Solutions

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1. Prove that the truth value of $\neg\Box A$ at a world is the same as that of $\Diamond\neg A$.

$v_w(\neg\Box A) = 0$

iff $v_w(\Box A) = 1$
iff for all w' such that wRw', $v_{w'}(A) = 1$
iff for all w' such that wRw', $v_{w'}(\neg A) = 0$
iff $v_w(\Diamond\neg A) = 0$

$\blacksquare$

2. Show the following. Where the tableau does not close, use it to define a countermodel, and draw this, as in 2.4.8

(a) $\vdash (\Box A \wedge \Box B) \supset \Box(A \wedge B)$

$$
\begin{array}{c}
(\Box A \wedge \Box B), 0 \\
\neg\Box(A \wedge B), 0 \\
\Box A, 0 \\
\Box B, 0 \\
\Diamond\neg(A \wedge B), 0 \\
0r1 \\
\neg(A \wedge B), 1 \\
\begin{array}{cc}
\neg A, 1 & \neg B, 1 \\
A, 1 & B, 1 \\
\otimes & \otimes
\end{array}
\end{array}
$$

(b) $\vdash (\Box A \wedge \Box B) \supset \Box(A \vee B)$

$$
\begin{array}{c}
(\Box A \vee \Box B), 0 \\
\neg\Box(A \vee B), 0 \\
\Diamond\neg(A \vee B), 0 \\
0r1 \\
\neg(A \vee B), 1 \\
\neg A, 1 \\
\neg B, 1 \\
\begin{array}{cc}
\Box A & \Box B \\
A, 1 & B, 1 \\
\otimes & \otimes
\end{array}
\end{array}
$$

(c) $\vdash \Box A \equiv \neg\Diamond\neg A$

$$
\begin{array}{c}
\neg(\Box A \equiv \neg\Diamond\neg A), 0 \\
\begin{array}{cc}
\Box A, 0 & \neg\Box A, 0 \\
\neg\neg\Diamond\neg A, 0 & \neg\Diamond\neg A, 0 \\
\Diamond\neg A, 0 & \Diamond\neg A, 0 \\
\neg\Box A, 0 & \otimes \\
\otimes &
\end{array}
\end{array}
$$

(d) $\vdash \Diamond A \equiv \neg\Box\neg A$

$$
\begin{array}{c}
\neg(\Diamond A \equiv \neg\Box\neg A), 0 \\
\begin{array}{cc}
\Diamond A, 0 & \neg\Diamond A, 0 \\
\neg\neg\Box\neg A, 0 & \neg\Box\neg A, 0 \\
\Box\neg A, 0 & \Box\neg A, 0 \\
\neg\Diamond A, 0 & \otimes \\
\otimes &
\end{array}
\end{array}
$$

(e) $\vdash \Diamond(A \wedge B) \supset (\Diamond A \wedge \Diamond B)$

$$
\begin{array}{c}
\neg(\Diamond(A \wedge B) \supset (\Diamond A \wedge \Diamond B)), 0 \\
\Diamond(A \wedge B), 0 \\
\neg(\Diamond A \wedge \Diamond B), 0 \\
\begin{array}{cc}
\neg\Diamond A, 0 & \neg\Diamond B, 0 \\
\Box\neg A, 0 & \Box\neg B, 0 \\
0r1 & 0r1 \\
A \wedge B, 1 & A \wedge B, 1 \\
A, 1 & A, 1 \\
B, 1 & B, 1 \\
\neg A, 1 & \neg B, 1 \\
\otimes & \otimes
\end{array}
\end{array}
$$

(f) $\vdash \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$

$$
\begin{array}{c}
\neg(\Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)), 0 \\
\Diamond(A \vee B), 0 \\
\neg(\Diamond A \vee \Diamond B), 0 \\
\neg\Diamond A, 0 \\
\neg\Diamond B, 0 \\
\Box\neg A, 0 \\
\Box\neg B, 0 \\
0r1 \\
A \vee B, 1 \\
\begin{array}{cc}
A, 1 & B, 1 \\
\neg A, 1 & \neg B, 1 \\
\otimes & \otimes
\end{array}
\end{array}
$$

(g) $\Box(A \supset B) \vdash \Diamond A \supset \Diamond B$

$$\begin{array}{c}
\Box(A \supset B), 0 \\
\neg(\Diamond A \supset \Diamond B), 0 \\
\Diamond A, 0 \\
\neg\Diamond B, 0 \\
\Box\neg B, 0 \\
0r1 \\
A, 1 \\
A \supset B, 1 \\
\neg B, 1 \\
\begin{array}{cc}
\neg A, 1 & B, 1 \\
\otimes & \otimes
\end{array}
\end{array}$$

