Dualising Intuitionistic Negation

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Abstract. One of Da Costa's motives when he constructed the paraconsistent logic C_ω was to dualise the negation of intuitionistic logic. In this paper I explore a different way of going about this task. A logic is defined by taking the Kripke semantics for intuitionistic logic, and dualising the truth conditions for negation. Various properties of the logic are established, including its relation to C_ω. Tableau and natural deduction systems for the logic are produced, as are appropriate algebraic structures. The paper then investigates dualising the intuitionistic conditional in the same way. This establishes various connections between the logic, and a logic called in the literature 'Brouwerian logic' or 'closed-set logic'.

Keywords: Da Costa, paraconsistency, intuitionism, C_ω, Kripke semantics, Brouwerian algebras, closed set logic, negation.

Introduction

Newton da Costa is justly famed for his epoch-making work on paraconsistent logic. At a time when the very idea of such a subject seemed to most logicians to be outrageous, he showed that the notion is perfectly coherent, and rich in mathematical and philosophical applications. Only the most hardened of troglodytes can now doubt this. Newton, through his own work, and that of his students and collaborators, has played a major role in the establishment of the subject. It is a pleasure to dedicate this article to him.

In some of his earliest work on paraconsistency, he invented a family of systems of logic now normally called 'the C systems'. In his seminal paper, 'On the Theory of Inconsistent Formal Systems' (da Costa 1974), he describes the systems and some of the motivating ideas. Though he never says as much, one of the ideas, it has always seemed to me, is this. Intuitionist logic is a logic that allows for "truth value gaps"; for example, the Law of Excluded Middle fails. It ought to be possible to construct a logic which is the same, except that negation behaves in a dual fashion, so that the Law of Non-Contradiction fails. The C systems, and particularly C_ω,[1] achieve this, though there are clear costs: for example the substitutivity of provable equivalents fails in negated contexts (which creates real problems for the algebraicisation of the systems; see Mortensen 1980).

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Newton proceeded axiomatically, preserving the positive part of intuitionist logic, and changing the axioms for negation. But the various semantics for intuitionist logic suggest other ways of pursuing da Costa's goal. In what follows I will show how this may be done, taking a lead from the Kripke semantics for intuitionist logic.[2]

1. Dualising Intuitionist Negation

Kripke semantics for intuitionist logic are well known, and require little by way of exposition (see, e.g., Priest 2008, ch. 6). A Kripke interpretation for intuitionist propositional logic is a structure $\langle W, R, v\rangle$, where W is a set of worlds, R is a binary relation on worlds which is reflexive and transitive, and v assigns a truth value, 1 or 0, to each propositional parameter, p, at each world, subject to the following constraint: For each parameter, p:

$$\text{if } wRw' \text{ and } v_w(p) = 1 \text{ then } v_{w'}(p) = 1.$$

This is called the Heredity Constraint. The truth conditions for the logical operators are as follows. For all $w \in W$:

$$v_w(\alpha \wedge \beta) = 1 \text{ iff } v_w(\alpha) = 1 \text{ and } v_w(\beta) = 1,$$
$$v_w(\alpha \vee \beta) = 1 \text{ iff } v_w(\alpha) = 1 \text{ or } v_w(\beta) = 1,$$
$$v_w(\alpha \rightarrow \beta) = 1 \text{ iff for all } w' \text{ such that } wRw', \text{ if } v_{w'}(\alpha) = 1 \text{ then } v_{w'}(\beta) = 1,$$
$$v_w(\neg\alpha) = 1 \text{ iff for all } w' \text{ such that } wRw', v_{w'}(\alpha) = 0.$$

Alternatively, for negation, we may take the language to contain a logical constant, $\bot$, such that for all $w \in W$:

$$v_w(\bot) = 0.$$

$\neg\alpha$ may then be defined as $\alpha \rightarrow \bot$. It is not difficult to show by induction that the Heredity Constraint extends to all formulas, not just propositional parameters.

An inference is intuitionistically valid ($\models_I$) if it preserves truth at all worlds of all interpretations. It is not difficult to show that $\models_I \alpha \rightarrow \neg\neg\alpha$, but $\not\models_I \neg\neg\alpha \rightarrow \alpha$; and $\models_I (\alpha \wedge \neg\alpha) \rightarrow \beta$, but $\not\models_I \beta \rightarrow (\alpha \vee \neg\alpha)$.

To produce a logic that is the same as intuitionist logic, except that the negation is dualised, everything remains the same, except that the truth conditions for negation are replaced by their dual:

$$v_w(\neg\alpha) = 1 \text{ iff for some } w' \text{ such that } w'Rw, v_{w'}(\alpha) = 0.$$

It is not difficult to check that the Heredity Constraint still generalises to all formulas. The case for $\neg$ in the inductive argument goes as follows. Suppose that wRw' and

$v_w(\neg\alpha) = 1$. Then for some w'' such that $w''Rw$, $v_{w''}(\alpha) = 0$. By the transitivity of R, $w''Rw'$. Hence $v_{w'}(\neg\alpha) = 1$.

Let us call this logic, in honour of Newton, *Da Costa Logic*, and write its semantic consequence relation as $\vDash_D$. As is to be expected, the properties of negation in da Costa logic are dual to those of intuitionist logic. In particular, as we shall see in a moment, we have $\nvDash_D \alpha \to \neg\neg\alpha$, but $\vDash_D \neg\neg\alpha \to \alpha$; and $\nvDash_D (\alpha \wedge \neg\alpha) \to \beta$, but $\vDash_D \beta \to (\alpha \vee \neg\alpha)$.

2. Tableaux

A tableaux system for intuitionist logic is as follows. Lines are of the form $\alpha, +i$, $\alpha, -i$, or irj, where i and j are natural numbers. The tableau for the inference with premises $\alpha_1, \ldots, \alpha_n$ and conclusion β, starts with lines of the form $\alpha_1, +0 \ldots \alpha_n, +0$, and $\beta, -0$. The tableaux rules are as follows:

$$\begin{array}{c} \alpha \wedge \beta, +i \\ \downarrow \\ \alpha, +i \\ \beta, +i \end{array} \qquad \begin{array}{c} \alpha \wedge \beta, -i \\ \swarrow \quad \searrow \\ \alpha, -i \quad \beta, -i \end{array}$$

$$\begin{array}{c} \alpha \vee \beta, +i \\ \swarrow \quad \searrow \\ \alpha, +i \quad \beta, +i \end{array} \qquad \begin{array}{c} \alpha \vee \beta, -i \\ \downarrow \\ \alpha, -i \\ \beta, -i \end{array}$$

$$\begin{array}{c} \alpha \to \beta, +i \\ irj \\ \swarrow \quad \searrow \\ \alpha, -j \quad \beta, +i \end{array} \qquad \begin{array}{c} \alpha \to \beta, -i \\ \downarrow \\ irk \\ \alpha, +k \\ \beta, -k \end{array}$$

$$\begin{array}{c} \neg\alpha, +i \\ irj \\ \downarrow \\ \alpha, -j \end{array} \qquad \begin{array}{c} \neg\alpha, -i \\ \downarrow \\ irk \\ \alpha, +k \end{array}$$

where k, in any of these rules, is a number new to the branch. We also have rules for r and the Heredity Condition:

$$\begin{array}{c} . \\ \downarrow \\ iri \end{array} \qquad \begin{array}{c} irj \\ jrk \\ \downarrow \\ irk \end{array} \qquad \begin{array}{c} irj \\ p, +i \\ \downarrow \\ p, +j \end{array}$$

A branch of the tableau is closed if it contains lines of the form $\alpha, +i$ and $\alpha, -i$. And an inference is tableau valid if all branches close.

