Solutions

Louis Barson
Kyoto University

April 29, 2009

1. Check the truth of each of the following using tableaux. If the inference is invalid, read off a counter-model from the tree, and check directly that it makes the premises true and the conclusion false, as in 1.5.4

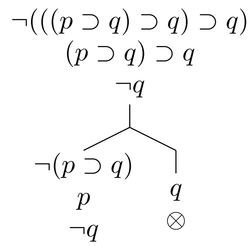
(a) $p \supset q, r \supset q \vdash (p \vee r) \supset q$

$$\begin{array}{c}
(p \supset q) \\
(r \supset q) \\
\neg((p \vee r) \supset q) \\
(p \vee r) \\
\neg q \\
\begin{array}{cc}
\begin{array}{c}
\neg p \\
\begin{array}{cc}
\begin{array}{c} q \\ \otimes \end{array} &
\begin{array}{c} \neg r \\ \begin{array}{cc} \begin{array}{c} p \\ \otimes \end{array} & \begin{array}{c} r \\ \otimes \end{array} \end{array} \end{array}
\end{array}
\end{array} &
\begin{array}{c} q \\ \otimes \end{array}
\end{array}
\end{array}$$

(b) $p \supset (q \wedge r), \neg r \vdash \neg p$

$$\begin{array}{c}
p \supset (q \wedge r) \\
\neg r \\
\neg\neg p \\
p \\
\begin{array}{cc}
\begin{array}{c} \neg p \\ \otimes \end{array} &
\begin{array}{c} q \wedge r \\ q \\ r \\ \otimes \end{array}
\end{array}
\end{array}$$

(c) $\nvdash ((p \supset q) \supset q) \supset q$

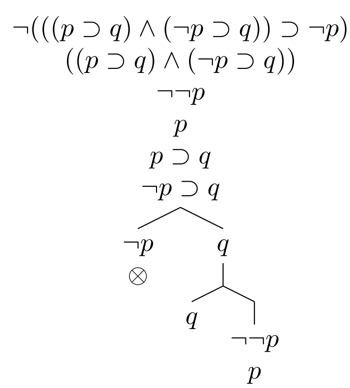


An interpretation which shows the inference to be invalid is

$$v(p) = 1, v(q) = 0$$

Applying this interpretation, we can see that it makes $p \supset q$ false. Since the antecedent is false, $(p \supset q) \supset q$ must be true. Thus, the interpretation makes the sentence $((p \supset q) \supset q) \supset q$ false, as required.

(d) $\nvdash ((p \supset q) \wedge (\neg p \supset q)) \supset \neg p$

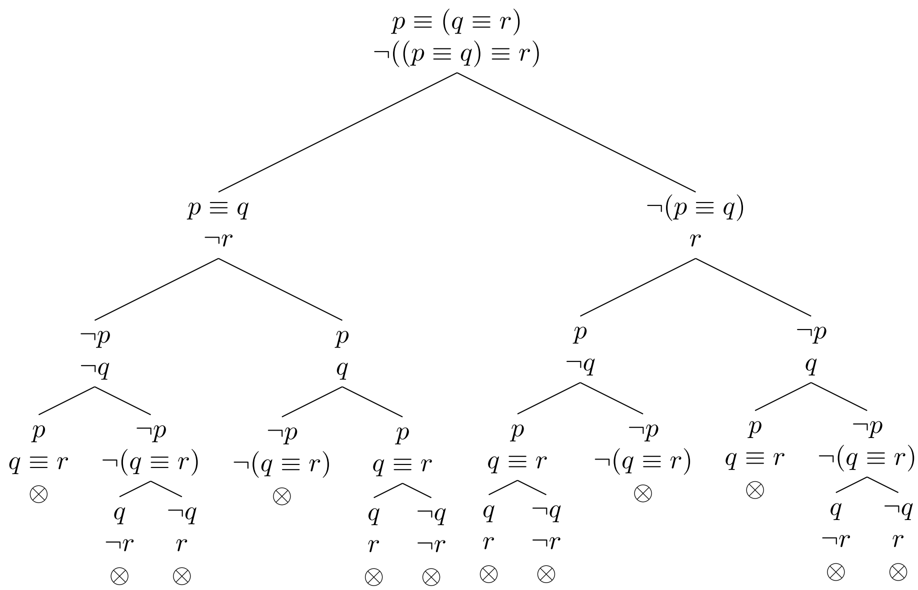


An interpretation which shows the inference to be invalid is

$$v(p) = v(q) = 1$$

Applying this interpretation, we can see that, because it makes q true, it makes both $p \supset q$ and $\neg p \supset q$ true. Thus it makes $(p \supset q) \wedge (\neg p \supset q)$ true. And so it makes the sentence $((p \supset q) \wedge (\neg p \supset q)) \supset \neg p$ false, as required.

