Non-Classical Logic: Philosophical Issues

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Abstract

So called classical logic is the logic invented/discovered largely by Frege and Russell around the turn of the 20th Century, and polished by subsequent logicians, including Hilbert, Tarski, Gentzen, and others. Non-classical logics are logics that were invented/discovered by logicians in response to various philosophical inadequacies perceived in classical logic. Though their origins are also in the start of the 20th century, they witnessed a spectacular—and continuing—development in the second half of the century. The point of the present paper is to survey the philosophical considerations which gave rise to non-classical logics, the logics to which they gave rise, and the philosophical issues to which they, in turn, gave rise.

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1 Introduction: Classical and Non-Classical Logic

The word 'logic' can be used in many ways. For the purpose of what follows, I will understand it as meaning a theory of what follows from what, and why. That is, an inference has premises and a conclusion (or, in the case of multiple-conclusion logics, conclusions). A valid inference is one where the premises provide a ground for the conclusion. A logic is an account of what conclusions follow validly from what premises—and what do not. Generally, it will do more than this, however: it will provide an account of why valid inferences are valid—and why invalid inferences are invalid. Of course, this is a very rough and ready characterisation of validity. It is the brief of a logical theory to render it more precise.

Theories of logic have been produced for well over two millennia. However, a significant revolution occurred in the subject around the turn of the 20th Century. Resulting from, amongst other things, the drive for rigour in mathematics, the techniques of modern mathematics were applied to the subject. The theory that then appeared is now standardly called 'classical logic'. This was produced initially and most notably, by Frege and Russell, and subsequently polished by Hilbert, Tarski, Gentzen, and others. The theory was not called 'classical logic' by these thinkers: it was just logic. However, early in the new century, a new kind of mathematics which rejected certain standard forms of mathematical reasoning was proposed by Brouwer. More of this in due course. The orthodox mathematical community required a name for the more traditional kind of mathematics, and 'classical' became standard. Somewhat later, the epithet was transferred to the logic that was taken by the orthodox to be used in such mathematics—that of Frege and Russell.

Be that as it may, this theory of logic was so powerful that it soon superseded the traditional logic that was standard in the 19th Century. Like all theories, however, it had clear problems, and no sooner had it appeared than logicians were applying the newly used mathematical techniques to produce different theories of logic, aimed at addressing these problems. Such theories have come to be known, naturally enough, as non-classical logics.

The theories are, loosely speaking, of two kinds. The first aimed to supplement perceived expressive inadequacies of classical logic with new devices which redress the deficiency. The second, and more radical kind, took certain features of classical logic simply to be wrong, and aimed to correct these features. The first kind may be called *supplements* to classical logic; the second, *rivals*.[1]

The worries that prompted non-classical logics were, generally speaking, of a philosophical kind, and these stimulated many technical novelties. These novelties produced, in turn, many new philosophical issues. That dialectical interplay is the subject of this essay. The relevant terrain is enormous. One could write a book on each of the subjects I discuss—and many more that I do not. Of necessity, then, the coverage in a single essay is selective; but my choices are not arbitrary. My aim is to cover what I see as the most central issues in the area. Different authors might reasonably have made different choices. However, any such selection would, I think, have to have at least a very substantial overlap with the material we will cover.

I will structure the discussion by moving through several different kinds of non-classical logic, charting the interplay between philosophical and technical considerations, throwing in a few historical details for context. (Those who want a glimpse ahead can consult the table of contents.) There are, of course, interactions between these different kinds of logic, as we shall see. Given the space, it is impossible to discuss each philosophical issue in any detail. I must content myself with little more than explaining what it is. At the end of every section I shall give references where discussion and further references may be found. (The titles tell their relevance.)

Let me finish this introduction by saying two things that this essay will *not* be about—though much of interest could be said about them. First, as noted, many important logical theories were produced by logicians in the Ancient and Medieval periods of Western philosophy and in Eastern Philosophy. By definition, these were non-classical logics. Indeed, generally speaking, they were incompatible with classical logic, even giving conflicting accounts of which inferences are valid. (We will have an example of this in due course.) These theories came with their own techniques and rafts of philosophical issues. However, such things

[1]The terminology is due to Haack (1974).

would have to be the subject of entirely different treatises.

Secondly, as is generally agreed, inferences come in two flavours: deductive and non-deductive (inductive). A deductive inference is one where the premises cannot (in some sense) be true without the conclusion being true. In a non-deductive inference, the premises provide some lesser ground. Hence, it can be the case that *A* follows from *B*, but additional premises may undercut the validity of the inference. Hence, *A* may not follow from *B* plus *C*. (Hence the modern name for such inferences: *non-monotonic*.) For reasons that are less than clear, historically, non-deductive validity has been much less theorised by logicians then deductive validity. However, some considerable attention has been paid to it in the last 100 years or so. Again, the various theorisations come with their own raft of techniques and philosophical problems, concerning probability, normality, and other matters. And again, such things would have to be the subject of entirely different treatises. This essay is concerned solely with deductive logic.[2]

2 Many-Valued Logics: Gaps, Gluts, and Fuzziness

The first family of logics we will look at are many-valued logics. These are rivals to classical logic.

Classical logic has two truth values: *true* (and true only), and *false* (and false only). But using the standard mathematical tools, one can construct semantics for logics in which there can be any number of values (finite or infinite). Logics where there are more than two values are *many-valued logics*. Given the values of the logic, an interpretation assigns one of these to each propositional parameter, and truth functions then assign values to all formulas. Some of the values are termed *designated*, and an inference is valid if it preserves designated values in all interpretations. (*true* is, of course, a designated value.)

The first person to construct a many-valued logic was Łukasiewicz in 1920. His was a three-valued logic. Motivated by Aristotle's arguments about future contingents, he thought of the third value as *possible*. However, this makes possibility behave in a very inappropriate way, so let us pass this over. (We will look at a much more adequate way in the next section.)

Setting the infinite case aside for the moment, it is very hard to find more than a handful of plausible meanings for the non-classical values. The following are the ones that are more commonly discussed. The philosophical issues in each case

[2] *Further reading and references:* Priest (2014); Kneale and Kneale (1962); Dutilh Novaes (2020); Gillon (2021); Hawthorne (2019); Strasser (2019).

concern whether there are statements with such values—or at least, whether such statements should be accommodated in an account of logical consequence.

Perhaps the most popular non-classical value is *neither true nor false*, as found in (strong) Kleene logic, K_3. Many candidates for having such a value have been suggested. The first was by Aristotle himself. Some statements about the future are already determined as true (e.g., 'the sun will rise tomorrow'); others are as yet indeterminate—e.g. (10/11/22), 'the war in Ukraine will still be going on in 2024'. In important respects, the future is metaphysically open: for some things, there is presently no fact of the matter. Statements about such "future contingents" are presently neither true nor false (though they will be so in due course).

A quite different rationale was proposed by Frege. Certain noun-phrases have no referent. One might think of names in fiction, such as 'Odysseus', or infelicitous definite descriptions, such as 'the present king of France'. The truth value of a sentence containing such a name is a function of the referent of the name, and since the name has no referent, such a sentence has no truth value. That is, it is neither true nor false.

Yet a third candidate for sentences with such a value (suggested, e.g., by Goddard and Routley) are meaningless sentence, such as 'this stone is thinking about Vienna', 'it is noon at the North pole', or '1/0'. One thing that is distinctive about meaningless sentences is that one might expect them to behave "infectiously", as found in Bochvar logic, B_3 (aka, *weak Kleene*) or Halldén logic, H_3 (aka *paraconsistent weak Kleene*). That is, if such a sentence is a sub-formula of another sentence, it, too, is meaningless. One might also think that one should just exclude meaningless sentences from logical consideration. The standard reply is that one cannot do this because (following, e.g., Wittgenstein), such nonsense may be "hidden".