A tableau system for Da Costa logic is exactly the same, except that the rules for negation are replaced by:

$$\begin{array}{cc} \neg\alpha, +i & \neg\alpha, -i \\ \downarrow & jri \\ kri & \downarrow \\ \alpha, -k & \alpha, +j \end{array}$$

wnere again, k is new to the branch. Let us use $\vdash_T$ to denote tableau validity in this system.

Here are tableaux to establish that $\vdash_T \neg\neg p \rightarrow p$ and $\vdash_T q \rightarrow (p \vee \neg p)$. (I double up on some lines to save space.)

$$\begin{array}{cc} \neg\neg p \rightarrow p, -0 & q \rightarrow (p \vee \neg p), -0 \\ 0r0 & 0r0 \\ 0r1, 1r1 & 0r1, 1r1 \\ \neg\neg p, +1 & q, +1 \\ p, -1 & p \vee \neg p, -1 \\ 2r1, 2r2 & p, -2 \\ \neg p, -2 & \neg p, -2 \\ p, +2 & p + 2 \\ p, +1 & \times \\ \times & \end{array}$$

In the left tableau, the penultimate line follows from the preceding one because $2r2$, and the last line is an application of the Heredity Rule. In the right tableau, the last line follows from the preceding one because $2r2$.

Here is a tableaux to show that $\nvdash_T (p \wedge \neg p) \rightarrow q$.

$$\begin{array}{c} (p \wedge \neg p) \rightarrow q, -0 \\ 0r0 \\ 0r1, 1r1 \\ p \wedge \neg p, +1 \\ q, -1 \\ p, +1 \\ \neg p, +1 \\ 2r1, 2r2 \\ p, -2 \end{array}$$

Counter-models are read off from an open branch of a tableau in the natural way. There is a world, w_i, for every i on the branch; $w_i R w_j$ iff irj is on the branch; and

$v_w(p) = 1$ iff $p, +i$ is on the branch. Thus the open tableau above determines a model which may be depicted as follows. I omit the arrows given by reflexivity and transitivity for simplicity:

$$\begin{array}{ccccc} -p, -q & w_0 & & w_2 & -p, -q \\ & & \searrow \quad \swarrow & & \\ & & w_1 & & \\ & & +p, -q & & \end{array}$$

It is routine to check that the counter-model works.

The tableau showing that $\nvdash_T p \to \neg\neg p$ is infinite. But here is a (diagram of a) finite counter-model.

$$\begin{array}{ccc} w_1 & \to & w_0 \\ -p & & +p \end{array}$$

$\neg p$ holds at w_1 and w_0. So $\neg\neg p$ fails at w_0, as then, does $p \to \neg\neg p$.

Soundness and completeness for finite sets of premises are established essentially as for intuionistic logic, as in Priest 2008, 6.7. The only differences are in the cases for negation.

For the Soundness Lemma. Suppose that the function f shows branch b to be faithful to the interpretation $\mathscr{I}$. If we apply the first rule for negation, then $v_{f(i)}(\neg\alpha) = 1$. So for some w such that $wRf(i)$, $v_w(\alpha) = 0$. Let f' be the same as f, except that $f'(k) = w$. Then f' shows the extended branch to be faithful to $\mathscr{I}$. If we apply the second rule for negation, then $v_{f(i)}(\neg\alpha) = 0$, and $f(j)Rf(i)$. So for every w such that $wRf(i)$, and in particular, $f(j)$, $v_w(\alpha) = 1$. So f shows the extended branch to be faithful to $\mathscr{I}$.

For the Completeness Lemma. Let $\mathscr{I}$ be the interpretation induced by branch b. Suppose that $\neg\alpha, +i$ is on b. Then for some k such that kri is on b, $\alpha, -k$ is on b. So $w_k R w_i$, and $v_{w_k}(\alpha) = 0$ by induction hypothesis (IH). Hence, $v_{w_i}(\neg\alpha) = 1$. Suppose that $\neg\alpha, -i$ is on b. Then for any j such that jri is on b, $\alpha, +j$ is on b. So for every w_j such that $w_j R w_i$, $v_{w_j}(\alpha) = 1$, by IH. Hence, $v_{wi}(\neg\alpha) = 0$.

Trees for infinite premise sets can be defined, and the soundness and completeness proofs extended to these, as in as Priest 2008, 12.7 and 12.10. Finally, I note that the semantics and tableaux can be extended to the first-order case, where the positive logic is positive intuitionist logic. The construction is exactly as in Priest 2008, ch. 20.

3. Facts about Da Costa Logic

In this section, I will establish some further facts about Da Costa logic.

1. First, consider the fragment of the logic with sentences containing just $\wedge$, $\vee$, and $\neg$. In the case of intuitionist logic, the logical truths of this language do not coincide with classical logic (since, e.g., $p \vee \neg p$ is not a logical truth). In da Costa logic they do (and so, in particular, $\models_D \neg(\alpha \wedge \neg\alpha)$). Any classical interpretation is effectively a one-world interpretation for da Costa logic. Hence, anything valid in da Costa logic is valid in classical logic. Conversely, suppose that $\not\models_D \gamma$. Then there is an interpretation, and a world, w, of the interpretation, such that $v_w(\gamma) \neq 1$. Let Γ_x be the set of propositional parameters in γ true at world x. If yRx, then $\Gamma_y \subseteq \Gamma_x$, by heredity. There must be a world, x, such that xRw and for all y such that yRw, $\Gamma_y = \Gamma_x$. For otherwise, we could find an infinite sequence $\ldots x_{n+1}Rx_nR\ldots Rw$ such that $\ldots\Gamma_{x_{n+1}} \subset \Gamma_{x_n} \subset \ldots \subset \Gamma_w$, which is impossible, since Γ_w is finite. Call this world w'. By heredity, $v_{w'}(\gamma) \neq 1$.

Now, for all α and x such that xRw', α is true at w' iff α is true at x. From right to left this holds by heredity. The converse is proved by induction. For propositional parameters it is true by the construction of w'. The cases for conjunction and disjunction are trivial. For negation, suppose that $\neg\alpha$ holds at w'. Then for some y such that yRw', α fails at y. By IH, α fails at w' and so x. So $\neg\alpha$ holds at x, as required.

Finally, define a classical interpretation, μ, which agrees with w' on the propositional parameters. If we can show that for all α, $v_{w'}(\alpha) = 1$ iff $\mu(\alpha) = 1$, we are home, since it follows that $\mu(\gamma) = 0$. The argument is by induction. The cases for propositional parameters, conjunction, and disjunction are trivial. For negation, suppose that $\mu(\neg\alpha) = 1$, then $\mu(\alpha) = 0$. By IH, $v_{w'}(\alpha) = 0$. Hence, $v_{w'}(\neg\alpha) = 1$. Conversely, suppose that $v_{w'}(\neg\alpha) = 1$. Then for some x such that xRw', $v_x(\alpha) = 0$. By the lemma of the last paragraph, $v_{w'}(\alpha) = 0$. By induction, $\mu(\alpha) = 0$, so $\mu(\neg\alpha) = 1$.