(e) $p \equiv (q \equiv r) \vdash (p \equiv q) \equiv r$



(f) $\neg(p \supset q) \wedge \neg(p \supset r) \vdash \neg q \vee \neg r$

$$
\begin{array}{c}
\neg(p \supset q) \wedge \neg(p \supset r) \\
\neg(\neg q \vee \neg r) \\
\neg(p \supset q) \\
\neg(p \supset r) \\
p \\
\neg q \\
p \\
\neg r \\
\neg\neg q \\
\neg\neg r \\
q \\
r \\
\otimes
\end{array}
$$

(g) $p \wedge (\neg r \vee s), \neg(q \supset s) \nvdash r$

$$\begin{array}{c}
p \wedge (\neg r \vee s) \\
\neg(q \supset s) \\
\neg r \\
q \\
\neg s \\
p \\
\neg r \vee s \\
\begin{array}{cc}
\neg r & s \\
 & \otimes
\end{array}
\end{array}$$

An interpretation which shows the inference to be invalid is

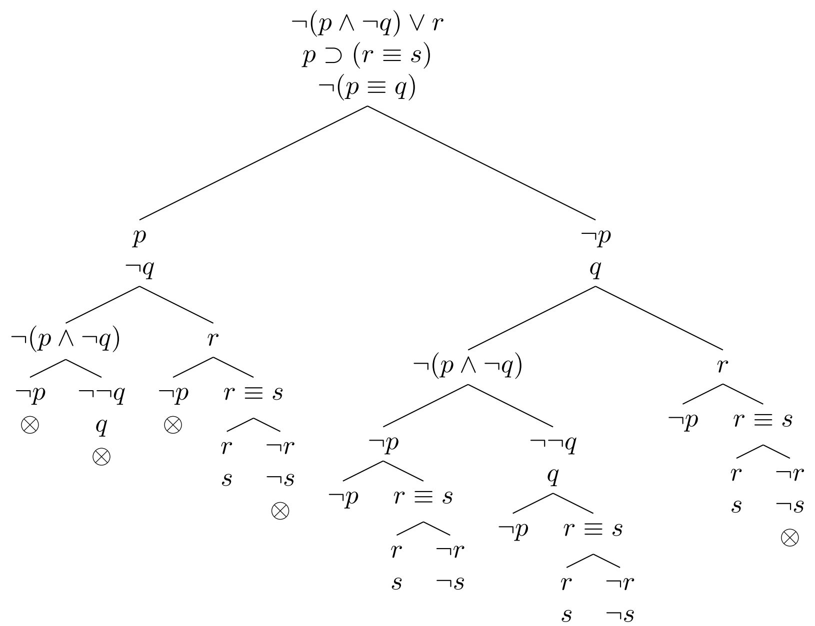
$$v(p) = v(q) = 1, v(r) = v(s) = 0$$

Applying this interpretation, we see that it makes $\neg r$, and hence $\neg r \vee s$ true. Accordingly it makes $p \wedge (\neg r \vee s)$ true. It also makes $q \supset s$ false; hence $\neg(q \supset s)$ true. Finally it makes r false. The premises are true, and conclusion false, as required.

(h) $\vdash (p \supset (q \supset r)) \supset (q \supset (p \supset r))$

$$\begin{array}{c}
\neg((p \supset (q \supset r)) \supset (q \supset (p \supset r))) \\
p \supset (q \supset r) \\
\neg(q \supset (p \supset r)) \\
q \\
\neg(p \supset r) \\
p \\
\neg r \\
\begin{array}{cc}
\neg p & (q \supset r) \\
\otimes & \begin{array}{cc} \neg q & r \\ \otimes & \otimes \end{array}
\end{array}
\end{array}$$

(i) $\neg(p \wedge \neg q) \vee r, p \supset (r \equiv s) \nvdash p \equiv q$



An interpretation which show the inference to be invalid is (fourth branch from the left)

$$v(p) = v(r) = v(s) = 1, v(q) = 0$$

Applying this interpretation, we see that, because it makes r true, it makes $(p \wedge \neg q) \vee r$ true. Because it makes $r \equiv s$ true, it also makes $p \supset (r \equiv s)$ true. Finally it makes $p \equiv q$ false, as required.