A less common, but increasing popular suggestion for a non-classical value is *both true and false*, as found in the Logic of Paradox, *LP*. Again, many candidates for sentences with such a value have been suggested. One concerns motion (Priest). Consider Zeno's Paradox of the Arrow. At an instant of its motion, the progress made by an arrow on its journey is zero. The progress made in the whole journey is the sum of progresses made at each instant; and a sum of zeros—even transfinitely many—is zero. So the arrow does not move. The solution (endorsed by Hegel) is that at an instant the arrow *does* make some progress. It is where it is, but since it is in motion, it has already gone a little bit further, and so is not there. In fact, the difference between an object in motion, and an object at rest at the same point is precisely that the former realises a contradictory state. It is there and not there. 'It is there' is both true and false.

One of the most frequently discussed suggestion for sentences with a non-classical value concerns some other paradoxes: those of self-reference. Take the liar sentence as an example: *this sentence is false*. If it is true, it is false; if it is false it is true. Some have suggested that the sentence is *neither* true *nor* false (Bochvar, Kripke, Field). Others have suggested that it is *both* true *and* false (Halldén, Routley, Priest.) A *prima facie* advantage of the latter suggestion is that the former does not escape paradox. Merely consider the "extended" paradox: *this sentence is either false or neither true nor false*. To claim that *this* sentence is neither true nor false, would be to claim that it is true. Contradiction. Dually, however, consider the sentence: *this sentence is false and not both true and false*. Reasoning in an obvious way, one establishes that it is false only and both true and false. This is, of course, also a contradiction. But unlike the *neither*-solution, the *both*-solution is not meant to eliminate contradiction, but to tame it.

Yet another paradox which has been invoked to justify non-classical values is the sorites paradox. These occur for vague predicates, such as 'is bald', 'is drunk'. Applications of such predicates appear to have clear cases of truth, clear cases of falsity, and borderline cases, symmetrically poised between the two. It is often suggested that in borderline cases the resulting sentence is *neither true nor false* (Tye, Field) or *both true and false* (Hyde, Colyvan, Priest). The obvious problem with either suggestion concerns "higher-order vagueness". What caused the original problem is that there is no clear cut-off between true applications and false applications. But we have now simply multiplied the problem, since there is no clear cut off point between the applications of each classical value and the non-classical value (whatever that is).

To handle this problem many (Zadeh, Hájek, Smith) have felt that one should use an infinite-valued many-valued logic, "fuzzy logic", where statements can have any value in the interval of real numbers $[0, 1]$. Non-integral values are those of things that are less than fully true or fully false. Using such values, it would appear that precise cut-off points can be avoided. However, this is not the case, since in a continuous transition from 0 to 1, there must be a precise cut-off point between taking the value 0 and taking a value greater than 0. (Similarly for value 1.)

Given any many-valued logic with a designated value which is a fixed point for negation then, provided that there are some values that are not designated, the inference of Explosion, $A, \neg A \vdash B$, is invalid. Logics in which this inference fails are called *paraconsistent*.[3] Hence some many-valued logics are paraconsistent.

[3]Derivatively, logics in which the dual inference, $A \vdash B, \neg B$, fails are sometimes called *para-*

There are also paraconsistent logics that are not many-valued. We will meet such in due course.[4]

3 Modal Logic: Intensionality and Intentionality

Let us now turn to modal logics, which are normally thought of as supplements to classical logic, since they incorporate it. (Though, one might note, it is perfectly possible for a modal logic to have any many-valued logic as its non-modal part, and so be a rival.)

Modal logics are so called, since they deal with the modes in which things may be true or false: possibly, impossibly, necessarily. The matter has been discussed in Western logic since its origins. However, contemporary discussions of the topic date back to the work of C. I. Lewis, starting just before the First World War. That is, just a couple of years after the appearance of Russell and Whitehead's *Principia Mathematica*. Lewis' concern was not modality as such, however. He was troubled by the account of conditionality to be found in *Principia*. This was the material conditional, $A \supset B$, defined as $\neg A \vee B$. If one takes this to represent the natural-language conditional, one gets such oddities as the validity of the inferences:

- I will have an egg for breakfast tomorrow. Hence if I die tonight I will have an egg for breakfast tomorrow.
- Canberra is the capital of Australia. Hence, if Sydney is the capital of Australia, the capital of Australia is in Germany.

Lewis' solution was to define a new kind of conditional, $\prec$. Writing $\Box$ for 'it is necessary that', he defined $A \prec B$ as $\Box(A \supset B)$. We will return to Lewis' definition in due course. The developments that ensued took up the investigation of $\Box$ and its cognates.

Lewis developed five systems of modal logic $S1$–$S5$, defined purely axiomatically. The very intelligibility of modal notions was influentially challenged by Quine and others in the 1950s and 1960s. But matters changed with the discovery of a systematic semantics for modal logics. The main player in this game was Kripke. In these semantics, interpretations are furnished with a bunch of objects

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complete.

[4]*Further reading and references:* Priest (2008), ch. 7 (many-valued logic); Priest (202a); Gottwald (2015); Cintula, Fermüller, and Noguera (2021).

called (and thought of as) possible worlds. What is possible at a world may change from world to world. (In a world where I am currently in Melbourne, it is physically possible, with current technology, for me to be in Sydney in a few hours. In the present world, where I am in New York, it is not.) Hence the semantics have a binary "accessibility relation", R. wRw' is taken to mean that at world w, world w' is possible. Then:

- $\Box A$ is true at w iff for all w' such that wRw', A is true.

Writing $\Diamond$ for the dual of $\Box$,'it is possible that', then:

- $\Diamond A$ is true at w if for some w' such that wRw', A is true

Different constraints on R define different modal logics. If no constraints are imposed, so that R is arbitrary, we have the logic K, which is not one of the Lewis systems, though $S4$ and $S5$ are extensions of it. Systems that are extensions of K are called *normal*.

The semantics generate a number of philosophical issues. For start, there are many clearly different notions of necessity, such as: logical, physical, temporal, epistemic, deontic, and—many add to the menagerie—metaphysical.

These have different properties. For example, the logic obtained by extending the semantics of K with the constraint of reflexivity on R (for all w, wRw) verifies the logical truth of $\Box A \supset A$. This holds for logical necessity, but not deontic necessary. (The fact that it is morally necessary (obligatory) for someone to bring it about that A notoriously does not imply that they do so.) There are philosophical issues about which modal logic captures a particular notion of necessity. For example, the characteristic of the Lewis System $S4$ is the transitivity of R (if xRy and yRz then xRz). This verifies the logical truth of $\Box A \supset \Box\Box A$. Does this hold of epistemic necessity? Is what is known always known to be known?

Next, as a piece of formal apparatus, possible worlds are unproblematic. They are just a bunch of objects with certain mathematical properties. But what do such things represent in reality (if anything). That is, what is the metaphysical status of possible worlds—other than the one which is actual? Some have taken them to be abstract objects, such as sets of propositions. Some have taken them to be non-existent objects. Perhaps the idea that has received the most airtime is that proposed by (David) Lewis, called, for better or for worse, *modal realism*. According to this, non-actual possible worlds are physical worlds, just like the actual world. They just exist in their own space and time, causally isolated from ours.