2. Next, we may define the logical constant $\top$ as $\alpha \vee \neg\alpha$, for any α we choose, since this is true at every world of every interpretation. We may define $\perp$ as $\neg\top$. This is true at no world of any interpretation; so for all β, $\perp \models_D \beta$. (In particular, then, da Costa logic is finitely trivialisable, as C_ω is not (da Costa and Guillaume 1964), since $\neg(\alpha \vee \neg\alpha) \models_D \beta$.) The addition of $\perp$ to positive intuitionistic logic produces full intuitionist logic. Hence da Costa logic is an extension of intuitionistic logic. In particular, we may define the intuitionist negation of α as $\alpha \to \perp$. It is not difficult to check that $\alpha \to \perp \models_D \neg\alpha$, but not the other way around.

However, intuitionist logic does not contain da Costa logic. Specifically, there is no formula in the language of intuionistic logic, $\alpha(p)$, which is logically equivalent to $\neg p$. To see this, consider a Kripke interpretation, $\mathcal{I}$, of intuitionist logic of the following form (again I omit the arrows of reflexivity):

$$\begin{array}{ccc} w_{-1} & \to & w_0 \\ -p & & +p \end{array}$$

Let $\mathcal{I}'$ be exactly the same, except that it contains only w_0. Every intuitionist formula $\alpha(p)$ has the same truth value at w_0 in both interpretations. (Since all truth conditions are "forward looking".) But if $\alpha(p)$ were logically equivalent to $\neg p$, it would have different truth values in the two interpretations, since $\neg p$ holds at w_0 in the first, and fails at w_0 in the second.

3. Finally, the connection between da Costa Logic and C_ω. Axiomatically, C_ω is obtained from positive intuitionist logic by adding the axiom schemas (Priest 2007, 4.3):

$$\neg\neg\alpha \rightarrow \alpha,$$
$$\alpha \vee \neg\alpha.$$

In other words, let I^+ be positive intuitionist logic. Let Δ_1 be the set of all formulas of the form $\neg\neg\alpha \rightarrow \alpha$; and let Δ_2 be all sets of formulas of the form $\alpha \vee \neg\alpha$. Then an inference is valid in C_ω iff the conclusion follows from the premises plus members of $\Delta_1 \cup \Delta_2$ in positive intuitionist logic. It is easy to check that da Costa Logic is at least as strong as C_ω, since it verifies all the members of $\Delta_1 \cup \Delta_2$ at every world, and the rules of positive intuitionist logic preserve truth at every world.

However, da Costa Logic is stronger than C_ω. Where $\longleftrightarrow$ is defined in the usual way, though $\alpha \longleftrightarrow (\alpha \wedge \alpha)$ is valid in C_ω, $\neg\alpha \longleftrightarrow \neg(\alpha \wedge \alpha)$ is not (Priest 2007, 4.3). It is easy to check that it is valid in da Costa logic. In fact, da Costa logic delivers the substitutivity of logical equivalents. In particular, if $\models_D \alpha \longleftrightarrow \beta$ then $\models_D \neg\alpha \longleftrightarrow \neg\beta$.[3] For suppose that $\models_D \alpha \longleftrightarrow \beta$, but not $\models_D \neg\alpha \longleftrightarrow \neg\beta$. Then, there is an interpretation with a world, w, where $\neg\alpha$ is true, but $\neg\beta$ is not (or vice versa, which is similar). Hence, there is a w' such that $w'Rw$, such that α is false at w', and β is true. This is impossible since $\alpha \longleftrightarrow \beta$ is a logical truth, and so holds at w'. It should be noted, however, that in da Costa Logic, it is not the case that $\alpha \longleftrightarrow \beta \models_D \neg\alpha \longleftrightarrow \neg\beta$. Here is a (diagram of a) counter-model for the instance $p \longleftrightarrow q \not\models_D \neg p \longleftrightarrow \neg q$:

$$\begin{array}{ccc} w_1 & \rightarrow & w_0 \\ +p, -q & & +p, +q \end{array}$$

It is easy to check that $p \longleftrightarrow q$ is true at w_0, but $\neg p \longleftrightarrow \neg q$ is not.

4. Natural Deduction

For theoretical investigations, rather than practical purposes, natural deduction systems or sequent calculi are often more useful than tableau systems. So let me formulate a natural deduction system (in the style of Prawitz 1965) for da Costa logic. A rule system for positive intuitionist logic has the following rules.

$$\frac{\alpha \wedge \beta}{\alpha} \qquad \frac{\alpha \wedge \beta}{\beta} \qquad \frac{\alpha \quad \beta}{\alpha \wedge \beta}$$

$$\frac{\alpha}{\alpha\vee\beta}\qquad\frac{\beta}{\alpha\vee\beta}\qquad\frac{\alpha\vee\beta\quad\begin{array}{c}\overline{\alpha}\\ \vdots\\ \gamma\end{array}\quad\begin{array}{c}\overline{\beta}\\ \vdots\\ \gamma\end{array}}{\gamma}$$

$$\frac{\alpha\quad\alpha\rightarrow\beta}{\beta}\qquad\qquad\frac{\begin{array}{c}\overline{\alpha}\\ \vdots\\ \beta\end{array}}{\alpha\rightarrow\beta}$$

The vertical dots indicate a sub-proof, and the bar over the assumption indicates that the rule discharges it. To these, we add the following rules for negation:

$$\frac{\cdot}{\alpha\vee\neg\alpha}\qquad\frac{\alpha\vee\beta\quad\neg\alpha}{\beta}$$

The first of these means that $\alpha\vee\neg\alpha$ can always be introduced, and when it is, it is not an undischarged assumption. In the second, it is important that the proof of $\alpha\vee\beta$ depends on no undischarged assumptions, i.e., that it is a logical truth. (If we drop this assumption, it is easy to show that we can prove Explosion, and hence that we have a rule system for classical logic.) Let us use $\vdash_N$ for natural-deduction derivability.

As an example to illustrate the rules, let us show that if $\alpha\vdash_N\beta$ then $\neg\beta\vdash_N\neg\alpha$. Suppose that we have a proof, Π, of β with only α as an undischarged assumption. We show that $\neg\beta\vdash_N\neg\alpha$ as follows:

$$\dfrac{\dfrac{\dfrac{}{\alpha\vee\neg\alpha}\quad\dfrac{\begin{array}{c}\overline{\alpha}\\ \Pi\\ \beta\end{array}}{\neg\alpha\vee\beta}\quad\dfrac{\overline{\neg\alpha}}{\neg\alpha\vee\beta}}{\neg\alpha\vee\beta}\qquad\neg\beta}{\neg\alpha}$$

It is straightforward to show that the rule system is sound with respect to the Kripke semantics. Specifically, we establish that if we have a proof of α, with undischarged assumptions in Σ, then $\Sigma\models_D\alpha$. This is proved by a recursion over the construction of proofs. The cases for the intuitionist rules are straightforward. The argument for the first negation rule is also straightforward. Here is the argument for the second. We suppose, by IH, that $\models_D \alpha \vee \beta$ and $\Sigma \models_D \neg\alpha$. Consider any interpretation, and world, w , where all the members of Σ are true. Then $v_w(\neg\alpha) = 1$. So for some w' such that $w'Rw$, $v_{w'}(\alpha) = 0$. Since $v_{w'}(\alpha \vee \beta) = 1$, $v_{w'}(\beta) = 1$. By heredity, $v_w(\beta) = 1$, as required.

Completeness is proved using the canonical model construction. Call a set of formulas, Σ, *deductively closed* if:

if $\Sigma \vdash_N \alpha$ then $\alpha \in \Sigma$,

and *prime*, if:

if $\alpha \vee \beta \in \Sigma$ then $\alpha \in \Sigma$ or $\beta \in \Sigma$.