There are other interpretations which show the inference to be invalid - try reading them from the tree diagram.

(j) $p \equiv \neg\neg q, \neg q \supset (r \wedge \neg s), s \supset (p \vee q) \vdash (s \wedge q) \supset p$

$$
\begin{array}{c}
p \equiv \neg\neg q \\
\neg q \supset (r \wedge \neg s) \\
s \supset (p \vee q) \\
\neg((s \wedge q) \supset p) \\
s \wedge q \\
\neg p \\
s \\
q \\
\begin{array}{cc}
p & \neg p \\
\neg\neg q & \neg\neg\neg q \\
\otimes & \neg q \\
 & \otimes
\end{array}
\end{array}
$$

2. Give an argument to show that $A \vDash B$ iff $\vDash A \supset B$. (Hint: split the argument into two parts: left to right, and right to left. Then just apply the definition of $\vDash$. You may find it easier to prove the contrapositives. That is, assume that $\nvDash A \supset B$ and deduce that $A \nvDash B$; then vice versa.)

$\boxed{A \vDash B \text{ iff } \vDash A \supset B}$

LTR $\rightarrow$

$\boxed{\text{If } A \vDash B \text{ then } \vDash A \supset B}$

Reductio proof: (Assume the antecedent is true, and consequent false - if this entails a contradiction, then the inference holds)

Assume $A \vDash B$ and $\nvDash (A \supset B)$.

If $\nvDash (A \supset B)$, then there is an interpretation such that $v(A \supset B) = 0$ i.e. v(A) = 1, and v(B) = 0. However by the definition of '$\vDash$', and the fact that $A \vDash B$, whenever v(A) = 1, v(B) = 1. Contradiction.

$\blacksquare$

RTL $\leftarrow$

$\boxed{\text{If } \vDash A \supset B, \text{ then } A \vDash B}$

Contrapositive proof:

Assume it is not the case that $A \vDash B$; then there must be an interpretation such that v(A) = 1 and v(B) = 0. But then $\nvDash (A \supset B)$.

■

4. *Check the details omitted in 1.11.2 and 1.11.5.

1.11.2: we consider the remaining cases.

$\boxed{A \equiv B}$

Suppose v is faithful to b and $A \equiv B$ occurs on b, and that we apply a rule to it. Then two branches eventuate - one extending b with A, B (the left branch) and one extending b with $\neg A, \neg B$ (the right branch). Since v is faithful to b it makes every formula on b true - in particular $v(A \equiv B) = 1$ so $v(A) = v(B)$. That is, either $v(A) = 1$ and $v(B) = 1$ or, $v(A) = 0$ and $v(B) = 0$. In the first case, v is faithful to the left branch, in the second case v is faithful to the right branch.

Suppose v is faithful to b and $\neg(A \equiv B)$ occurs on b, and that we apply a rule to it. Then two branches eventuate - one extending b with $A, \neg B$ (the left branch) and one extending b with $\neg A, B$ (the right branch). Since v is faithful to b it makes every formula on b true - in particular $v(\neg(A \equiv B)) = 1$ so $v(A) \neq v(B)$. That is, either $v(A) = 1$ and $v(B) = 0$ or, $v(A) = 0$ and $v(B) = 1$. In the first case, v is faithful to the left branch, in the second case v is faithful to the right branch.

$\boxed{A \vee B}$

Suppose v is faithful to b and $A \vee B$ occurs on b, and that we apply a rule to it. Then two branches eventuate - one extending b with A (the left branch) and one extending b with B (the right branch). Since v is faithful to b it makes every formula on b true - in particular $v(A \vee B) = 1$ so either $v(A) = 1$ or, $v(B) = 1$ or both. In the first case, v is faithful to the left branch, in the second case v is faithful to the right branch. (In the third case, it is faithful to both.)

Suppose v is faithful to b and $\neg(A \vee B)$ occurs on b, and that we apply a rule to it. Then one branch eventuates - extending b with $\neg A$ and$\neg B$. Since v is faithful to b it makes every formula on b true - in particular $v(\neg(A \vee B)) = 1$ so $v(A) = 0$ and, $v(B) = 0$, making v faithful to the extended branch.

$\boxed{A \wedge B}$

Suppose v is faithful to b and $A \wedge B$ occurs on b, and that we apply a rule to it. Then one branch eventuates - extending b with A and B. Since v is faithful to b it makes every formula on b true - in particular $v(A \wedge B) = 1$ so $v(A) = 1$ and, $v(B) = 1$, making v faithful to the extended branch.

