R w_1 \\
 v_{w_1}(p) &= 1, v_{w_1}(q) = 0
 \end{aligned}$$

This can be represented in the following diagram:



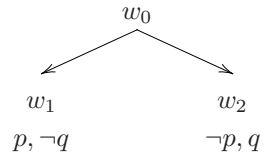
(n) $\Diamond p, \Diamond q \not\vdash \Diamond(p \wedge q)$



The following interpretation shows this inference to be invalid:

$$\begin{aligned}
 W &= \{w_0, w_1, w_2\} \\
 w_0 R w_1, w_0 R w_2 \\
 v_{w_1}(p) &= 1, v_{w_1}(q) = 0, v_{w_2}(p) = 0, v_{w_2}(q) = 1
 \end{aligned}$$

This can be represented in the following diagram:



$$(o) \not\models \Box p \supset p$$

$$\begin{array}{l} \neg(\Box p \supset p), 0 \\ \Box p, 0 \\ \neg p, 0 \end{array}$$

The following interpretation shows this inference to be invalid:

$$\begin{array}{l} W = \{w_0\} \\ v_{w_0}(p) = 0 \end{array}$$

This can be represented in the following diagram:

$$\begin{array}{c} w_0 \\ \neg p \end{array}$$

$$(p) \not\models \Box p \supset \Diamond p$$

$$\begin{array}{l} \neg(\Box p \supset \Diamond p), 0 \\ \Box p, 0 \\ \neg \Diamond p, 0 \\ \Box \neg p, 0 \end{array}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0\}$$

This can be represented in the following diagram:

$$w_0$$

(q) $p \not\vdash \Box p$

$$\begin{array}{c} p, 0 \\ \neg \Box p, 0 \\ \Diamond \neg p, 0 \\ 0r1 \\ \neg p, 1 \end{array}$$

The following interpretation shows this inference to be invalid:

$$\begin{array}{c} W = \{w_0, w_1\} \\ w_0 R w_1 \\ v_{w_0}(p) = 1, v_{w_1}(p) = 0 \end{array}$$

This can be represented in the following diagram:

$$\begin{array}{c} w_0 \ p \\ \downarrow \\ w_1 \\ \neg p \end{array}$$

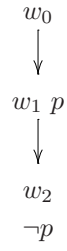
(r) $\not\vdash \Box p \supset \Box \Box p$

$$\begin{array}{c} \neg(\Box p \supset \Box \Box p), 0 \\ \Box p, 0 \\ \neg \Box \Box p, 0 \\ \Diamond \neg \Box p, 0 \\ 0r1 \\ \neg \Box p, 1 \\ \Diamond \neg p, 1 \\ p, 1 \\ 1r2 \\ \neg p, 2 \end{array}$$

The following interpretation shows this inference to be invalid:

$$\begin{array}{c} W = \{w_0, w_1, w_2\} \\ w_0 R w_1, w_1 R w_2 \\ v_{w_1}(p) = 1, v_{w_2}(p) = 0 \end{array}$$

This can be represented in the following diagram:



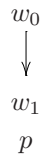
$$(s) \not\models \Diamond p \supset \Diamond \Diamond p$$

$$\begin{array}{c}
 \neg(\Diamond p \supset \Diamond \Diamond p), 0 \\
 \Diamond p, 0 \\
 \neg \Diamond \Diamond p, 0 \\
 \Box \neg \Diamond p, 0 \\
 0r1 \\
 p, 1 \\
 \neg \Diamond p, 1 \\
 \Box \neg p, 1
 \end{array}$$

The following interpretation shows this inference to be invalid:

$$\begin{array}{c}
 W = \{w_0, w_1\} \\
 w_0 R w_1 \\
 v_{w_1}(p) = 1
 \end{array}$$

This can be represented in the following diagram:



$$(t) \not\models p \supset \Box \Diamond p$$

$$\begin{array}{c} \neg(p \supset \Box \Diamond p), 0 \\ p, 0 \\ \neg \Box \Diamond p, 0 \\ \Diamond \neg \Diamond p, 0 \\ \text{Or1} \\ \neg \Diamond p, 1 \\ \Box \neg p, 1 \end{array}$$