Σ is *prime deductively close* (pdc) if it is both. It is easy to check that if Σ is pdc then:

$\alpha \wedge \beta \in \Sigma$ iff $\alpha \in \Sigma$ and $\beta \in \Sigma$;
$\alpha \vee \beta \in \Sigma$ iff $\alpha \in \Sigma$ or $\beta \in \Sigma$.

Next, define $\Sigma \vdash_N \Pi$ to mean that for some disjunction, π, of members of Π, $\Sigma \vdash_N \pi$. We have the following:

Fundamental Lemma. *If* $\Sigma \nvdash_N \Pi$, *there is a pdc,* Δ, *such that* $\Sigma \subseteq \Delta$ *and* $\Delta \nvdash_N \Pi$.

Proof. Enumerate the formulas of the language, $\alpha_0, \alpha_1, \ldots$ Define by recursion:

$\Delta_0 = \Sigma$;
$\Delta_{n+1} = \Delta_n \cup \{\alpha_n\}$ if $\Delta_n \cup \{\alpha_n\} \nvdash_N \Pi$; otherwise $\Delta_{n+1} = \Delta_n$;
$\Delta = \bigcup_{i<\omega} \Delta_i$.

Clearly, for each n, $\Delta_n \nvdash_N \Pi$. It follows that $\Delta \nvdash_N \Pi$. Δ is deductively closed. For suppose that $\Delta \vdash_N \alpha$, but $\alpha \notin \Delta$. Then for some n, and disjunction of members of Π, π, $\Delta_n \cup \{\alpha\} \vdash_N \pi$. But then $\Delta_n \vdash_N \pi$. Hence, $\Delta \vdash_N \Pi$, which is impossible. Δ is prime. For suppose that $\alpha \vee \beta \in \Delta$, but $\alpha \notin \Delta$ and $\beta \notin \Delta$. Then for some n, m, π_1, and π_2, $\Delta_n \cup \{\alpha\} \vdash_N \pi_1$ and $\Delta_m \cup \{\beta\} \vdash_N \pi_2$. It follows that $\Delta \cup \{\alpha\} \vdash_N \pi_1 \vee \pi_2$ and $\Delta \cup \{\beta\} \vdash_N \pi_1 \vee \pi_2$. Hence, $\Delta \cup \{\alpha \vee \beta\} \vdash_N \Pi$, which is impossible since $\alpha \vee \beta \in \Delta$.

Define the canonical model, $\langle W, R, v \rangle$ as follows.

$W = \{\Gamma : \Gamma \text{ is pdc}\}$,
$v_\Gamma(p) = 1$ iff $p \in \Gamma$,
$\Gamma R \Delta$ iff $\Gamma \subseteq \Delta$.

The model is indeed an interpretation. The Heredity Condition is obviously satisfied, and R is reflexive and transitive.

To finish the proof, we need two further lemmas.

Lemma 1. *If Γ is pdc and $\gamma \rightarrow \delta \notin \Gamma$, there is a pdc Θ such that $\Gamma R \Theta$, $\gamma \in \Theta$, and $\delta \notin \Theta$.*

Proof. Let $\Sigma = \Gamma \cup \{\gamma\}$, and $\Pi = \{\delta\}$. If $\Sigma \nvdash_N \Pi$, we can let Θ be the Δ of the Fundamental Lemma. This obviously has all the right properties. So suppose that $\Gamma \cup \{\gamma\} \vdash_N \delta$. Then $\Gamma \vdash_N \gamma \rightarrow \delta$, which is impossible.

Lemma 2. *If Δ is pdc, and $\neg\delta \in \Delta$, there is a pdc Θ such that $\Theta R \Delta$, $\delta \notin \Theta$.*

Proof. Let $\Sigma = \phi$, and $\Pi = \{\gamma : \gamma \notin \Delta\} \cup \{\delta\}$. If $\Sigma \nvdash_N \Pi$, we can let Θ be the Δ of the Fundamental Lemma. This clearly has the right properties. So suppose that $\Sigma \vdash_N \Pi$. Then there is some disjunction of members of Π, $\gamma = \gamma_1 \vee \ldots \vee \gamma_n$, such that $\vdash_N \delta \vee \gamma$. Thus, $\neg\delta \vdash_N \gamma$. Since $\neg\delta \in \Delta$, $\gamma \in \Delta$. So for some i, $\gamma_i \in \Delta$, which is impossible.

We can now show by induction that for every formula, α, and every $\Gamma \in W$:

$$v_\Gamma(\alpha) = 1 \text{ iff } \alpha \in \Gamma.$$

The atomic case is true by definition. The cases for $\wedge$ and $\vee$ are straightforward.

For $\rightarrow$: Suppose that $\alpha \rightarrow \beta \in \Gamma$. Then for all Γ such that $\Gamma R \Delta$, $\alpha \rightarrow \beta \in \Delta$, so if $\alpha \in \Delta$, $\beta \in \Delta$. That is, by IH, if $v_\Delta(\alpha) = 1$, $v_\Delta(\beta) = 1$. So $v_\Gamma(\alpha \rightarrow \beta) = 1$. Conversely, suppose that $\alpha \rightarrow \beta \notin \Gamma$. Then by Lemma 1, there is a pdc Δ, such that $\Gamma R \Delta$, $\alpha \in \Delta$ and $\beta \notin \Delta$. By IH, $v_\Delta(\alpha) = 1$ and $v_\Delta(\beta) = 0$. So $v_\Gamma(\alpha \rightarrow \beta) = 0$.

For $\neg$: Suppose that $\neg\alpha \in \Gamma$. By Lemma 2, there is a pdc Δ such that $\Delta R \Gamma$, and $\alpha \notin \Gamma$. By IH, $v_\Delta(\alpha) = 0$. Hence, $v_\Gamma(\neg\alpha) = 1$. Conversely, suppose that $\neg\alpha \notin \Gamma$. Then for every Δ such that $\Delta R \Gamma$, $\neg\alpha \notin \Delta$. Since $\alpha \vee \neg\alpha \in \Delta$, $\alpha \in \Delta$. By IH, $v_\Delta(\alpha) = 1$, and $v_\Gamma(\neg\alpha) = 0$.

Finally, suppose that $\Sigma \nvdash_N \alpha$. Then by the Fundamental Lemma, there is a pdc Δ such that $\Sigma \subseteq \Delta$, and $\alpha \notin \Delta$. The result follows.

5. Da Costa Algebras

Let us now turn to the algebraic structures associated with da Costa logic. The algebraic structures corresponding to intuitionistic logic are Heyting algebras (HA). Unsurprisingly, the algebraic structures corresponding to da Costa logic have the same positive part as Heyting algebras, but a dual treatment of negation.

The standard definition of a HA is as follows.[4] The algebra is a structure $\langle A, \vee, \wedge, \rightarrow, \perp \rangle$, where $\langle A, \vee, \wedge \rangle$ is a distributive lattice. In particular, we can define a partial order, $\leq$, on the lattice ($a \leq b$ iff $a \wedge b = a$). $\wedge$ is a residual with respect to the ordering, that is: $a \wedge b \leq c$ iff $a \leq b \rightarrow c$; and $\bot$ is a minimal element, that is: $\bot \leq a$. The relative pseudo-complement of a, $\neg a$, is defined as $a \rightarrow \bot$. To dualise this, we need to reformulate slightly. In particular, a Heyting algebra can be taken to be a structure $\langle A, \vee, \wedge, \rightarrow, \neg \rangle$ where everything is as before, except that mention of $\bot$ is dropped, and $\neg$ satisfies the conditions:

(1) $a \wedge \neg a \leq b$,
(2) $a \wedge b \leq c \wedge \neg c$ then $a \leq \neg b$.