Taking matters in a different direction: many have held that world semantics can be used for non-modal notions. One concerns counterfactuals; we will come to that matter in due course. Another concerns intentional propositional states; that is, those mental states, such as believing, fearing, hoping, which take as their objects states of affairs. This raises a whole new bunch of issues. To see what these are, let us have a couple of quick definitions. A context, C, is *extensional* if it satisfies the condition:

- if $A \equiv B$ then $C(A) \equiv C(B)$

where $A \equiv B$ is $(A \supset B) \wedge (B \supset A)$. Classical logic is extensional, in that if C is any context provided by the language, it is extensional. Modal logic is not. Let A be 'Nothing accelerates through the speed of light', and B be 'The capital of Australia is Canberra'. Then if $\Box$ is physical necessity, $A \equiv B$, but it is not the case that $\Box A \equiv \Box B$. A context, C, is *intensional* if it satisfies the condition:

- if $\models A \equiv B$ then $C(A) \equiv C(B)$

where $\models$ is logical truth. Normal modal logics are intentional, in that if C is any context provided by the language, it is intensional.

Contexts that are not (extensional or) intensional are called *hyperintensional*. And intentional contexts are hyperintensional. Clearly, for example, if A is some simple logical truth, such as 'If the sun is shining, it is shining', and B is some immensely complex propositional logical truth, with a million independent atomic sentences, I can believe that A without believing that B. Hence such operators cannot be accommodated by a normal system of modal logic.

Now, in the Lewis systems $S1$, $S2$, and $S3$, modal operators deliver hyperintensional contexts. For example, in all of these, $\Diamond(p \vee \neg p)$ and $\Box(p \vee \neg p)$ are logical truths. So $\models \Diamond(p \vee \neg p) \equiv \Box(p \vee \neg p)$; but $\Box\Diamond(p \vee \neg p)$ is a logical truth and $\Box\Box(p \vee \neg p)$ is not. Hence $\not\models \Box\Diamond(p \vee \neg p) \equiv \Box\Box(p \vee \neg p)$. In Kripke's semantics for $S2$ and $S3$ this matter is handled by having a special class of non-normal worlds where logical truths may fail.[5] Specifically, at such worlds modal operators work differently from how they work at normal worlds. All things of the form $\Box A$ are false, and all things of the form $\Diamond A$ are true. Since non-normal worlds are such that logical truths may fail at them, it is natural to think of them as impossible worlds (though Kripke himself never called them this).

[5] Kripke never produced a semantics for $S1$, for which somewhat different semantic techniques are required.

In a world semantics, logically impossible worlds would seem to be required to handle intentional operators. Thus, suppose that Ψ is any such operator. Take 'Mary believes that' as an example. And let R_Ψ be the accessibility relation that corresponds to it, so that if w is a normal world:

- ΨA is true at w iff for all w' such that $wR_\Psi w'$, A is true at w'

Even rational agents may believe things that are not logically true. And, if so, these w' must be impossible worlds. (Well, not quite. If w accesses no worlds then, for all A, ΨA is true vacuously. But let us assume that Mary does not believe everything.) Hence, intentionality occasions consideration of a semantics in which there are impossible worlds.[6] The advent of impossible world semantics has put on the table a whole new bunch of problems.

Perhaps the most obvious is whether it is possible to deliver an adequate semantics for hyperintensional operators which do not use worlds—at least impossible worlds. There are many other questions. One that immediately looms is what to make of the metaphysical status of impossible worlds. Do they have the same status as possible (non-actual) worlds, or must they be supposed to have a different kind of status? The answer to that question may (or may not) depend on what one takes the metaphysical status of possible worlds to be.

The philosophical ideas generate, in turn, various technical questions. Impossible worlds of Kripke's kind are not general enough to handle intentional contexts, if only because non-modal logical truths hold in all worlds, normal and non-normal; but even rational (non-ideal) agents do not believe all logical truths of this kind. And once other intentional operators are on the table, the matter is patent. If A is of the form $p \vee \neg p$, Mary does not have to fear that A, wonder whether A, hope that A. In other words such worlds are far too constrained. Once intentional states are on the table, much more logical anarchy is required. Such anarchy may come by degrees. Thus, a relatively controlled form of anarchy suffices for the impossible worlds of relevant logic. (See below.) But once intentional states are on the table, complete anarchy would seem to be required, since real people may believe or fail to believe pretty much any collection of statements. How should one make philosophical sense of such anarchic worlds?[7]

[6] In the early days of investigation, these were sometimes called impossible possible worlds. It hardly needs to be said that this is unfortunate terminology.

[7] *Further reading and references:* Ballarin (2021); Priest (2008), chs. 2-4 (modal logic); Hylton and Kemp (2019); Berto and Jago (2019), Priest (2008), ch. 11a (many-valued modal logic).

4 Intuitionistic Logic: Realism and Anti-Realism

Let us turn to our third non-classical logic, which is clearly a rival to classical logic. This is intuitionist logic. Beginning in the first decade of the 20th century, Brouwer took issue with certain standard principles of mathematical reasoning, and started to develop a kind of mathematics which did not use them. Taking the name from Kant, he called this *intuitionism*.

Brouwer was motivated by the thought that mathematical objects do not exist in some abstract platonic realm. They are simply mental constructions. For such an object to exist, then, is for there to be a mental operation for constructing it. Now, suppose that we wish to prove that $\exists xPx$, and assume for *reductio* that $\neg\exists xPx$. We deduce a contradiction, and so establish that $\neg\neg\exists xPx$. We have not constructed an object satisfying Px. Hence we have not shown that $\exists xPx$. The law of double negation in one direction, $\neg\neg A \vdash A$, therefore fails. Other logical failures follow, notably the law of excluded middle. For suppose that $\exists xPx \vee \neg\exists xPx$. We have proved $\neg\neg\exists xPx$. So by the disjunctive syllogism, $\exists xPx$ would follow.

Brouwer himself did not formalise a system of logic. Indeed he was skeptical of formal logic. But an appropriate system of logic, now termed *intuitionistic logic*, was produced by Heyting in the 1920s. (And note that this is provably not a finitely many-valued logic.) This has a number of different semantics. One of these is a world semantics. What is true at a world is thought of as what has been proved there; and wRw' means that what has been proved at w' extends what has been proved at w. It is assumed that once something is established, it stays established. The fact that something fails at w does not imply that it it not true, since a proof may be found at some "later" w'. And the domain of objects at w' may extend that of w, as new objects are constructed. The truth conditions for both negation and the universal quantifier therefore involve a world-shift:[8]

- $\neg A$ holds at w iff for no w' such that wRw', A holds at w'
- $\forall xA$ holds at w iff for all w' and all objects, a, in the domain of w', $A_x(a)$ holds at w'

The most obvious philosophical issue raised by all of this is whether existence in mathematics really is to be understood in terms of construction. To state one obvious problem: many apparently legitimate mathematical constructions clearly

[8]Notation: $A_x(a)$ is A with every free occurrence of x replaced by the constant a; and I use any object in the domain as its own name.

outrun actual human ability, for example the decimal expansion of the number $10^{10^{10^{10}}}$. If we restrict ourselves to constructions that people can *actually* do, we are forced into an ultra finitism incompatible with intuitionist mathematics. We must, then, take ourselves to be dealing with constructions that are possible in principle. But what is the principle? Why can there not be a proof, for some P, of $P0$, $P1$, $P2$, ... and so of $\forall x Px$? Maybe each proof would take half the time of the previous one, the impossibility of such a sequence of proofs being (as Russell put it in a different context[9]), a mere medical impossibility. But such a construction would make every true sentence of standard natural-number arithmetic verifiable, which is again incompatible with intuitionist mathematics.