(h) $\Box A, \Diamond B \vdash \Diamond(A \wedge B)$

$$\begin{array}{c}
\Box A, 0 \\
\Diamond B, 0 \\
\neg\Diamond(A \wedge B), 0 \\
\Box\neg(A \wedge B), 0 \\
0r1 \\
B, 1 \\
A, 1 \\
\neg(A \wedge B), 1 \\
\begin{array}{cc}
\neg A, 1 & \neg B, 1 \\
\otimes & \otimes
\end{array}
\end{array}$$

(i) $\vdash \Box A \equiv \Box(\neg A \supset A)$

$$\begin{array}{c}
\neg(\Box A \equiv \Box(\neg A \supset A)), 0 \\
\begin{array}{cc}
\begin{array}{c}
\Box A, 0 \\
\neg\Box(\neg A \supset A), 0 \\
\Diamond\neg(\neg A \supset A), 0 \\
0r1 \\
\neg(\neg A \supset A), 1 \\
A, 1 \\
\neg A, 1 \\
\neg A, 1 \\
\otimes
\end{array}
&
\begin{array}{c}
\neg\Box A, 0 \\
\Box(\neg A \supset A), 0 \\
\Diamond\neg A, 0 \\
0r1 \\
\neg A, 1 \\
\neg A \supset A, 1 \\
\begin{array}{cc}
\begin{array}{c} \neg\neg A, 1 \\ A, 1 \\ \otimes \end{array} & \begin{array}{c} A, 1 \\ \otimes \\ \end{array}
\end{array}
\end{array}
\end{array}
\end{array}$$

(j) $\vdash \Box A \supset \Box(B \supset A)$

$$
\begin{gathered}
\neg(\Box A \supset \Box(B \supset A)), 0 \\
\Box A, 0 \\
\neg\Box(B \supset A), 0 \\
\Diamond\neg(B \supset A), 0 \\
0\mathrm{r}1 \\
\neg(B \supset A), 1 \\
B, 1 \\
\neg A, 1 \\
A, 1 \\
\otimes
\end{gathered}
$$

(k) $\vdash \neg\Diamond B \supset \Box(B \supset A)$

$$
\begin{gathered}
\neg(\neg\Diamond B \supset \Box(B \supset A)), 0 \\
\neg\Diamond B, 0 \\
\neg\Box(B \supset A), 0 \\
\Diamond\neg(B \supset A), 0 \\
\Box\neg B, 0 \\
0\mathrm{r}1 \\
\neg(B \supset A), 1 \\
B, 1 \\
\neg A, 1 \\
\neg B, 1 \\
\otimes
\end{gathered}
$$

(l) $\nvdash \Box(p \vee q) \supset (\Box p \vee \Box q)$

$$\begin{array}{c}
\neg(\Box(p \vee q) \supset (\Box p \vee \Box q)), 0 \\
\Box(p \vee q), 0 \\
\neg(\Box p \vee \Box q), 0 \\
\neg\Box p, 0 \\
\neg\Box q, 0 \\
\Diamond\neg p, 0 \\
\Diamond\neg q, 0 \\
0r1 \\
\neg p, 1 \\
p \vee q, 1 \\
\begin{array}{cc}
\begin{array}{c} p, 1 \\ \otimes \end{array} &
\begin{array}{c} q, 1 \\ 0r2 \\ \neg q, 2 \\ p \vee q, 2 \\
\begin{array}{cc} p, 2 & \begin{array}{c} q, 2 \\ \otimes \end{array} \end{array}
\end{array}
\end{array}
\end{array}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_1, w_0 R w_2$$

$$v_{w_1}(p) = 0, v_{w_1}(q) = 1, v_{w_2}(p) = 1, v_{w_2}(q) = 0$$

This can be represented in the following diagram:

$$\begin{tikzcd}
& w_0 \arrow[dl] \arrow[dr] & \\
w_1 & & w_2 \\
\neg p, q & & p, \neg q
\end{tikzcd}$$