It is not difficult to see that this is an equivalent formulation. For a start, the standard formulation implies these two conditions:

	$a \rightarrow b \leq a \rightarrow b$
So	$a \wedge (a \rightarrow b) \leq b$
In particular	$a \wedge (a \rightarrow \bot) \leq \bot$
And since	$\bot \leq b$
(1) follows.	

Now, suppose that $a \wedge b \leq c \wedge (c \rightarrow \bot)$. As we have just seen, $c \wedge (c \rightarrow \bot) \leq \bot$; so $a \wedge b \leq \bot$, and $a \leq b \rightarrow \bot$, which is (2).

Conversely, by (1), $a \wedge \neg a = b \wedge \neg b \leq c$. So we can define $\bot$ as $a \wedge \neg a$ for any a. Since $a \wedge \neg a \leq \bot$, $\neg a \leq a \rightarrow \bot$. And since $a \wedge (a \rightarrow \bot) \leq \bot$, it follows from (2) that $a \rightarrow \bot \leq \neg a$. So $a \rightarrow \bot = \neg a$.

Having established this, it is clear what the dual of a Heyting algebra is. A *da Costa algebra* is a structure $\langle A, \vee, \wedge, \rightarrow, \neg \rangle$ where everything is the same, except that:

(1d) $a \leq b \vee \neg b$,
(2d) $c \vee \neg c \leq a \vee b$ then $\neg a \leq b$.

Clearly, we can define a top element, $\top$, as $b \vee \neg b$, for any b; and $a \leq \top$. (2d) can therefore be written as: if $\top \leq a \vee b$ then $\neg a \leq b$.

Given a da Costa algebra, we can define an interpretation as a map, μ, from the language into the algebra which is a homomorphism with respect to the operators. (So that $\mu(\alpha \rightarrow \beta) = \mu(\alpha) \rightarrow \mu(\beta)$, etc.) An inference with a finite set of premises is algebraically valid, $\alpha_1, \ldots, \alpha_n \models_A \beta$, iff for every algebra, and every interpretation into the algebra, $\mu(\alpha_1) \wedge \ldots \wedge \mu(\alpha_n) \wedge \top \leq \mu(\beta)$, i.e., $\top \leq \mu(\alpha_1) \wedge \ldots \wedge \mu(\alpha_n) \rightarrow \mu(\beta)$, i.e., $\top \leq \mu((\alpha_1 \wedge \ldots \wedge \alpha_n) \rightarrow \beta)$.

For finite premise sets, algebraic validity is equivalent to validity in the Kripke semantics. We show this by proving that the algebraic semantics are sound and complete with respect to the rule system of the previous section.

For soundness, we show that if we have a deduction with undischarged assumptions Σ and conclusion β then $\Sigma \models_A \beta$. The proof is by recursion over the construction of proofs. The positive cases are as in intuitionist logic, and the cases for the rules for negation just deploy (1d) and (2d) in the obvious way. Here is the case for (2d). Suppose that we have a proof from zero undischarged assumptions of $\alpha \vee \beta$. Then by IH, for any algebra and interpretation μ, $\top \leq \mu(\alpha \vee \beta)$. So $\top \leq \mu(\alpha) \vee \mu(\beta)$. By (2d), $\neg\mu(\alpha) \leq \mu(\beta)$, and $\top \wedge \mu(\neg\alpha) \leq \mu(\beta)$. By IH, $\Sigma \models_A \neg\alpha$. It follows that $\Sigma \models_A \beta$.

For finite sets of premises, we prove the converse by constructing the Lindenbaum algebra. Define a relationship on formulas $\alpha \sim \beta$ to mean that $\alpha \dashv\vdash_N \beta$ (where this indicates bi-deducibility). It is easy to see that $\sim$ is an equivalence relation. Moveover, it is a congruence relation with respect to the logical constants. Thus, if $\alpha \sim \beta$ then $\alpha \vee \gamma \sim \beta \vee \gamma$, $\gamma \wedge \alpha \sim \gamma \wedge \beta$, $\gamma \rightarrow \alpha \sim \gamma \rightarrow \beta$, $\neg\alpha \sim \neg\beta$, etc. Showing this for the positive connectives is as in intuitionist logic. The case for negation follows from the fact that if $\alpha \vdash_N \beta$ then $\neg\beta \vdash_N \neg\alpha$, which we have already seen. Let $[\alpha]$ be the equivalence class of α under $\sim$. $\top = [\gamma \vee \neg\gamma]$.

We define the algebra $\langle A, \vee, \wedge, \rightarrow, \neg\rangle$ where $A = \{[\alpha] : \alpha$ a formula of the language$\}$; and $[\alpha] \vee [\beta] = [\alpha \vee \beta]$, $\neg[\alpha] = [\neg\alpha]$, etc. (This is well defined, since $\sim$ is a congruence relation.) $[\alpha] \leq [\beta]$ iff $[\alpha \wedge \beta] = [\alpha]$ iff $\alpha \wedge \beta \dashv\vdash_N \alpha$ iff $\alpha \vdash_N \beta$. It is easy to check that the algebra is a da Costa algebra. Here, for example, is the verification of (2d). Suppose that $\top \leq [\alpha] \vee [\beta]$. Then $\gamma \vee \neg\gamma \vdash_N \alpha \vee \beta$. Hence, $\vdash_N \alpha \vee \beta$, $\neg\alpha \vdash_N \beta$, and $[\neg\alpha] \leq [\beta]$. Now, suppose that $\alpha_1, \ldots, \alpha_n \nvdash_N \beta$. Then $[\alpha_1 \wedge \ldots \wedge \alpha_n] \nleq [\beta]$. Consider the function, μ, such that $\mu(\alpha) = [\alpha]$. This is an evaluation, since it is a homomorphism into the algebra. We have $\mu(\alpha_1 \wedge \ldots \wedge \alpha_n) \nleq \mu(\beta)$, so $\alpha_1, \ldots, \alpha_n \nvDash_A \beta$.

To extend the result to arbitrary sets of premises we can go a couple of ways. First, we can restrict ourselves to complete da Costa algebras (where every set of members of the algebra has a meet and join), and define $\Sigma \models_A \alpha$ to mean that for all algebras and evaluation functions, μ, $\top \wedge \bigwedge\{\mu(\sigma) : \sigma \in \Sigma\} \leq \mu(\alpha)$. We then proceed as in the finite case, though we have to show that the Lindenbaum algebra can be embedded in a complete Lindenbaum algebra. More simply, we can just define $\Sigma \models_A \alpha$ to mean that for some finite subset $\Sigma' = \{\sigma_1, \ldots, \sigma_n\}$ of Σ, $\mu(\sigma_1) \wedge \ldots \wedge \mu(\sigma_n) \wedge \top \leq \mu(\alpha)$. The result then follows from the finite case. I will adopt this definition for all algebraic semantics which follow, and so restrict myself to the finite premise sets.