The following interpretation shows this inference to be invalid:

$$\begin{array}{l} W = \{w_0, w_1\} \\ w_0 R w_1 \\ v_{w_0}(p) = 1 \end{array}$$

This can be represented in the following diagram:

$$\begin{array}{c} w_0 \ p \\ \downarrow \\ w_1 \end{array}$$

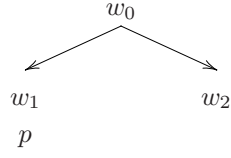
$$(u) \not\models \Diamond p \supset \Box \Diamond p$$

$$\begin{array}{c} \neg(\Diamond p \supset \Box \Diamond p), 0 \\ \Diamond p, 0 \\ \neg \Box \Diamond p, 0 \\ \Diamond \neg \Diamond p, 0 \\ \text{Or1} \\ p, 1 \\ \text{Or2} \\ \neg \Diamond p, 2 \\ \Box \neg p, 2 \end{array}$$

The following interpretation shows this inference to be invalid:

$$\begin{array}{l} W = \{w_0, w_1, w_2\} \\ w_0 R w_1, w_0 R w_2 \\ v_{w_1}(p) = 1 \end{array}$$

This can be represented in the following diagram:



$$(v) \not\models \Diamond(p \vee \neg p)$$

$$\neg \Diamond(p \vee \neg p), 0$$

$$\Box \neg(p \vee \neg p), 0$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0\}$$

This can be represented in the following diagram:

$$w_0$$

4. *Check the details omitted in 2.9.3 and 2.9.6.

2.9.3 Soundness Lemma: Let b be any branch of a tableau, and $I = \langle W, R, v \rangle$ be any interpretation. If I is faithful to b , and a tableau rule is applied to it, then it produces at least one extension, b' , such that I is faithful to b' .

Proof:

The proof proceeds by a case-by-case consideration of the tableau rules. $\boxed{A \vee B}$, $\boxed{A \supset B}$, $\boxed{A \equiv B}$, $\boxed{\neg \neg A}$, and $\boxed{\neg \Box A}$ have not been explicitly dealt with.

$$\boxed{A \vee B}$$

Suppose I is faithful to b and $A \vee B, i$ occurs on b , and that we apply a rule to it. Then two branches eventuate - one extending b with A, i (the left branch) and one extending b with B, i (the right branch). Since I is faithful to b it makes every formula on b true - in particular $A \vee B$ is true at $f(i)$, that is, $v_{w_i}(A \vee B) = 1$, so either A or B is true at $f(i)$: i.e. $v_{w_i}(A) = 1$ or, $v_{w_i}(B) = 1$.

In the first case, I is faithful to the left branch; in the second case I is faithful to the right branch.

Suppose I is faithful to b and $\neg(A \vee B), i$ occurs on b , and that we apply a rule to it. Then one branch eventuates - extending b with $\neg A, i$ and $\neg B, i$. Since I is faithful to b it makes every formula on b true - in particular $v_{w_i}(\neg(A \vee B)) = 1$ so $v_{w_i}(A) = 0$ and, $v_{w_i}(B) = 0$, making I faithful to the extended branch.

$A \equiv B$

Suppose I is faithful to b and $A \equiv B, i$ occurs on b , and that we apply a rule to it. Then two branches eventuate - one extending b with A, i and B, i (the left branch) and one extending b with $\neg A, i$ and $\neg B, i$ (the right branch). Since I is faithful to b it makes every formula on b true - in particular $v_{w_i}(A \equiv B) = 1$ so $v_{w_i}(A) = v_{w_i}(B)$. That is, either $v_{w_i}(A) = 1$ and $v_{w_i}(B) = 1$ or, $v_{w_i}(A) = 0$ and $v_{w_i}(B) = 0$. In the first case, I is faithful to the left branch, in the second case I is faithful to the right branch.

Suppose I is faithful to b and $\neg(A \equiv B), i$ occurs on b , and that we apply a rule to it. Then two branches eventuate - one extending b with A, i and $\neg B, i$ (the left branch) and one extending b with $\neg A, i$ and B, i (the right branch). Since I is faithful to b it makes every formula on b true - in particular $v_{w_i}(\neg(A \equiv B)) = 1$ so $v_{w_i}(A) \neq v_{w_i}(B)$. That is, either $v_{w_i}(A) = 1$ and $v_{w_i}(B) = 0$ or, $v_{w_i}(A) = 0$ and $v_{w_i}(B) = 1$. In the first case, I is faithful to the left branch, in the second case I is faithful to the right branch.

$\neg\neg A$

Suppose I is faithful to b and $\neg\neg A, i$ occurs on b , and that we apply a rule to it. Then just one branch eventuates: extending b with A, i . Since I is faithful to b it makes every formula on b true - in particular $v_{w_i}(\neg\neg A) = 1$ so $v_{w_i}(A) = 1$. Thus I is faithful to the extended branch.