(m) $\Box p, \Box\neg q \nvdash \Box(p \supset q)$

$$\begin{array}{c} \Box p, 0 \\ \Box\neg q, 0 \\ \neg\Box(p \supset q), 0 \\ \Diamond\neg(p \supset q), 0 \\ 0\mathrm{r}1 \\ \neg(p \supset q), 1 \\ p, 1 \\ \neg q, 1 \\ p, 1 \\ \neg q, 1 \end{array}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

$$w_0 R w_1$$

$$v_{w_1}(p) = 1, v_{w_1}(q) = 0$$

This can be represented in the following diagram:

$$\begin{tikzcd} w_0 \arrow[d] \\ w_1 \\[-20pt] p, \neg q \end{tikzcd}$$

(n) $\Diamond p, \Diamond q \nvdash \Diamond(p \wedge q)$

$$
\begin{array}{c}
\Diamond p, 0 \\
\Diamond q, 0 \\
\neg\Diamond(p \wedge q), 0 \\
\Box\neg(p \wedge q), 0 \\
0r1 \\
p, 1 \\
\neg(p \wedge q), 1 \\
\begin{array}{cc}
\begin{array}{c} \neg p, 1 \\ \otimes \end{array} &
\begin{array}{c}
\neg q, 1 \\
0r2 \\
q, 2 \\
\neg(p \wedge q), 2 \\
\begin{array}{cc}
\neg p, 2 & \begin{array}{c} \neg q, 2 \\ \otimes \end{array}
\end{array}
\end{array}
\end{array}
\end{array}
$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_1, w_0 R w_2$$

$$v_{w_1}(p) = 1, v_{w_1}(q) = 0, v_{w_2}(p) = 0, v_{w_2}(q) = 1$$

This can be represented in the following diagram:

$$
\begin{tikzcd}
 & w_0 \arrow[dl] \arrow[dr] & \\
w_1 & & w_2 \\
p, \neg q & & \neg p, q
\end{tikzcd}
$$

(o) $\nvdash \Box p \supset p$

$$\begin{gathered} \neg(\Box p \supset p), 0 \\ \Box p, 0 \\ \neg p, 0 \end{gathered}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0\}$$

$$v_{w_0}(p) = 0$$

This can be represented in the following diagram:

$$\begin{gathered} w_0 \\ \neg p \end{gathered}$$

(p) $\nvdash \Box p \supset \Diamond p$

$$\begin{gathered} \neg(\Box p \supset \Diamond p), 0 \\ \Box p, 0 \\ \neg \Diamond p, 0 \\ \Box \neg p, 0 \end{gathered}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0\}$$

This can be represented in the following diagram:

$$w_0$$

(q) $p \nvdash \Box p$

$$\begin{array}{c} p, 0 \\ \neg\Box p, 0 \\ \Diamond\neg p, 0 \\ 0\text{r}1 \\ \neg p, 1 \end{array}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

$$w_0 R w_1$$

$$v_{w_0}(p) = 1, v_{w_1}(p) = 0$$

This can be represented in the following diagram:

$$\begin{tikzcd} w_0\ p \arrow[d] \\ w_1 \\ \neg p \end{tikzcd}$$

(r) $\nvdash \Box p \supset \Box\Box p$

$$\begin{array}{c} \neg(\Box p \supset \Box\Box p), 0 \\ \Box p, 0 \\ \neg\Box\Box p, 0 \\ \Diamond\neg\Box p, 0 \\ 0\text{r}1 \\ \neg\Box p, 1 \\ \Diamond\neg p, 1 \\ p, 1 \\ 1\text{r}2 \\ \neg p, 2 \end{array}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_1, w_1 R w_2$$

$$v_{w_1}(p) = 1, v_{w_2}(p) = 0$$

This can be represented in the following diagram:

$$\begin{tikzcd} w_0 \arrow[d] \\ w_1 \; p \arrow[d] \\ w_2 \\ \neg p \end{tikzcd}$$

(s) $\nvdash \Diamond p \supset \Diamond\Diamond p$

$$\begin{gathered} \neg(\Diamond p \supset \Diamond\Diamond p), 0 \\ \Diamond p, 0 \\ \neg\Diamond\Diamond p, 0 \\ \Box\neg\Diamond p, 0 \\ 0\text{r}1 \\ p, 1 \\ \neg\Diamond p, 1 \\ \Box\neg p, 1 \end{gathered}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

$$w_0 R w_1$$

$$v_{w_1}(p) = 1$$

This can be represented in the following diagram:

$$\begin{tikzcd} w_0 \arrow[d] \\ w_1 \\ p \end{tikzcd}$$

(t) $\nvdash p \supset \Box\Diamond p$

$$\begin{array}{c}
\neg(p \supset \Box\Diamond p), 0 \\
p, 0 \\
\neg\Box\Diamond p, 0 \\
\Diamond\neg\Diamond p, 0 \\
0r1 \\
\neg\Diamond p, 1 \\
\Box\neg p, 1
\end{array}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1\}$$

$$w_0 R w_1$$

$$v_{w_0}(p) = 1$$

This can be represented in the following diagram:

$$\begin{tikzcd}
w_0 \; p \arrow[d] \\
w_1
\end{tikzcd}$$

(u) $\nvdash \Diamond p \supset \Box\Diamond p$

$$\begin{array}{c}
\neg(\Diamond p \supset \Box\Diamond p), 0 \\
\Diamond p, 0 \\
\neg\Box\Diamond p, 0 \\
\Diamond\neg\Diamond p, 0 \\
0r1 \\
p, 1 \\
0r2 \\
\neg\Diamond p, 2 \\
\Box\neg p, 2
\end{array}$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0, w_1, w_2\}$$

$$w_0 R w_1, w_0 R w_2$$

$$v_{w_1}(p) = 1$$

This can be represented in the following diagram:

$$\begin{tikzcd}
 & w_0 \arrow[dl] \arrow[dr] & \\
w_1 & & w_2
\end{tikzcd}$$

$$p$$

(v) $\nvdash \Diamond(p \vee \neg p)$

$$\neg\Diamond(p \vee \neg p), 0$$
$$\Box\neg(p \vee \neg p), 0$$

The following interpretation shows this inference to be invalid:

$$W = \{w_0\}$$

This can be represented in the following diagram:

$$w_0$$

4. *Check the details omitted in 2.9.3 and 2.9.6.

2.9.3 Soundness Lemma: Let b be any branch of a tableau, and $I =< W, R, v >$ be any interpretation. If I is faithful to b, and a tableau rule is applied to it, then it produces at least one extension, b', such that I is faithful to b'.

Proof:

The proof proceeds by a case-by-case consideration of the tableau rules. $\boxed{A \vee B}$, $\boxed{A \supset B}$, $\boxed{A \equiv B}$, $\boxed{\neg\neg A}$, and $\boxed{\neg\Box A}$ have not been explicitly dealt with.

$\boxed{A \vee B}$

Suppose I is faithful to b and $A \vee B, i$ occurs on b, and that we apply a rule to it. Then two branches eventuate - one extending b with A, i (the left branch) and one extending b with B, i (the right branch). Since I is faithful to b it makes every formula on b true - in particular $A \vee B$ is true at $f(i)$, that is, $v_{w_i}(A \vee B) = 1$, so either A or B is true at $f(i)$: i.e. $v_{w_i}(A) = 1$ or, $v_{w_i}(B) = 1$.

In the first case, I is faithful to the left branch; in the second case I is faithful to the right branch.

Suppose I is faithful to b and $\neg(A \vee B), i$ occurs on b, and that we apply a rule to it. Then one branch eventuates - extending b with $\neg A, i$ and$\neg B, i$. Since I is faithful to b it makes every formula on b true - in particular $v_{w_i}(\neg(A \vee B)) = 1$ so $v_{w_i}(A) = 0$ and, $v_{w_i}(B) = 0$, making I faithful to the extended branch.

$\boxed{A \equiv B}$

Suppose I is faithful to b and $A \equiv B, i$ occurs on b, and that we apply a rule to it. Then two branches eventuate - one extending b with A, i and B, i (the left branch) and one extending b with $\neg A, i$ and$\neg B, i$ (the right branch). Since I is faithful to b it makes every formula on b true - in particular $v_{w_i}(A \equiv B) = 1$ so $v_{w_i}(A) = v_{w_i}(B)$. That is, either $v_{w_i}(A) = 1$ and $v_{w_i}(B) = 1$ or, $v_{w_i}(A) = 0$ and $v_{w_i}(B) = 0$. In the first case, I is faithful to the left branch, in the second case I is faithful to the right branch.

Suppose I is faithful to b and $\neg(A \equiv B), i$ occurs on b, and that we apply a rule to it. Then two branches eventuate - one extending b with A, i and $\neg B, i$ (the left branch) and one extending b with $\neg A, i$ B, i (the right branch). Since I is faithful to b it makes every formula on b true - in particular $v_{w_i}(\neg(A \equiv B)) = 1$ so $v_{w_i}(A) \neq v_{w_i}(B)$. That is, either $v_{w_i}(A) = 1$ and $v_{w_i}(B) = 0$ or, $v_{w_i}(A) = 0$ and $v_{w_i}(B) = 1$. In the first case, I is faithful to the left branch, in the second case I is faithful to the right branch.

$\boxed{\neg\neg A}$

Suppose I is faithful to b and $\neg\neg A, i$ occurs on b, and that we apply a rule to it. Then just one branch eventuates: extending b with A, i. Since I is faithful to b it makes every formula on b true - in particular $v_{w_i}(\neg\neg A) = 1$ so $v_{w_i}(A) = 1$. Thus I is faithful to the extended branch.