6. Dualising $\rightarrow$

Just as $\neg$ dualises intuitionistic negation, so may we dualise the intuitionistic $\rightarrow$. Let us start by returning to the Kripke semantics. We add a new binary connective, $\leftarrow$, to the language, with the conditions dual to those for $\rightarrow$:

$$v_w(\alpha \leftarrow \beta) = 1 \text{ iff for some } w' \text{ such that } w'Rw,\ v_{w'}(\alpha) = 1 \text{ and } v_{w'}(\beta) = 0.$$

It is simple to check that the Heredity Condition still obtains. If we take the language to contain a logical constant, $\top$, such that for all $w \in W$:

$$v_w(\top) = 0,$$

then it is easy to see that $\neg\alpha$ may then be defined as $\top \leftarrow \alpha$. I will use $\vDash_D$ for the consequence relation of the extended semantics as well.

Tableaux for the new connective can be obtained by adding the rules:

$$\begin{array}{c} \alpha \leftarrow \beta, +i \\ \downarrow \\ kri \\ \alpha, +k \\ \beta, -k \end{array} \qquad \begin{array}{c} \alpha \leftarrow \beta, -i \\ jri \\ \swarrow \searrow \\ \alpha, -j \quad \beta, +i \end{array}$$

where, in the first rule, k is new to the branch. It is a simple matter to show that the extended rule system is sound and complete with respect to the new semantics, and I leave it as an exercise.

A natural deduction system for the extended language is obtained by adding the rules:

$$\frac{\begin{array}{c}\overline{\alpha} \\ \vdots \\ \beta \vee \gamma\end{array} \quad \alpha \leftarrow \beta}{\gamma} \qquad \frac{\alpha}{(\alpha \leftarrow \beta) \vee \beta}$$

where, in the first, α is the only undischarged assumption in the proof of $\beta \vee \gamma$.

I will use $\vdash_N$ for the extended natural deduction system as well. To prove soundness for the extended natural deduction system, we simply have to check the new rules. For the first, suppose that $\alpha \vDash_D \beta \vee \gamma$, and that $\alpha \leftarrow \beta$ is true at some world, w, of some interpretation. Then for some w' such that $w'Rw$, α is true at w', and β is false. Since α is true at w', so is $\beta \vee \gamma$. And since β fails at w', γ holds there. By heredity, γ holds at w.

For the second, suppose that α holds at world w. Then either β holds there, and so the disjunction does, or β fails there, in which case $\alpha \leftarrow \beta$ holds there, and so does the disjunction.

In the completeness theorem we need an additional lemma.

Lemma 3. *If Γ is pdc and $\alpha \leftarrow \beta \in \Gamma$, there is a pdc Θ such that $\Theta R \Gamma$, $\alpha \in \Theta$, and $\beta \notin \Theta$.*

Proof. Let $\Sigma = \{\alpha\}$, and $\Pi = \{\beta\} \cup \{\gamma : \gamma \notin \Gamma\}$. If $\Sigma \nvdash \Pi$, we can let Θ be the Δ of the Fundamental Lemma. This obviously has all the right properties. So suppose that for some $\gamma_1, \ldots, \gamma_n \notin \Gamma, \{\alpha\} \vdash_N \beta \vee \gamma_1 \vee \ldots \vee \gamma_n$. Then $\{\alpha \leftarrow \beta\} \vdash_N \gamma_1 \vee \ldots \vee \gamma_n$. Since $\Gamma \vdash_N \alpha \leftarrow \beta$, $\Gamma \vdash_N \gamma_1 \vee \ldots \vee \gamma_n$, which is impossible.

In the proof of the theorem, we now have to check an additional case for $\leftarrow$, namely:

$$\nu_\Gamma(\alpha \leftarrow \beta) = 1 \text{ iff } \alpha \leftarrow \beta \in \Gamma.$$

Suppose that $\alpha \leftarrow \beta \in \Gamma$. Then by Lemma 3, there is a pdc Θ such that $\Theta R \Delta$, $\alpha \in \Theta$, and $\beta \notin \Theta$. By IH, $\nu_\Theta(\alpha) = 1$, and $\nu_\Theta(\beta) = 0$. Hence $\nu_\Gamma(\alpha \leftarrow \beta) = 1$. Conversely, suppose that $\alpha \leftarrow \beta \notin \Gamma$. Then for all Δ such that $\Delta R \Gamma$, $\alpha \leftarrow \beta \notin \Delta$. Now suppose that $\nu_\Delta(\alpha) = 1$. Then by IH, $\alpha \in \Delta$. Since $\alpha \vdash_N (\alpha \leftarrow \beta) \vee \beta$, and $\alpha \leftarrow \beta \notin \Delta$, $\beta \in \Delta$; and so by IH, $\nu_\Delta(\beta) = 1$. Hence there is no Δ such that $\nu_\Delta(\alpha) = 1$ and $\nu_\Delta(\beta) = 0$. That is, $\nu_\Gamma(\alpha \leftarrow \beta) = 0$.

Turning to the corresponding algebras, an extended da Costa algebra is a structure $\langle A, \vee, \wedge, \rightarrow, \neg, \leftarrow \rangle$, where $\langle A, \vee, \wedge, \rightarrow, \neg \rangle$ is a da Costa algebra, and $\leftarrow$ behaves dually to $\rightarrow$. That is:

(*) $a \leq b \vee c$ iff $a \leftarrow b \leq c$.

The extended rule system is sound and complete with respect to the extended algebraic semantics.

For the soundness of the first rule for $\leftarrow$, suppose that we have a proof of $\beta \vee \gamma$ from just α. Then by IH, $\mu(\alpha) \wedge \top \leq \mu(\beta \vee \gamma)$, So $\mu(\alpha) \leq \mu(\beta) \vee \mu(\gamma)$, $\mu(\alpha) \leftarrow \mu(\beta) \leq \mu(\gamma)$, and $\mu(\alpha \leftarrow \beta) \wedge \top \leq \mu(\gamma)$, as required. For the second, $\mu(\alpha) \leftarrow \mu(\beta) \leq \mu(\alpha \leftarrow \beta)$, so $\mu(\alpha) \leq \mu(\alpha \leftarrow \beta) \vee \mu(\beta)$, and $\mu(\alpha) \wedge \top \leq \mu(\alpha \leftarrow \beta) \vee \mu(\beta)$, as required.

For completeness, we have to check that the Lindenbaum algebra satisfies condition (*). If $[\alpha] \leq [\beta] \vee [\gamma]$ then $\alpha \vdash_N \beta \vee \gamma$. It follows that $\alpha \leftarrow \beta \vdash_N \gamma$. That is, $[\alpha \leftarrow \beta] \leq [\gamma]$. Conversely, if $[\alpha \leftarrow \beta] \leq [\gamma]$ then $\alpha \leftarrow \beta \vdash_N \gamma$. Let Π be a proof from $\alpha \leftarrow \beta$ to γ. Consider the following deduction:

$$\dfrac{\dfrac{\alpha}{(\alpha \leftarrow \beta) \vee \beta} \quad \dfrac{\begin{array}{c}\overline{\alpha \leftarrow \beta}\\ \Pi \\ \gamma\end{array}}{\beta \vee \gamma} \quad \dfrac{\overline{\beta}}{\beta \vee \gamma}}{\beta \vee \gamma}$$

We see that $\alpha \vdash_N \beta \vee \gamma$. That is, $[\alpha] \leq [\beta] \vee [\gamma]$, as required.