The concerns of Brouwer and Heyting were specifically about mathematics, but their views were clearly a form of verificationism. In mathematics verification is proof. Verificationism of a more general kind was extended to all of language by Dummett in the 1970s. Dummett argued, on generally Wittgensteinian grounds, that if someone understands the meaning of a sentence, they must be able to demonstrate a grasp of that meaning. In particular, we demonstrate our understanding of the meaning of a sentence by being prepared to assert it in those conditions under which it obtains (and just those). If truth is the kind of thing that may transcend our ability to recognise it, this is impossible. Hence, meanings must be specified in terms of something which we can recognise as obtaining, namely the conditions under which a sentence is shown to be true, that is, verified. Hence Dummett advocated using intuitionist logic for all domains of discourse, not just mathematics.

This motivation for intuitionist logic clearly raises many philosophical issues. The first is whether Dummett's arguments are cogent. One may doubt this. For a start, why must it always be necessary to be able to manifest a grasp of meaning? Some aspects of meaning might simply be innate, or hard-wired into us, as Chomsky has argued. And even granting that the grasp of meaning must be manifestable, why does it have to be manifestable in a way as strong as the argument requires? Why is it not sufficient simply to assent to a sentence when the state of affairs it describes is manifest, and not when it isn't?

If Dummett's arguments are not cogent, the next issue is whether verificationism about truth is correct, and whether, indeed, it is a coherent view of the nature of truth at all. For a notorious objection, consider the claim that something exists only if its existence can be verified. How could this, itself, be verified? (Note that this is not a problem for Brouwer, since such is a philosophical claim, not a

[9]Russell (1935–1936), p. 143.

mathematical one.)

Other problems are of a more technical kind. As I noted, the world-semantics assume that once something is verified it stays verified. Perhaps this is a plausible assumption when the verification is mathematical proof; but it is not so for any form of empirical verification, which is all too fallible. So the use of intuitionist logic (or at least its Kripke semantics) is not justified.[10]

5 Conditionals: Relevant Logics and "Variably Strict" Conditionals

Let us now return to the subject of conditionals. Arguably, conditionality is and always has been the most contentious of all the logical notions, where it is seemingly impossible to get all the moving pieces to fit together coherently.

As we noted, classical logic verifies highly counter-intuitive principles of conditionality, such as:

- $A \models B \supset A$
- $A \models \neg A \supset B$

(And so does intuitionist logic.) Lewis' solution was, as we noted, to define conditionality using a modal operator. $A \dashv B$ is $\Box(A \supset B)$. Substituting $\dashv$ for $\supset$ in the above produces invalid inferences. However, the inferences have valid modal analogues:

- $\Box A \models B \dashv A$
- $\Box A \models \neg A \dashv B$

Given the premise, the conclusions hold, since the material conditional is vacuously true at all (possible) worlds. But these examples seem just as problematic. Where the $\Box$ is logical necessity, merely consider:

- $\Box$(If the sun is shining, then the sun is shining). Hence if all instances of the law of identity are false, if the sun is shining, the sun is shining.

[10] *Further reading and references:* Priest (2008), chs. 6, 20 (intuitionist logic); Iemhoff (2019); Wright (2018); Glanzberg (2018), esp. §4.

- $\Box$(If the sun is shining, then the sun is shining). Hence if it is not the case that (if the sun is shining the sun is shining), then the classical account of the conditional is correct.

Lewis himself, perhaps reluctantly, accepted the validity of these inferences. However, some later logicians balked.

Writing $\to$ for the conditional, a propositional logic is *relevant* iff:

- whenever $\models A \to B$, A and B have at least one propositional parameter in common

Starting in the late 1950s, and drawing on earlier work by Ackermann and Church, Anderson and Belnap put forward a number of relevant logics. These were specified in purely proof-theoretic form (with axiom systems or certain kinds of natural deduction-systems). A decade or two later, various semantics were discovered. Of these, the most versatile was developed by Routley and Meyer. The semantics are world-semantics, and the key idea is to have worlds where logical truths fail, and where logical falsehoods hold. So even if $A \to A$ is a logical truth, there may be impossible worlds where B is true, yet $A \to A$ fails. A conditional is logically true if it preserves truth at *all* worlds of all interpretations. Hence, $A \to A$ is a logical truth, $B \to (A \to A)$ fails to be a logical truth.

The impossible worlds are not completely anarchic. Conjunction and disjunction work standardly. Hence, e.g., $A \wedge B$ is true at a world iff A and B are true there. However, $\neg$ and $\to$ must come in for different treatments. Negation can be handled by taking the logic of a world to be 4-valued, where the values are *true* (only), *false* (only), *both*, and *neither*. However, it turns out that it is very difficult to provide semantics in this way for the Anderson/Belnap logics when one has to worry about nested conditionals. So Routley and Meyer took another tack. Each world, w, comes with a "mate", w^*, and:

- $\neg A$ is true at w iff A is not true at w^*

Not, *NB*, at w, as in classical logic—though there is nothing to prevent w^* from being w in an interpretation. For the semantics of $\to$, a *ternar*y relation, R is used. And:

- $A \to B$ is true at world w iff for all x and y such that $Rwxy$, if (materially) A is true at x, B is true at y.

This generalises the semantics of $⥽$, where a binary relation would be used, and x and y are the same.

The most obvious philosophical question concerning these semantics is what the machinery $*$ and R mean. It can be agreed that they do what is necessary as purely technical devices; but more than that is needed if what they are to provide is a *semantics*, properly so called—that is, an intelligible account of the meanings of *not* and *if* (in their appropriate senses).

Another question follows quickly. Various constraints on R deliver an enormously wide variety of systems of relevant logic, with quite diverse properties. Which one is appropriate for the semantics of a natural language *if*? This is a crucial question, since some relevant logics are not appropriate for some applications. For example, those logics which validate the contraction principle, $A \to (A \to B) \models A \to B$. reduce the naive theory of truth or of sets to triviality, due to Curry paradoxes. (See below.)

Relevant logics are most naturally thought of as rivals to classical logic. True, they extend the vocabulary of classical logic with the new connective $\to$. But the treatment of negation naturally ensures the invalidity of the classically valid principle of Explosion—though one can construct relevant logics where it holds. (It can be the case that $A \wedge \neg A \models B$ without it being the case that $\models (A \wedge \neg A) \to B$.) So relevant logics are another kind of paraconsistent logic.

A rather different matter concerning conditionality centres round some other apparently aberrant inferences, such as:

- If there is a subway strike tomorrow, I will walk to work. So if there is a subway strike tomorrow and I die tonight, I will walk to work.

- If the other candidates die before the election, Biden will be re-elected. If Biden is re-elected, the other candidates will be disappointed. So if the other candidates die before the election, they will be disappointed.

These inference are valid if *if* is interpreted as $\supset$ (classically or intuitionistically), $\prec$, or $\to$.

To handle the invalidity of this kind of inference, a class of logics called *variably strict* (VS) *logics* (or sometimes simply *conditional logics*) was proposed about 1970, first by Stalnaker and then by (David) Lewis. There are some technical niceties which we may slide over here; the basic idea is as follows. To evaluate the truth of $A \prec B$ at w, we look at all the worlds where A is true. If B is true at all of them, $A \prec B$ is true at w; if not, not. In VS logics, things work exactly the same, except that we do not look at *all* the worlds where A holds, but just those of a certain kind, determined by A. Technically these can be picked out by a function, $f_A(w)$ from worlds to subsets of worlds (where, of course, A is true at worlds in $f_A(w)$). Let us write this new kind of conditional as >. Then:

- $A > B$ is true at w iff for all $w' \in f_A(w)$, B is true at w'.