Suppose v is faithful to b and $\neg(A \wedge B)$ occurs on b, and that we apply a rule to it. Then two branches eventuate - one extending b with $\neg A$ (the left branch) and one extending b with $\neg B$ (the right branch). Since v is faithful to b it makes every formula on b true - in particular $v(\neg(A \wedge B)) = 1$ so either $v(A) = 0$ or, $v(B) = 0$ or both. In the first case, v is faithful to the left branch, in the second case v is faithful to the right branch. (In the third case, it is faithful to both.)

$\boxed{\neg\neg A}$

Suppose v is faithful to b and $\neg\neg A$ occurs on b, and that we apply a rule to it. Then just one branch eventuates: extending b with A. Since v is faithful to b it makes every formula on b true - in particular $v(\neg\neg A) = 1$ so $v(A) = 1$. Thus v is faithful to the extended branch.

■

1.11.5

Finish proof of the Completeness Lemma:

CL: Let b be an open complete branch of a tableau. Let v be the interpretation induced by b. Then:

if A is on b, $v(A) = 1$
if $\neg A$ is on b, $v(A) = 0$

$\neg$ and $\wedge$ have already been done

$\boxed{B \vee C}$

Suppose $B \vee C$ occurs on b; since b is complete, the appropriate rule has already been applied, hence either B or C are on the branch. By induction hypothesis, either $v(B) = 1$ or $v(C) = 1$. In either case, $v(B \vee C) = 1$ as required.

Suppose $\neg(B \vee C)$ occurs on b; since b is complete, the appropriate rule has already been applied, hence $\neg B$ and $\neg C$ are on the branch. By induction hypothesis, $v(B) = 0$ and $v(C) = 0$. Hence $v(B \vee C) = 0$ as required.

$\boxed{B \supset C}$

Suppose $B \supset C$ occurs on b; since b is complete, the appropriate rule has already been applied, hence either $\neg B$ or C are on the branch. By induction hypothesis, either $v(B) = 0$ or $v(C) = 1$. In either case, $v(B \supset C) = 1$ as required.

Suppose $\neg(B \supset C)$ occurs on b; since b is complete, the appropriate rule has already been applied, hence B and $\neg C$ are on the branch. By induction hypothesis, $v(B) = 1$ and $v(C) = 0$. Hence $v(B \supset C) = 0$ as required.

$\boxed{B \equiv C}$

Suppose $B \equiv C$ occurs on b; since b is complete, the appropriate rule has already been applied, hence either B and C, or $\neg B$ and $\neg C$ are on the branch. By induction hypothesis, either $v(B) = v(C) = 1$ or $v(B) = v(C) = 0$. In either case, $v(B \equiv C) = 1$ as required.

Suppose $\neg(B \equiv C)$ occurs on b; since b is complete, the appropriate rule has already been applied, hence either B and $\neg C$, or $\neg B$ and C are on the branch. By induction hypothesis, either $v(B) = 1$ and $v(C) = 0$ or $v(B) = 0$ and $v(C) = 1$. In either case, $v(B \equiv C) = 0$ as required.

■

5. Use the Soundness and Completeness lemmas to show that if one completed tableau for an inference is open, they all are. Infer that the result of a tableau test is indifferent to the order in which one lists the premises of the argument and applies the tableau rules.

Soundness Lemma:

If v is faithful to a branch of a tableau, b, and a tableau rule is applied to b, then v is faithful to at least one of the branches generated.

Completeness Lemma:

Let b be an open complete branch of a tableau. Let v be the interpretation induced by b. Then:

if A is on b, $v(A) = 1$
if $\neg A$ is on b, $v(A) = 0$

Suppose we have a completed open tableau for an inference $\Sigma \nvdash D$.

By the completeness theorem, for v' induced by (complete) open branch c, if A is on c, then $v(A) = 1$ and if $\neg A$ is on c, then $v(A) = 0$.

Thus, there is an interpretation v' that is faithful to the whole branch c. Clearly v' is also faithful to the initial list of c - the premises, and the negation of the conclusion. It is faithful to the initial list regardless of the order of the theorems in the initial list - this follows from the definition of faithfulness.

Now let d be any other tableau for $\Sigma \nvdash D$. v' is faithful to the initial list of d (since the initial lists of c and d are functionally identical.) By the Soundness Lemma, since v' is faithful to the initial list, it is faithful to at least one complete branch. Hence at least one branch is open. Hence all tableau for the inference are open.

■

This implies that the result of a tableau test is indifferent to the order in which one lists the premises of the argument and applies the tableau rules.