7. Brouwerian Algebras

An extended da Costa algebra is of the form $\langle A, \vee, \wedge, \rightarrow, \neg, \leftarrow\rangle$, or equivalently, $\langle A, \vee, \wedge, \rightarrow, \top, \leftarrow\rangle$, since $\top$ and $\neg$ are inter-definable, given $\leftarrow$. If we drop the component for $\rightarrow$, we get what is often called in the literature a Brouwerian algebra (Tarski and McKinsey 1948). If we restrict ourselves to a language without $\rightarrow$, then a notion of validity may be defined using Brouwerian algebras, as for da Costa algebras. Let us call this *Brouwerian logic*, and write it as $\models_B$.[5] We may show that for sentences not containing $\rightarrow$, validity in (extended) da Costa logic and Brouwerian logic coincide. It suffices to show this for finite sets of premises, since the algebraic definition of validity for arbitrary sets of premises ensures compactness. So assume Σ to be finite.

If $\Sigma \not\models_A \alpha$ then there is a da Costa algebra which invalidates the inference. The algebra obtained by dropping the component $\rightarrow$ is a Brouwerian algebra, and also, obviously, invalidates the inference. Hence $\Sigma \not\models_B \alpha$.

To show the converse, we need a couple of Lemmas.

Lemma 4. *Any finite distributive lattice has an operator, $\leftarrow$, satisfying the condition that $a \leq b \vee c$ iff $a \leftarrow b \leq c$.*

Proof. Let $\langle M, \vee, \wedge\rangle$ be a finite distributive lattice. Let $X_B = \{b : a \leq b \vee c\} = \{b_1, \ldots, b_n\}$. Define $a \leftarrow c$ as $b_1 \wedge \ldots \wedge b_n$. Clearly, if $a \leq b \vee c$ then $a \leftarrow c \leq b$. Conversely, suppose that $a \leftarrow c \leq b$. Then $b_1 \wedge \ldots \wedge b_n \leq b$. So $(b_1 \wedge \ldots \wedge b_n) \vee c \leq b \vee c$, i.e., $(b_1 \vee c) \wedge \ldots \wedge (b_n \vee c) \leq b \vee c$. But for each b_i, $a \leq b_i \vee c$. So $a \leq b \vee c$.

Lemma 5. *Any finite distributive lattice has an operator, $\rightarrow$, satisfying the condition that $a \wedge b \leq c$ iff $a \leq b \rightarrow c$.*

Proof. Let $\langle M, \vee, \wedge\rangle$ be a finite distributive lattice. Let $X_B = \{b : a \wedge b \leq c\} = \{b_1, \ldots, b_n\}$. Define $a \rightarrow c$ as $b_1 \vee \ldots \vee b_n$. The proof then just dualises the proof in the preceding lemma.

We can now prove the converse. Suppose that $\Sigma \not\models_B \alpha$. Then there is a Brouwerian algebra, $\langle A, \vee, \wedge, \top, \leftarrow\rangle$, and a map μ, which invalidates the inference. We show, first, that there is a finite Brouwerian algebra with the same property.

Let $S = \{\mu(\gamma) : \gamma$ is a sub-formula of any formula in $\Sigma \cup \{\alpha\}\} \cup \{\top, \bot\}$ and let A' be the closure of S under $\vee$ and $\wedge$. Consider the sub-lattice $\langle A', \vee, \wedge\rangle$. This is a distributive lattice, and it is finite. (By distribution, every member of A' is equivalent to something of the form $C_1 \vee \ldots \vee C_n$, where each C_i is a conjunction of members of S, $\top$, and $\bot$. There is only a finite number of these.) By Lemma 4, there is an operator, $\leftarrow'$ such that $\langle A', \vee, \wedge, \top, \leftarrow'\rangle$ is a Brouwerian algebra.

Let μ' be an evaluation of formulas into the new algebra which agrees with μ on all propositional parameters. If we can show that for every sub-formula, γ, of a formula in $\Sigma \cup \{\alpha\}$, $\mu(\gamma) = \mu'(\gamma)$, then it follows that the new algebra is a counter-model for the inference, and we have what we need. This is proved by recursion. The cases for parameters, conjunction, and disjunction are trivial. For $\leftarrow$, it suffices to show that if $a \leftarrow b \in A'$, $a \leftarrow b = a \leftarrow' b$. Note that if $a \leftarrow b \in A'$, then $a, b \in A'$. We have the following:

1. if $c \in A'$ then $a \leftarrow' b \leq c$ iff $a \leq b \vee c$,

2. if $c \in A$ then $a \leftarrow b \leq c$ iff $a \leq b \vee c$.

By 2, $a \leq b \vee (a \leftarrow b)$; so by 1, $a \leftarrow' b \leq a \leftarrow b$. Conversely, by 1, $a \leq b \vee (a \leftarrow' b)$; so by 2, $a \leftarrow b \leq a \leftarrow' b$. Hence, $a \leftarrow b = a \leftarrow' b$.

To finish the proof: the algebra we have just constructed is finite; hence, by Lemma 5, it has an operator, $\rightarrow$ satisfying $a \wedge b \leq c$ iff $a \leq b \rightarrow c$. Hence it is a da Costa algebra, and so $\Sigma \not\models_A \alpha$.

Goodman (1981) proves that there is no connective, $\rightarrow$, definable in terms of the connectives of Brouwerian algebra, such that $\models_B \alpha \rightarrow \beta$ iff $\alpha \models_B \beta$. What we have now seen is that Brouwerian logic can be embedded in da Costa logic, which has such a connective.

Finally, it follows from the equivalence that da Costa logic and Brouwerian logic have the same $\wedge$-$\vee$-$\neg$ fragment.[6] Where $\langle A, \wedge, \vee, \leftarrow, \neg \rangle$ is Brouwerian algebra, Mortensen and Lavers call $\langle A, \wedge, \vee, \neg \rangle$ a *paraconsistent algebra*.[7] The logic generated by this class of algebras is exactly the same fragment. If there is a Brouwerian algebra which invalidates an inference, there is certainly a paraconsistent algebra. Conversely, if there is a paraconsistent algebra, we can produce a Brouwerian algebra by essentially the same construction we have just been through.[8]

8. Topological Semantics

Dual intuitionist negation also has a topological semantics that is worth noting. Intuitionist logic can be given a topological interpretation. Specifically, given any topological space, an interpretation for intuitionist logic is given by a map, μ, of the propositional parameters into the open sets of the space, extended to all formulas by the following clauses:

$$\mu(\alpha \wedge \beta) = \mu(\alpha) \cap \mu(\beta),$$
$$\mu(\alpha \vee \beta) = \mu(\alpha) \cup \mu(\beta),$$
$$\mu(\alpha \rightarrow \beta) = (\overline{\mu(\alpha)} \vee \mu(\beta))^i,$$

$$\mu(\bot) = \phi,$$

where $\overline{a}$ is the complement of a, and a^i is the interior of a. Taking negation to be defined, $\mu(\neg\alpha) = \mu(\alpha \rightarrow \bot) = \overline{\mu(\alpha)}^i$. A formula, α, holds in the interpretation iff $\mu(\alpha)$ is the whole space. It is well known that intuitionist logic is sound and complete with respect to these semantics (Dummett 1977, ch. 5).