Our two inferences are invalidated by the fact that, in each example, the two conditionals involved have different antecedents, A_1 and A_2, so when evaluating at the actual world, @, $f_{A_1}(@)$ and $f_{A_2}(@)$ are different. Thus, in the first example, $f_{A_1}(@)$ is the set of worlds where there is a subway strike tomorrow, but otherwise things go on pretty much as normal; and $f_{A_2}(@)$ is the world there is a subway strike tomorrow, I die tonight, but otherwise things go on pretty much as normal.

The crucial philosophical issue is now how one picks out the worlds in question. For a start, the selection is going to be contextually dependent, as standard examples show:

- If this car were an photon, some cars would travel at the speed of light.
- If this car were a photon, some photons would not travel at the speed of light.

Which of these is true depends on what, exactly, the context of discussion is.

Lewis and Stalnaker took $f_A(w)$ to be the worlds most similar to w where A is true—and Stalnaker took there to be a unique such world. (Similarity is, of course, a contextual matter.) But this gives very strange results. Consider the conditional concerning the Cuban missile crisis:

- If Kennedy had pressed the button, a nuclear armageddon would have ensued.

This is almost certainly true. But on almost any plausible understanding of similitude, a world after a nuclear holocaust is absolutely *nothing* like the present world. So what is in fact true is that:

- If Kennedy had pressed the button, something would have happened to prevent a nuclear armageddon.

You have to be a *real* optimist to believe that.

Setting similarity aside, $f_A(w)$ would seem to contain those worlds where A is true and where certain contextually determined truths of w hold. But how one determines what carries over from w is very hard to specify.

At this point, the matter intersects with another thorny issue. Consider another very standard pair of examples:

- If Oswald were not to have shot (or had not shot) Kennedy at Dealey Plaza on November 22, 1963, someone else would have.
- If Oswald did not shoot Kennedy at Dealey Plaza on November 22, 1963, someone else did.

The second is true, but assuming that Oswald did, in fact, act alone, the first is false.

The first kind of conditional is sometimes called a counterfactual conditional. The name is a poor one. The antecedents of both of our conditionals are contrary to fact. Perhaps a bit better, the first conditional is sometimes called a subjunctive conditional, since the verb of its antecedent is in the subjunctive mood (to the extent that English still has one). The second is called an indicative conditional, due to the mood of the verb of the antecedent. One has to be careful even here, however. It is very difficult to obtain a contrast of this kind for all subjunctive/indicative pairs, especially when the subjunctive is a present one. Witness:

- If I go to the party I will enjoy myself
- If I were to go the the party I would enjoy myself.

But setting niceties of terminology aside. Some (Jackson, Lewis himself) have argued that the example shows that English has two kinds of conditionals, that indicative conditionals are captured by $\supset$, and that subjunctive conditionals are captured by $>$. That does not look very plausible: even setting aside the paradoxes of the material conditional, our subway examples above contain only indicative conditionals.

Others have held that English has only one kind of conditional (Stalnaker, Priest), in which case, the difference between the two Kennedy examples must by accounted for, not by two meanings of *if*, but by the differences of tense and mood involved. Loosely, the first Oswald example is really the conditional:

- If Oswald does not shoot Kennedy at Dealey Plaza on November 22, 1963, someone else will.

where the point of evaluation is shifted back to that fateful morning.

One final comment. Notwithstanding this matter, standard VS logics may clearly be thought of as supplements to classical logic, since one just takes a modal logic based on classical logic and adds the connective $>$. However, if one proceeds in this way, conditionals with necessarily false antecedents come out as vacuously true, just as for strict conditionals. This seems just as absurd. Merely consider (and assuming, for the sake of illustration that classical logic is correct):

- If intuitionist logic were correct, the law of Excluded Middle world fail
- If intuitionist logic were correct, the law of Explosion world fail

These are true and false, respectively. But since there are no possible worlds where the antecedent is true, they are both vacuously true. Some (such as Williamson) have bitten the bullet. But given an approach to conditionals of the kind we have been discussing, it would seem necessary to take into account worlds where the antecedent is true; and these (given our assumption) are impossible worlds. In fact, it is perfectly straightforward to construct a VS logic based on a relevant logic, with its supply of possible and impossible worlds. In such a logic, conditionals with logically false antecedents are not vacuously true. Since such logics are relevant logics, they are rivals to classical logic.[11]

6 Contraclassical Logics: Connexivism and Robustly Paraconsistent Logics

All the non-classical logics we have considered so far are sub-logics of classical logic (when the various conditionals are identified with $\supset$). However, there are non-classical logics for which this is not the case. Such logics are called *contra-classical*, and they are clearly rivals to classical logic. There are a number of such logics, but perhaps one of the most interesting kind are connexive logics.

Consider the two principles:[12]

Aristotle: $\models \neg(A \to \neg A)$

Boethius: $A \to B \models \neg(A \to \neg B)$

where $\to$ is some generic sort of conditional. (Clearly Boethius entails Aristotle provided $\models A \to A$.) As is easy to check, these are not classically valid principles (when $\to$ is $\supset$). Despite this, they have a certain *prima facie* plausibility. Indeed, the philosophers they are named after appear to have endorsed them, and both principles were fairly orthodox until about the 12th Century. They then fell out of

[11]*Further reading and references:* Priest (2008), chs. 5 (conditional logic), 10 (relevant logic); Restall (1999); Beall *et al* (2012); Priest (2018); Edgington (2020); Starr (2019); Berto, French, Priest, and Ripley (2018).

[12]One finds a number of different articulations and variations of these in the contemporary literature, but these will do for present purposes.

favour and were largely forgotten. They were rescued from oblivion by Angell and McCall in the 1960s, who constructed formal systems in which these inferences are valid. Such systems are now called *connexive logics*.

If connexive logics are to avoid inconsistency, they must have a non-standard account of conjunction, giving up the principle that $(A \wedge B) \to B$. For this gives us $(A \wedge \neg A) \to A$ and $(A \wedge \neg A) \to \neg A$. **Boethius** then delivers $\neg((A \wedge \neg A) \to \neg A)$.[13] For this reason, the systems of Angell and McCall are rather ugly. However, a very simple and elegant way of producing a connexive logic was noted by Wansing in the early years of this century. We simply specify the *falsity* condition of $A \to B$ (and so the truth condition of $\neg(A \to B)$) as that of the truth of $A \to \neg B$. Given $A \to B$ and double negation, we then have $A \to \neg\neg B$; and so $\neg(A \to \neg B)$ follows immediately. Moreover, the falsity conditions are very natural, since we often negate conditionals in this way—'If you go to the movies, will you take me?' 'No. If I go to the movies I will *not* take you.'