$\boxed{\neg\Box A}$

Suppose I is faithful to b and $\neg\Box A, i$ occurs on b, and that we apply a rule to it. Then just one branch eventuates: extending b with $\Diamond\neg A, i$. Since I is faithful to b it makes every formula on b true - in particular $v_{w_i}(\neg\Box A) = 1$. By the proof in question 1, this implies that $v_{w_i}(\Diamond\neg A) = 1$. Thus I is faithful to the extended branch.

■

2.9.6 Finish the proof of the Completeness Lemma: The atomic case, $B \vee C$, $\Box B$, and $\neg\Box B$ have already been shown.

Completeness Lemma: Let b be any open complete branch of a tableau. Let $I = < W, R, v >$ be the interpretation induced by b. Then:

if A, i is on b then A is true at w_i
if $\neg A$, i is on b then A is false at w_i

Proof:
The proof is by recursion on the complexity of A.

$\boxed{\neg(B \vee C)}$

If A occurs on b, and is of the form $\neg(B \vee C)$, then the rule for disjunction has been applied to $\neg(BvC), i$. Thus, $\neg B, i$ and $\neg C, i$ are on b. By induction hypothesis, both $\neg B$ and $\neg C$ are true at w_i. Hence $(B \vee C)$ is false at w_i, as required.

$\boxed{B \wedge C}$

If A occurs on b, and is of the form $B \wedge C$, then the rule for conjunction has been applied to $B \wedge C, i$. Thus, B, i and C, i are on b. By induction hypothesis, both B and C are true at w_i. Hence $B \wedge C$ is true at w_i, as required.

$\boxed{\neg(B \wedge C)}$

If A occurs on b, and is of the form $\neg(B \wedge C)$, then the rule for conjunction has been applied to $\neg(B \wedge C), i$. Thus, $\neg B, i$ or $\neg C, i$ is on b. By induction hypothesis, either $\neg B$ or $\neg C$ is true at w_i. Hence $B \wedge C$ is false at w_i, as required.

$\boxed{B \supset C}$

If A occurs on b, and is of the form $B \supset C$, then the rule for the conditional has been applied to $B \supset C, i$. Thus, $\neg B, i$ or C, i is on b. By induction hypothesis, either $\neg B$ or C is true at w_i. Hence $B \supset C$ is true at w_i, as required.

$\boxed{\neg(B \supset C)}$

If A occurs on b, and is of the form $\neg(B \supset C)$, then the rule for the conditional has been applied to $\neg(B \supset C), i$. Thus, B, i and $\neg C, i$ are on b. By induction hypothesis, both B and $\neg C$ are true at w_i. Hence $B \supset C$ is false at w_i, as required.

$\boxed{B \equiv C}$

If A occurs on b, and is of the form $B \equiv C$, then the rule for equivalence has been applied to $B \equiv C, i$. Thus, B, i and C, i, or $\neg B, i$ and $\neg C, i$ are on b. By induction hypothesis, either B and C, or $\neg B$ and $\neg C$ are true at w_i. Hence $B \equiv C$ is true at w_i, as required.

$\boxed{\neg(B \equiv C)}$

If A occurs on b, and is of the form $\neg(B \equiv C)$, then the rule for equivalence has been applied to $\neg(B \equiv C), i$. Thus, B, i and $\neg C, i$, or $\neg B, i$ and C, i are on b. By induction hypothesis, either B and $\neg C$, or $\neg B$ and C are true at w_i. Hence $(B \equiv C)$ is false at w_i, as required.

$\boxed{\neg\neg B}$

If A occurs on b, and is of the form $\neg\neg B$, then the rule for double negation has been applied to $\neg\neg B, i$. Thus, B, i is on b. By induction hypothesis, B is true at w_i. Hence $\neg B$ is false at w_i, as required.

$\boxed{\Diamond B}$

If A occurs on b, and is of the form $\Diamond B$, then the rule for possibility has been applied to $\Diamond B, i$. Thus, for some j such that iRj is on b, B, j is on b. By construction and the induction hypothesis, for some w_j such that $w_i R w_j$, B is true at w_j. Hence $\Diamond B$ is true at w_i, as required.

$\boxed{\neg\Diamond B}$

If A occurs on b, and is of the form $\neg\Diamond B$, then the rule for possibility has been applied to $\neg\Diamond B, i$. Thus, $\Box\neg B, i$ is on b. So, for all j such that iRj, $\neg B, j$ is on b. By induction hypothesis, for all j such that $w_i R w_j$, B is false at w_j. Hence $\Diamond B$ is false at w_i, as required.

■