We can dualise these semantics by operating with closed sets, instead of open sets. Specifically, a topology is a structure $\langle S, c\rangle$, where S is a set and c is an operator from subsets of S to subsets of S (a closure operator[9]) satisfying the conditions that for all $X, Y \subseteq S$:

$$X \subseteq X^c,$$
$$X^{cc} = X^c,$$
$$\phi^c = \phi,$$
$$(X \cup Y)^c = X^c \cup Y^c.$$

A set, X, is *closed* if $X^c = X$. It is not difficult to establish that S is closed; that if X and Y are closed, so are $X \cup Y$ and $X \cap Y$; and if $X \subseteq Y$, $X^c \subseteq Y^c$. (Proof of the last of these: if $X \subseteq Y$ then $X \cup Y = Y$. So $(X \cup Y)^c = X^c$, $X^c \cup Y^c = X^c$, and $X^c \subseteq Y^c$.)

Given a topological space, an evaluation is a map, μ, from propositional parameters to closed sets. It is extended to all $\rightarrow$ -free formulas by the clauses:

$$\mu(\alpha \vee \beta) = \mu(\alpha) \cup \mu(\beta),$$
$$\mu(\alpha \wedge \beta) = \mu(\alpha) \cap \mu(\beta),$$
$$\mu(\alpha \leftarrow \beta) = (\mu(\alpha) \cap \overline{\mu(\beta)})^c,$$
$$\mu(\top) = S.$$

Taking negation to be defined, $\mu(\neg\alpha) = \mu(\top \leftarrow \alpha) = \overline{\mu(\alpha)}^c$. Note that we cannot extend these semantics to $\rightarrow$, since even if $\mu(\alpha)$ and $\mu(\beta)$ are closed, $(\overline{\mu(\alpha)} \vee \mu(\beta))^i$ may not be. An inference is topologically valid, $\alpha_1, \ldots, \alpha_n \models_T \beta$, iff for every topology, and every interpretation into the topology, $\mu(\alpha_1) \cap \ldots \cap \mu(\alpha_n) \cap S \subseteq \mu(\beta)$.

McKinsey and Tarski (1947, Theorem 1.19.) established that the closed sets of a topology, under the defined operations, is a Brouwerian algebra, and conversely, that every Brouwerian algebra is isomorphic to a sub-algebra of the closed sets of some topological space. Hence, the algebraic and the topological semantics are equivalent. As we saw in the last section, Brouwerian logic is equivalent to the $\rightarrow$-free fragment of da Costa logic. It follows that this fragment is sound and complete with respect to the topological semantics.

9. The Meaning of $\neg$

Let us end by addressing the question of the meaning of $\neg$ in da Costa logic. Return again to the Kripke semantics. The interpretation of Kripke models in terms of proof conditions is well known. Essentially, the truths at a world are things of which we have a proof at some time; wRw' iff w' is the set of those things proved at a later time, at which we have some number (possibly zero) of further proofs. The truth conditions of the connectives can then be interpreted in terms of proofs. In particular, $\alpha \rightarrow \beta$ is proved at a certain time if we have a construction which applies to any proof of α to give a proof of β. If we have such a construction, then at any later time at which α becomes proved, β will automatically become proved. Conversely, if we have no such construction, it is possible that there will be a future time at which we have a proof of α, but still no proof of β (see Priest 2008, 6.2, 6.3). For its dual, a proof of $\alpha \leftarrow \beta$ is a proof that there is no proof of β from α. If we have such a proof then there was, in principle, a time at which α had been proved and β had not. Conversely, if we have no such proof then it is possible that there is a proof of β from α, and hence that at all past times at which α was proved, so was β.

Take the intuitionist negation of α to be defined as $\alpha \rightarrow \bot$. One normally thinks of $\bot$ as something of which there can be no proof. A proof of $\alpha \rightarrow \bot$ is a construction which turns any proof of α into a proof of $\bot$. Since there can be no proof of $\bot$, this construction is effectively a proof that α cannot be proved. But one might think of $\bot$, slightly differently, as expressing the things which are antithetical to whatever inquiry we are in. A proof of $\alpha \rightarrow \bot$ is then a construction to the effect that if we have a proof of α, we are in Trouble. We might have neither of $\alpha \rightarrow \bot$ nor $\neg\alpha \rightarrow \bot$ at some stage of our investigation: neither may have been shown to get us into Trouble. Dually, one may think of $\top$ as the set of things that are fundamental to our inquiry (that is, whatever theory it is that is a taken as a given at any stage of our inquiry). Then a proof of $\top \leftarrow \alpha$ will be a construction that α cannot be demonstrated on the basis of $\top$. It is then clear why we may have both $\top \leftarrow \alpha$ and $\top \leftarrow \neg\alpha$: α may simply be independent of $\top$.

One might ask whether, in the light of this, $\neg$ in da Costa logic really is negation. Exactly the same question may be asked, of course, of the negation of intuitionist logic. This raises interesting questions appropriate for another occasion. I end simply by noting that the technical duality between the two negations provides a new philosophical perspective from which to view such questions.

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Resumo. Uma das razões de da Costa para erigir a lógica paraconsistente C_ω foi a dualização da negação intuicionista. Neste artigo eu exploro um modo alternativo de chegar a este resultado. Uma lógica é definida a partir de uma semântica de Kripke para a lógica intuicionista, e então dualizando as condições de verdade para a negação. Várias propriedades desta lógica são estabelecidas, incluindo suas relações com C_ω. Sistemas de tableau e de dedução natural são apresentados, bem como as estruturas algébricas apropriadas. O artigo investiga então a dualização do condicional intuicionista seguindo o mesmo procedimento. Isso estabelece várias conexões entre a lógica, e uma lógica denominada na literatura de 'lógica Brouweriana' ou lógica 'closed-set'.

Palavras-chave: da Costa, paraconsistência, intuicionismo, C_ω, semântica de Kripke, álgebras Brouwerianas, lógica *closed-set*, negação.

Notes

[1] The sytems C_i, for finite i, have a "consistency operator", $\circ$, which allows for the definition of a classical negation. This makes the positive part of these logics classical, not intuitionist. See Priest 2007, 4.3.

[2] That one can proceed in this way is explained without details in Priest 2007, Section 3. The Kripke semantics is also given by Rauszer 1977, where the logic is called Heyting-Brouwer Logic.
[3] The general case of substitutivity can be proved by an induction on the complexity of the formula into which one is substituting. The proof is routine, given that the case for negation is unproblematic.
[4] See Dummett 1977, 172ff. I use the same symbols for the logical opertors and the algebraic operations. Context will suffice to distinguish which is at issue.
[5] Goodman 1981 formulates this logic, and calls it anti-intuitionistic logic.
[6] This proves the conjecture of Priest 2007, 156, fn. 86.
[7] Mortensen 1995, 103ff. They include a top element, $\top$, in the algebra. As we have noted, this is redundant.
[8] Let $\langle A, \wedge, \vee, \neg \rangle$ be a paraconsistent algebra, with top element $\top$. We construct a finite Brouwerian algebra $\langle A', \vee, \wedge, \top, \leftarrow' \rangle$ from this as before. The algebra has a negation $\neg' a$, defined as $\top \leftarrow' a$. Suppose that $\neg a \in A'$ (and so $a \in A'$). We have:

1. if $c \in A'$ then $\top \leq a \vee c$ iff $\neg' a \leq c$,
2. if $c \in A$ then $\top \leq a \vee c$ iff $\neg a \leq c$.

Since $\top \leq a \vee \neg a$, $\neg' a \leq \neg a$, by 1. And since $\top \leq a \vee \neg a$, $\neg a \leq \neg' a$, by 2. Hence $\neg a = \neg' a$.
[9] Interior and closure operators are inter-definable. Specifically, $X^i = \overline{\overline{X}^c}$.