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JC BEALL, ROSS T. BRADY, A. P. HAZEN, GRAHAM PRIEST
and GREG RESTALL

RELEVANT RESTRICTED QUANTIFICATION

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ABSTRACT. The paper reviews a number of approaches for handling restricted quantification in relevant logic, and proposes a novel one. This proceeds by introducing a novel kind of enthymematic conditional.

1. INTRODUCTION: RESTRICTED QUANTIFICATION

Quantifiers rarely occur in a bare form (as in "Everything is a mortal", "Something is happy") in vernacular reasoning. They usually occur in a restricted way (as in "All men are mortal", "Some person in the room is happy"). So, when formalising vernacular reasoning, an understanding of such restricted quantification is essential.

Three forms of restricted quantification are particularly important: *universal*, *particular* and *null*. We can write these, neutrally for the moment, as follows:

- All αs are βs: $Av[\alpha(v), \beta(v)]$
- Some αs are βs: $Sv[\alpha(v), \beta(v)]$
- No αs are βs: $Nv[\alpha(v), \beta(v)]$

The reading of these quantifiers in classical logic takes them to be, respectively:

- $\forall v(\alpha(v) \supset \beta(v))$
- $\exists v(\alpha(v) \wedge \beta(v))$
- $\neg\exists v(\alpha(v) \wedge \beta(v))$.

These translations of the quantifiers are less than fully satisfying when one moves from classical logic to relevant logic. For example, the inference $\alpha(a), Av[\alpha(v), \beta(v)] \vdash \beta(a)$ would seem to be unproblematically valid, but $\alpha(a), \forall v(\alpha(v) \supset \beta(v)) \vdash \beta(a)$ is not valid in a paraconsistent (relevant) setting: it is a generalisation of the disjunctive syllogism. Thus, one might admit a situation that is inconsistent about whether or not $\alpha(a)$ obtains, but in which $\neg\alpha(v)$ is true for every other object. So, in this circumstance $\alpha(a)$ is true, and $\forall v(\alpha(v) \supset \beta(v))$ is true (since $\neg\alpha(v)$ for every v), but it need not follow that $\beta(a)$ is true in this circumstance: we have said nothing of the behaviour of β at all. So, once we admit inconsistent situations we have counter-examples to $\alpha(a)$, $Av[\alpha(v), \beta(v)] \vdash \beta(a)$ if the A form is read classically. On the other hand, such counter-examples do not seem to be reasons to reject this principle *tout court*; rather, they lead us to look for another way to understand restricted universal quantification. The point of this paper is to establish what an adequate understanding of restricted quantification in a relevant logic should be like.

The discussion will be conducted with an eye on the applications of relevant logic, and in particular its use in formalising naive set theory and truth theory. These theories are liable to be inconsistent, but in relevant logic this does not matter, since Explosion ($\alpha, \neg\alpha \vdash \beta$) is invalid, so contradictions are quarantined. But not all relevant logics are suitable for the purpose. Relevant logics that contain the principle of Contraction ($\alpha \rightarrow (\alpha \rightarrow \beta) \vdash \alpha \rightarrow \beta$) trivialise these theories, due to Curry Paradoxes.[1] It is therefore important that restricted quantification does not introduce Explosion or Contraction through the back door, so to speak.[2] (We will see how this might happen in a moment.)

We will proceed as follows. We will start by considering some desiderata for accounts of relevant restricted quantification. Whilst these may certainly be contested, they provide a yardstick against which accounts may be measured. We will then consider some possible accounts and measure them. By the end, we will have found a plausible account.[3]

2. Some Constraints

Let us start with some inferences that one might expect restricted quantifiers to validate. All of the following are very natural. ($\rightarrow$, here, is a standard relevant conditional, which we will assume to satisfy *modus ponens*, in the form $\alpha \rightarrow \beta, \alpha \vdash \beta$, throughout).

A1 $\alpha(a), Av[\alpha(v), \beta(v)] \vdash \beta(a)$
A2 $\forall v\beta(v) \vdash Av[\alpha(v), \beta(v)]$
A3 $\forall v(\alpha(v) \rightarrow \beta(v)) \vdash Av[\alpha(v), \beta(v)]$
A4 $\alpha(a), \beta(a) \vdash Sv[\alpha(v), \beta(v)]$
A5 $\neg\exists v\beta(v) \vdash Nv[\alpha(v), \beta(v)]$

Inferences of this kind seem to be applied all the time when reasoning with restricted quantifiers, e.g., in set theory.[4] So, any account of restricted quantification would do well to render these principles *valid*, or to provide some kind of informative *diagnosis* as to why they are not, despite appearances. To fall short on this score would be a mark against any theory purporting to codify principles of inference.