Of course, this procedure makes sense only if we are in a semantics where we can specify truth and falsity conditions of formulas (maybe at a world) independently. But this is required in any logic where sentences can be neither true nor false or both true and false. In such logics, if one knows whether something is true, nothing follows about whether or not it is false: this has to be specified separately. (And if the truth and falsity conditions overlap we have a truth value glut. If they underlap we have a truth value gap.)[14]

If one constructs a connexive logic in this way, and conjunction behaves normally, so that we have the logical truth of $(A \wedge B) \to A$, then our logical truths will be inconsistent, as we have seen. Assuming that the semantics guarantees that the set of logical truths is non-trivial (i.e., does not contain all sentences), the logic must be paraconsistent. Most standard paraconsistent logics are quite consistent. That is, their set of logical truths is consistent. As far as I know, logics in which this is not the case have no standard name, so let us call a paraconsistent logic where the set of logical truths is inconsistent *robustly paraconsistent*. The view that some contradictions are true is termed *dialetheism*. Dialetheism is not a view of logic, but of metaphysics. However, since logical truths are, presumably, true, robustly paraconsistent logics are committed to dialetheism. That even logic itself is dialetheic is a thought that invites much further philosophical reflection.[15]

[13] Indeed, these principles appear to have disappeared from the history of logic, about the time that an extensional account of conjunction was becoming orthodox.

[14] Wansing first applied the idea to Nelson's logic of constructible negation—essentially positive intuitionist logic, but based on an underlying 4-valued logic.

[15] *Further reading and references*: Priest (2008), 9.7a (connexive logic), Wansing (2022).

7 Sub-Structural Logics: Contraction and Cut

Let us return to sub-logics of classical logic. In the first half of the 20th century, the systems of proof for logics were usually axiomatic. However, a powerful and much more perspicuous method of proof was introduced for classical and intuitionist logic by Gentzen in the 1930s: sequent calculi. Sequents are objects of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are multisets of formulas,[16] meaning, intuitively, that if all the members of Γ are true, some of the members of Δ are true. For each connective there is a pair of rules, one introducing the connective on the left of $\Rightarrow$ and one introducing it on the right. In addition, there are *structural* rules, which do not involve connectives, but manipulate sequents in other ways. Two of the most important are Contraction left and right:

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

and Cut:

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

The logics obtained by dropping structural rules are *sub-structural* logics.

Dropping the rule of Contraction has been advocated mainly by logicians who are interested in resource-boundedness. Thus, it may be possible to derive a conclusion using A twice, but not using it only once. However, as a way of solving the paradoxes of self-reference, dropping Contraction has more recently been advocated by some logicians (Paoli, Zardini). And for the same reason, dropping Cut has been advocated by others (Cobreros, Égré, Ripley, van Rooij). The attraction of such a solution is that it gives a uniform solution to at least the Liar and Curry's paradox. Here, for example, is a derivation of Curry's paradox, with the applications of Cut and Contraction marked. Let A be any sentence one wishes, and let C be a sentence of the form $T\langle C\rangle \rightarrow A$. Then we have:

$$\frac{\begin{array}{c} \vdots \\ T\langle C\rangle \Rightarrow A \\ \Rightarrow T\langle C\rangle \rightarrow A \\ \Rightarrow T\langle C\rangle \end{array} \qquad \begin{array}{c} \dfrac{T\langle C\rangle \Rightarrow T\langle C\rangle \quad A \Rightarrow A}{T\langle C\rangle, T\langle C\rangle \rightarrow A \Rightarrow A} \\ T\langle C\rangle, T\langle C\rangle \Rightarrow A \\ T\langle C\rangle \Rightarrow A \end{array}}{\Rightarrow A} \qquad \begin{array}{c} \text{Contraction} \\ \text{Cut} \end{array}$$

[16] I.e., sets with repetitions, so that $\{A, A, B\}$ is not the same as $\{A, B\}$.

The ellipsis on the left hand proof represents the first three lines of the right hand proof. Since Cut and Contraction are valid in classical logic, the logics produced by dropping these principles are rivals to classical logic.

Contraction-free logics do not have a simple semantics, but Cut-free logics may be given one using a 3-valued logic. A valid sequent is one that preserves designated values; but the designated values are different on the left and right side of $\Rightarrow$. In particular, a sequent is valid if whenever all the antecedents have value *true*, some consequent has either the value *true* or the non-classical value—whatever one takes this to mean. The standards of acceptability are, then, more strict on the left hand side of $\Rightarrow$, and more tolerant on the right hand side. Such logics may therefore be called Strict/Tolerant (ST).

Employing sub-structural logics in this way raises a number of important philosophical questions, notably:

- What independent reasons (if any) are there for believing the structural principles in question not to be valid? In particular for the Cut-free approach, why should we change designated values between antecedents and consequents?

- If one drops these principles, how can one account for the massive amount of apparently legitimate and non-paradoxical reasoning which uses them? For example, Cut is just another name for the transitivity of deduction; and this is used all the time in, say, mathematics, when we use lemmas to deduce theorems.

- Are there other paradoxes of self-reference that cannot be blocked in this way? For example, Contraction-free logics standardly have operators called *exponentials*, which allow for contractions of certain kinds; and these may be used to deliver versions of the liar and Curry paradoxes.

Another important question to which this suggestion gives arise is as follows. As proved by Gentzen, if one takes a sequent calculus for classical logic, any sequent that can be proved with Cut, can be proved without it (provided that there are no other rules of inference). It is sometimes claimed on this ground that ST is not a rival to classical logic, since it has the same consequence relation. However one can really claim this only if one ignores higher order inferences.

For think of the sequents we have been dealing with as of level 1, and write $\Rightarrow$ as $\Rightarrow_1$. Then we may think of Cut as a (meta-)inference sequent, of level 2, and write it as:

- $[(\Gamma \Rightarrow_1 \Delta, A),\ (A, \Gamma' \Rightarrow_1 \Delta')] \Rightarrow_2 [\Gamma, \Gamma' \Rightarrow_1 \Delta, \Delta']$

This sequent will hold in classical logic, but fail in ST.

It follows that two logics may agree at level 1 and disagree at level 2. Next, we can clearly consider sequents of level greater than 2—indeed of arbitrarily high level. And it has been shown (by Barrio, and his co-workers) that the phenomenon repeats itself all the way up (even into the transfinite). That is, two logical may agree on their inferences up to some level, α, and then diverge at level $\alpha + 1$.

In the first half of the 20th century it was common to identify logics with sets of logical truths. Once it was realised that different logics may have the same set of logical truths, it became standard to identify a logic with a level 1 consequence relation. It might well now be suggested that even this is not good enough. To specify a logic, we need to specify the consequence relation at *every* level.[17]

8 Free Logics: Existence and Quantification

With the exception of the discussion of intuitionist logic, quantification has not featured till now, but quantification is central to the logics of our last two sections. The first of these concerns free logics.

Classical first-order logic verifies the inference of existential generalisation:

EG: $A_x(a) \models \exists x A$

If one reads $\exists x$ as *there exists an x such that*, EG does not seem correct. All the following appear true:

- Sherlock Holmes is a fictional character
- I am reading about Sherlock Holmes
- The tourist visiting Baker St admired Sherlock Holmes.

But Sherlock Holmes does not exist, and never did.

Free logics are logics where EG fails. They were pioneered by Leonard and Lambert, starting around the late 1950s. They contain an existence predicate (primitive or defined—possibly in terms of identity), $\mathfrak{E}$. EG is invalid; what is valid is its restriction:

[17] *Further reading and references*: Priest (202b); Barrio, Rosenblatt, and Tajer (2015); Priest (202c).

EG: $A_x(a), \mathfrak{E}a \models \exists x A$

Free logics are, then, rivals to classical logic.

In the semantics, there is a domain, D, of objects (at each world, if we have a world-semantics), thought of as containing the existent objects. Terms in the language may have a denotation in D, or not. When not, matters can be handled in different ways. In some free logics such terms simply have no denotation. That is, the denotation function is a partial function. In others, there is a second ("outer") domain, D', thought of as containing non-existent objects; and terms without a denotation in D find one in D'. Quantifiers range over D; and if a has a denotation in D, $\mathfrak{E}a$ is true (at a world, if we have a world semantics). Otherwise it is false. In many cases, there is no technical difference between the two approaches, since a partially defined function can always be thought of as a total function where the undefined case is filled in with a "dummy" value—one of the members of D'.