$\neg\Box A$

Suppose I is faithful to b and $\neg\Box A, i$ occurs on b , and that we apply a rule to it. Then just one branch eventuates: extending b with $\Diamond\neg A, i$. Since I is faithful to b it makes every formula on b true - in particular $v_{w_i}(\neg\Box A) = 1$. By the proof in question 1, this implies that $v_{w_i}(\Diamond\neg A) = 1$. Thus I is faithful to the extended branch.

■

2.9.6 Finish the proof of the Completeness Lemma: The atomic case, $B \vee C$, $\Box B$, and $\neg\Box B$ have already been shown.

Completeness Lemma: Let b be any open complete branch of a tableau. Let $I = \langle W, R, v \rangle$ be the interpretation induced by b . Then:

if A, i is on b then A is true at w_i
if $\neg A, i$ is on b then A is false at w_i

Proof:

The proof is by recursion on the complexity of A .

$\neg(B \vee C)$

If A occurs on b , and is of the form $\neg(B \vee C)$, then the rule for disjunction has been applied to $\neg(B \vee C), i$. Thus, $\neg B, i$ and $\neg C, i$ are on b . By induction hypothesis, both $\neg B$ and $\neg C$ are true at w_i . Hence $(B \vee C)$ is false at w_i , as required.

$B \wedge C$

If A occurs on b , and is of the form $B \wedge C$, then the rule for conjunction has been applied to $B \wedge C, i$. Thus, B, i and C, i are on b . By induction hypothesis, both B and C are true at w_i . Hence $B \wedge C$ is true at w_i , as required.

$\neg(B \wedge C)$

If A occurs on b , and is of the form $\neg(B \wedge C)$, then the rule for conjunction has been applied to $\neg(B \wedge C), i$. Thus, $\neg B, i$ or $\neg C, i$ is on b . By induction hypothesis, either $\neg B$ or $\neg C$ is true at w_i . Hence $B \wedge C$ is false at w_i , as required.

$B \supset C$

If A occurs on b , and is of the form $B \supset C$, then the rule for the conditional has been applied to $B \supset C, i$. Thus, $\neg B, i$ or C, i is on b . By induction hypothesis, either $\neg B$ or C is true at w_i . Hence $B \supset C$ is true at w_i , as required.

$\neg(B \supset C)$

If A occurs on b , and is of the form $\neg(B \supset C)$, then the rule for the conditional has been applied to $\neg(B \supset C), i$. Thus, B, i and $\neg C, i$ are on b . By induction hypothesis, both B and $\neg C$ are true at w_i . Hence $B \supset C$ is false at w_i , as required.

$$\boxed{B \equiv C}$$

If A occurs on b , and is of the form $B \equiv C$, then the rule for equivalence has been applied to $B \equiv C, i$. Thus, B, i and C, i , or $\neg B, i$ and $\neg C, i$ are on b . By induction hypothesis, either B and C , or $\neg B$ and $\neg C$ are true at w_i . Hence $B \equiv C$ is true at w_i , as required.

$$\boxed{\neg(B \equiv C)}$$

If A occurs on b , and is of the form $\neg(B \equiv C)$, then the rule for equivalence has been applied to $\neg(B \equiv C), i$. Thus, B, i and $\neg C, i$, or $\neg B, i$ and C, i are on b . By induction hypothesis, either B and $\neg C$, or $\neg B$ and C are true at w_i . Hence $(B \equiv C)$ is false at w_i , as required.

$$\boxed{\neg\neg B}$$

If A occurs on b , and is of the form $\neg\neg B$, then the rule for double negation has been applied to $\neg\neg B, i$. Thus, B, i is on b . By induction hypothesis, B is true at w_i . Hence $\neg\neg B$ is true at w_i , as required.

$$\boxed{\Diamond B}$$

If A occurs on b , and is of the form $\Diamond B$, then the rule for possibility has been applied to $\Diamond B, i$. Thus, for some j such that iRj is on b , B, j is on b . By construction and the induction hypothesis, for some w_j such that w_iRw_j , B is true at w_j . Hence $\Diamond B$ is true at w_i , as required.

$$\boxed{\neg\Diamond B}$$

If A occurs on b , and is of the form $\neg\Diamond B$, then the rule for possibility has been applied to $\neg\Diamond B, i$. Thus, $\Box\neg B, i$ is on b . So, for all j such that iRj , $\neg B, j$ is on b . By induction hypothesis, for all j such that w_iRw_j , B is false at w_j . Hence $\neg\Diamond B$ is true at w_i , as required. ■