Next, we note some inferences that should *not* be forthcoming. The first concerns Explosion. Consider the following deduction, where α and β do not contain v. (If you think that quantifiers must always and everywhere bind at least some variable, then replace α by $(\alpha \wedge \gamma(v)) \vee \alpha$ – where $\gamma(v)$ is any formula that contains v free, such as $v = v$ – and similarly for β. These replacements contain v free, and are logically equivalent to the original formulas.)

$$\dfrac{\neg\beta \quad \dfrac{\dfrac{\dfrac{\beta}{\forall v\beta}}{Av[\alpha,\beta]}\ \text{A2}}{Av[\neg\beta,\neg\alpha]}\ \text{Contraposition for A}}{\neg\alpha}\ \text{A1}$$

This shows that, if Explosion is to be avoided, then in the presence of our desiderata, Contraposition for 'all' must fail:[5]

B1 $Av[\alpha, \beta] \nvdash Av[\neg\beta, \neg\alpha]$

The second inference that must fail concerns Contraction. Define $\alpha \rightsquigarrow \beta$ as $Av[\alpha, \beta]$ (again, add vacuous variables if desired). Then we cannot have $\alpha \rightsquigarrow (\alpha \rightsquigarrow \beta) \vdash \alpha \rightsquigarrow \beta$, or else we would be able to reason as follows.[6] Let γ be of the form $T\langle\gamma\rangle \rightsquigarrow \bot$, where $\bot$ is a constant – often written as F in the relevant-logic literature – such that $\bot \vdash \alpha$, for all α.

$$\dfrac{\dfrac{\dfrac{\dfrac{T\langle\gamma\rangle \leftrightarrow (T\langle\gamma\rangle \rightsquigarrow \bot)}{T\langle\gamma\rangle \rightsquigarrow (T\langle\gamma\rangle \rightsquigarrow \bot)}}{T\langle\gamma\rangle \rightsquigarrow \bot}}{T\langle\gamma\rangle}}{\bot}$$

T–schema
A3
Contraction for $\rightsquigarrow$
Modus ponens for $\rightarrow$
A1

Hence, Contraction for $\rightsquigarrow$ must fail:

B2 $Av[\alpha, Av[\alpha, \beta]] \nvdash Av[\alpha, \beta]$

3. The Square of Opposition

The final constraints concern a number of standard relations that hold between sentences with restricted quantifiers. We would like to have as many of these as possible. The precise formulation of the constraint is, however, not entirely obvious, since there is reasonable disagreement about what these relations are. In traditional logic, the restricted quantifiers satisfy the square of opposition, which is as follows:

$$\begin{array}{ccc} Av[\alpha(v),\beta(v)] & \leftarrow cr \rightarrow & Nv[\alpha(v),\beta(v)] \\ \downarrow & \nwarrow \ \nearrow & \downarrow \\ sa & cd & sa \\ \downarrow & \swarrow \ \searrow & \downarrow \\ Sv[\alpha(v),\beta(v)] & \leftarrow sc \rightarrow & Sv[\alpha(v),\neg\beta(v)] \end{array}$$

(*Key*: *cd*, contradictories; *cr* contraries; *sc*, subcontraries; *sa* subalternates). In classical logic, all that remains of the square of opposition is this:

$$\begin{array}{ccc} \forall v(\alpha(v) \supset \beta(v) & & \forall v(\alpha(v) \supset \neg\beta(v)) \\ & \nwarrow \ \nearrow & \\ & cd & \\ & \swarrow \ \searrow & \\ \exists v(\alpha(v) \wedge \beta(v)) & & \exists v(\alpha(v) \wedge \neg\beta(v)) \end{array}$$

Only the contradictories remain. We can, however, restore the relations that have disappeared if we assume that $\exists v\alpha(v)$ is always true.

It is natural to suppose that we should have at least what is common ground. For the vertical relationship, this gives:

C1 $\exists v\alpha(v),\ Av[\alpha(v),\ \beta(v)] \vdash Sv[\alpha(v),\ \beta(v)]$
C2 $\exists v\alpha(v),\ Nv[\alpha(v),\ \beta(v)] \vdash Sv[\alpha(v),\ \neg\beta(v)]$

The top two lines of the square are contraries (assuming that we have $\exists v\alpha(v)$). This is to say that to have the two together is a contradiction. We can write this as:

C3 $\exists v\alpha(v),\ Av[\alpha(v),\ \beta(v)] \wedge Nv[\alpha(v),\ \beta(v)] \vdash \#$

where # can be thought of as the disjunction of all contradictions (things of the form $\gamma \wedge \neg\gamma$). (In classical logic any one will do. But contradictions are not necessarily equivalent in relevant logic.) Dually (and again assuming that $\exists v\alpha(v)$), for the subcontraries on the bottom line of the square, we have:

C4 $\exists v\alpha(v),\ \flat \vdash Sv[\alpha(v),\ \beta(v)] \vee Sv[\alpha(v),\ \neg\beta(v)]$

where $\flat$ is the conjunction of all things of the form $\gamma \vee \neg\gamma$. (Of course, this premise is unnecessary if the logic contains the Law of Excluded Middle).