These semantics leave various matters open. The most important concerns the truth of atomic sentences. Consider an atomic sentence, Pa. (The generalisation to many-place predicates is straightforward.) If the interpretation satisfies the constraint:

- if a does not have a denotation in D, Pa is false (at a world)

the logic is called a *negative* free logic. If we have an underlying logic with truth value gaps, and the logic satisfies the condition:

- if a does not have a denotation in D, Pa is neither true nor false (at a world)—except, perhaps when P is $\mathfrak{E}$

the logic is called a *neutral* free logic. Otherwise it is called a *positive* free logic. Unless all atomic sentences are given the same value, the partial function semantics cannot be used for positive free logics, since denotations for terms are required to draw distinctions between the values of different atomic sentences.

The first important philosophical issue is which of these three is the correct approach. At least *prima facie*, a free logic needs to be positive. Our three examples above (and many like them) all appear true. Moreover, we do appear to need to draw distinctions between the truth values of atomic sentences. It is true that the Ancient Greeks worshipped Zeus. It is not true that they worshipped Sherlock Holmes.

Next, if we have a semantics with an "outer domain", why should we not let quantifiers range over $D \cup D'$? Of course, if one does so, one must precisely not read the particular (as opposed to universal) quantifier as *there exists*.

Some will do nicely. Let us write this quantifier as $\mathfrak{S}$. Then our old existential quantifier—now properly so-called—can be defined in the obvious way: $\exists xA$ is $\mathfrak{S}x(\mathfrak{E}x \wedge A)$. And there are plenty of examples of non-existentially-loaded quantification in English:

- You and I are thinking of something—the same thing: Sherlock Holmes.
- I wanted to buy you something for Christmas, but I found out that it does not exist: an actual photograph of Sherlock Holmes.

Quine argued that the particular quantifier *means*, *there exists*. But his arguments are, to put it mildly, underwhelming. Quine's view still, however, finds its philosophical defenders.

Moreover, once we have gone this far, there would appear to be nothing wrong with EG, provided we read the $\exists$ as $\mathfrak{S}$. Indeed, if none of the other machinery is at issue, there would appear to be no bar to taking over classical logic wholesale, simply understanding the particular quantifier as $\mathfrak{S}$. Free logic then is just a supplement of classical logic, not a rival. The supplement is the predicate $\mathfrak{E}$, and D has nothing to do with quantification. It is simply the extension of $\mathfrak{E}$.

There are, of course, other issues concerning terms that denote non-existent objects. One concerns how their denotations are fixed, since such objects are not in the causal web. The same issue concerns abstract mathematical objects, of course, even if one takes these to be existent objects (though the similarity here is rarely mentioned).

Another issue concerns descriptions (characterisations). It is natural to think that at least some of these refer to non-existent objects. The Greeks, after all, worshipped the head of the pantheon on Mt. Olympus. One may also think that an object characterised in a certain way indeed has its characterising properties:

- $A_x(\iota xA)$

No one can subscribe to this principle in full generality, however. It leads to triviality in two steps. Let B be an arbitrary sentence, and let us write t for $\iota x(B \wedge x = x)$. Then the principle gives us $B \wedge t = t$, from which B follows. So when (and where) do instances of the principle hold? Discussions about this loom large in debates between contemporary noneists (those who hold that some objects do not exist).[18]

[18] *Further reading and references*: Nolt (2020); Priest (2016), chs. 17, 18 (non-existence); Reicher (2019).

9 Second-Order and Plural Logics: Higher Order Matters

The last topic we will look at concerns second-order logic (and higher order logic more generally, but most of the philosophical issues are already raised at the second-order level, so let us stick with that). In this, there are distinctive variables, of every adicity, n, which can occupy syntactically the place of an n-place predicate. These can bound by appropriate quantifiers. There is then a comprehension scheme of the form:

- $\exists X_n \forall x_1 ... \forall x_n (X_n x_1 ... x_n \leftrightarrow A)$

where A is any formula not containing X_n. Notice that this includes the case where $n = 0$. Variables of the form X_0 are usually called 'propositional variables'.

The first thing to note here is that when Frege and Russell invented/discovered their logic, it was second-order. So in a sense, standard second-order logic is still classical. However, Frege/Russell logic was stripped down to first-order, largely by Hilbert and his school; and the term 'classical logic' is now more commonly applied to first-order logic only. In this case, second-order logic—if it a logic at all (see below)—is a supplement of classical logic. Note, however, that all the logics we have looked at can be extended to second-order. In that sense, there is nothing specifically classical about second-order logic.

The most obvious philosophical question concerning second-order logic is 'what do the variables range over, and what are the identity conditions of these entities?'. In the usual model theory, the second-order entitles are sets (or set-theoretic relations), with their usual extensional identity conditions. But there are other possibilities: one might take them to be propositions, or propositional functions. The identity conditions for such entities are more contentions, but one natural suggestion is that two are identical if they are *necessarily* co-instantiated.

Another issue is whether second-order logic is really logic at all. If one thinks of second-order entitles as sets, one may take second-order logic to be simply a fragment of set-theory; and this is not logic, one may argue, since it has a specific ontology. As Quine put it, it is simply 'set theory in sheep's clothing'.[19] Quine also argued that second-order logic is not logic since it is not axiomatisable. Whether or not this is a good argument, it raises the question of the role that axiomatisability ought to play in logic.

[19] Quine (1970), p. 66.

A logic closely connected with second-order logic is plural logic. This was pioneered by Boolos in the 1980s. Pluralities are things like:

- John, Paul, George, and Ringo
- the authors of *Principia Mathematica*
- prime numbers greater than 10^{10}

Plural logic has variables that range over such things—usually written as double lower case romans, as in *xx*—and corresponding quantifiers. It can also have names for pluralities, and predicates which apply to them. There is also a binary predicate, $\prec$, where $x \prec yy$ means 'x is one of the yy. So one can say things such as:

- Russell$\prec$ the authors of *Principia Mathematica*
- $\exists xx$ I am going to the party with xx (there are some people such that I am going to the party with them)
- Elizabeth Windsor and Philip Mountbatten married

Note that the last of these does not mean the same as:

- Elizabeth Windsor married and Philip Mountbatten married

(That would be true if each had had a different spouse.)

Standardly, plural logics come with a comprehension principle:

- $\exists xx \forall y (y \prec xx \leftrightarrow A)$

where xx does not occur in A—provided something satisfies A: standardly, there are no empty pluralities. And pluralities have extensional identity conditions:

- $xx = yy \leftrightarrow \forall z (z \prec xx \leftrightarrow z \prec yy)$

Clearly, then, plural logics are supplements of classical logic. But note that, as for second-order logic, any of the logics we have met can be extended to plural logics.

A major philosophical issue to which plural logic gives rise is this. Clearly, both plural logic and second-order are doing the same kind of thing in some sense: they both allow quantification over aggregates of objects—to use a neutral term.

Some advocates of plural logic have argued that plural logic is preferable, however, since it is committed to only first-order objects, and so has no "higher order" ontological commitments. Others have replied that plural logic is just "second-order logic in sheep's clothing"; and so it is committed to higher order entities. (My own view, for what it is worth, is that since there are names for pluralities, and the names refer, there are things to which they refer, however one conceptualises these things. Since *Naming and Necessity*, it is very hard to sustain the thought that names are really definite descriptions of some kind.)

A closely related issue concerns Cantor's paradox. In set theory, Cantor's theorem states that there are more subsets of a set than members of the set. That is, given any set, x, there is no injective function from the subsets of x to x. Problems then arise if there is a set of all sets, V. For the identity function is an injective function from the subsets of V to V.[20]

Given the close similarity between sets and pluralities, it is unsurprising that there is an analogue of Cantor's theorem for pluralities. There are more sub-pluralities of a plurality then objects in the plurality. That is, given any plurality, xx, there is no injective function from the sub-pluralities of xx to the xx.[21] There is then a problem if there is a plurality of all objects, VV, and an injective function from sub-pluralities of this (i.e., any plurality) to VV. Moreover, there do seem to be such functions. For example, we can map the plurality xx to the mereological sum of the objects, $\{y\}$, such that $y \prec xx$. (The singletons are necessary, since if x and y are proper parts of z, then the mereological sums of x and z, and y and z, are both z.) So either there is no plurality of all objects or there is something wrong with these function—or the theory must be formalised with a contradiction-tolerant, i.e. paraconsistent, logic. (In orthodox set theory, the the solution to the paradox is to reject the existence of V; but there is no obvious reason why one must make the same move in plural logic.)[22]

10 A Brief Look Back

The application of contemporary tools of mathematics to logic caused a revolution in the subject. However, I think it fair to say that it took logicians many more decades to start to explore the full riches of what they make possible. The dialectic

[20]This is just Russell's Paradox. See Priest (2005), 9.1.

[21]See Florio and Linnebo (2021), ch. 3.

[22]*Further reading and references*: Väänänen (2019); Florio and Linnebo (2021).

between philosophical issues and the application of these tools has been one of the most exciting aspects of logic in the last 50 years or so.

In this essay, we have been through some aspects of this dialectic. The coverage of each issue has perforce been superficial. And there is a wealth of issues it has not been possible to touch on at all—spreading into metaphysics, the philosophy of language, and epistemology. Many of all of these matters are still half-finished business. And many entirely new ones will surely emerge as time goes on. We live in perhaps the most fruitful and creative period in logic which there has ever been.[23]

References

[1] *SEP*:= E. Zalta (ed.), *Stanford Encyclopedia of Philosophy*.

[2] Ballarin, R. (2021), 'Modern Origins of Modal Logic', *SEP*, https://plato.stanford.edu/entries/logic-modal-origins/.

[3] Barrio, E., Rosenblatt, L., and Tajer, D. (2015), 'The Logics of Strict-Tolerant Logic', *Journal of Philosophical Logic* 44: 551–571.

[4] Beall, J., Brady, R., Dunn, J. M., Hazen, A., Mares, E., Meyer, R., Priest, G., Restall, G., Ripley, D., Slaney, J., and Sylvan, R. (2012), 'On the Ternary Relation and Conditionality', *Journal of Philosophical Logic* 41: 595–612.

[5] Berto, F., French, R., Priest, G., and Ripley, D. (2018), 'Williamson on Counterpossibles', *Journal of Philosophical Logic* 47: 693–713.

[6] Berto, F., and Jago, M. (2019), 'Impossible Worlds', *SEP*, https://plato.stanford.edu/entries/impossible-worlds/.

[7] Cintula, P., Fermüller, G., and Noguera, C. (2021), 'Fuzzy Logic', *SEP*, https://plato.stanford.edu/entries/logic-fuzzy/.

[8] Dutilh Novaes, C. (2020), 'Medieval Theories of Consequence', *SEP*, https://plato.stanford.edu/entries/consequence-medieval/.

[9] Edgington, D. (2022), 'Indicative Conditionals', *SEP*, https://plato.stanford.edu/entries/conditionals/.

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[10] Florio, S., and Linnebo, Ø. (2021), *The Many and the One: a Philosophical Study of Plural Logics*, Oxford: Oxford University Press.

[11] Gillon, B. (2021), 'Logic in Classical India', *SEP*, https://plato.stanford.edu/entries/logic-india/.

[12] Glanzberg, M. (2018), 'Truth', in E. Zalta (ed.), *SEP,* https://plato.stanford.edu/entries/truth/.

[13] Gottwald, S. (2015), 'Many-Valued Logic', *SEP,* https://plato.stanford.edu/entries/logic-manyvalued/.

[14] Haack, S. (1974), *Deviant Logic*, Cambridge: Cambridge University Press; 2nd (extended) edn, *Deviant Logic, Fuzzy Logic,* Chicago, IL: Chicago University Press, 1996.

[15] Hawthorne, J. (2019), 'Inductive Logic', *SEP,* https://plato.stanford.edu/entries/logic-inductive/.

[16] Hylton, P., and Kemp, G. (2019), 'Willard van Orman Quine', *SEP,* https://plato.stanford.edu/entries/quine/.

[17] Iemhoff, R. (2019), 'Intuitionism in the Philosophy of Mathematics', *SEP,* https://plato.stanford.edu/entries/intuitionism/.

[18] Kneale, W., and Kneale, M. (1962), *The Development of Logic*, Oxford: Oxford University Press.

[19] Nolt, J. (2020), 'Free Logic', *SEP*, https://plato.stanford.edu/entries/logic-free/.

[20] Priest, G. (2005), *Beyond the Limits of Thought*, 2nd edn, Oxford: Oxford University Press.

[21] Priest. G. (2008), *Introduction to Non-Classical Logic*, 2nd edn, Oxford: Oxford University Press.

[22] Priest, G. (2014), 'Revising Logic', ch. 12 of P. Rush (ed.), *The Metaphysics of Logic*, Cambridge: Cambridge University Press.

[23] Priest, G. (2016), *Towards Non-Being*, 2nd edn, Oxford: Oxford University Press.

[24] Priest, G. (2018), 'Some New Thoughts on Conditionals', *Topoi* 27: 369–377.

[25] Priest, G. (202a), 'What is the Third Value?', forthcoming.

[26] Priest, G. (202b), 'Substructural Solutions to the Semantic Paradoxes: a Dialetheic Perspective', forthcoming.

[27] Priest, G. (202c), 'Some Notes on Variable Designated Value Logics', to appear.

[28] Quine, W. (1970), *Philosophy of Logic*, Inglewood Cliffs, NJ: Prentice Hall.

[29] Reicher, M. (2019), 'Nonexistent Objects', *SEP*, https://plato.stanford.edu/entries/nonexistent-objects/.

[30] Restall, G. (1999), 'Negation in Relevant Logics (How I Stopped Worrying and Learned to Love the Routley Star)', pp. 53–76 of D. Gabbay and H. Wansing (eds.), *What is Negation?,* Dordrecht: Kluwer Academic Publishers.

[31] Russell, B. (1935–1936), 'The Limits of Empiricism', *Proceedings of the Aristotelian Society* 36: 131–150.

[32] Starr, W. (2019), 'Counterfactuals', *SEP,* https://plato.stanford.edu/entries/counterfactuals/.

[33] Strasser, C. (2019), 'Non-Monotonic Logic', *SEP*, https://plato.stanford.edu/entries/logic-nonmonotonic/.

[34] Väänänen, J. (2019), 'Second-Order and Higher-Order Order Logic', *SEP*, https://plato.stanford.edu/entries/logic-higher-order/.

[35] Wansing, H. (2022), 'Connexive Logic', *SEP,* https://plato.stanford.edu/entries/logic-connexive/.

[36] Wright, C. (1982), 'Strict Finitism', *Synthese* 51: 203–282.