The formulas at the ends of the diagonals are contradictories, that is, contraries and subcontraries (and now we may dispense with the condition that $\exists v\alpha(v)$). This gives:

C5 $Av[\alpha(v), \beta(v)] \wedge Sv[\alpha(v), \neg\beta(v)] \vdash \#$
C6 $Nv[\alpha(v), \beta(v)] \vee Sv[\alpha(v), \beta(v)] \vdash \#$

and:

C7 $\flat \vdash Av[\alpha(v), \beta(v)] \vee Sv[\alpha(v), \neg\beta(v)]$
C9 $\flat \vdash Nv[\alpha(v), \beta(v)] \vee Sv[\alpha(v), \beta(v)]$

This is our final set of constraints.

4. Some and No

We have now formulated our desiderata, A1–A5, B1–B2, C1–C8. Plausibly, one might add to (or subtract from) these. However, they provide, at least, a starting point from which to evaluate different accounts of restricted quantification. To this we now turn.[7]

Let us start with $Sv[\alpha(v), \beta(v)]$. There seems little alternative but to treat this as $\exists v(\alpha(v) \wedge \beta(v))$. The only other possibility that suggests itself is to treat it as $\exists v(\alpha(v) \circ \beta(v))$, where $\circ$ is the fusion operator of relevant logic.[8] But this fails A4 straight away, since for the logics in our sights $\alpha, \beta \nvdash \alpha \circ \beta$.

Treating $Sv[\alpha(v), \beta(v)]$ as $\exists v(\alpha(v) \wedge \beta(v))$ gives A4. C1 holds, since if $\exists v\alpha(v)$, then for some a, $\alpha(a)$; so $\beta(a)$ by A1; $\exists v(\alpha(v) \wedge \beta(v))$ follows by existential generalisation. C4 holds, since if $\exists v\alpha(v)$, then for some a, $\alpha(a)$. Given $\beta(a) \vee \neg\beta(a)$, the result follows. C5 becomes:

$$Av[\alpha(v), \beta(v)], \exists v(\alpha(v) \wedge \neg\beta(v)) \vdash \#$$

This holds given existential instantiation and A1.

There are two natural choices for $Nv[\alpha(v), \beta(v)]$ (which could, for all we have said so far, turn out to be the same thing):

Option 1 $\neg\exists v(\alpha(v) \wedge \beta(v))$
Option 2 $Av[\alpha(v), \neg\beta(v)]$

Under Option 1, A5 holds as in classical logic. C3 becomes:

$$\exists v\alpha(v), Av[\alpha(v), \beta(v)], \neg\exists v(\alpha(v) \wedge \beta(v)) \vdash \#$$

which holds in virtue of A1, existential instantiation and generalisation. And C6 and C8 are straightforward. However, C2 becomes:

$$\exists v\alpha(v), \neg\exists v(\alpha(v) \wedge \beta(v)) \vdash \exists v(\alpha(v) \wedge \neg\beta(v))$$

This fails in relevant logic since it uses a version of the disjunctive syllogism. (Given $\alpha(a)$ and $\neg\alpha(a) \wedge \neg\beta(a)$, we cannot get to $\neg\beta(a)$ without the syllogism).

Under Option 2, A5 reduces to A2. C3 becomes:

$$\exists v\alpha(v), Av[\alpha(v), \beta(v)], Av[\alpha(v), \neg\beta(v)] \vdash \#$$

This holds in virtue of existential instantiation and A1. C6 becomes:

$$\exists v(\alpha(v) \wedge \beta(v)), Av[\alpha(v), \neg\beta(v)] \vdash \#$$

which also holds in virtue of existential instantiation and A1. C8 becomes:

$$\flat \vdash Av[\alpha(v), \neg\beta(v)] \vee \exists v(\alpha(v) \wedge \beta(v))$$

At this point in the proceedings it is not clear what to make of this. It is clear that C8 reduces to C7, however.

But now C2 becomes:

$$\exists v\alpha(v), Av[\alpha(v), \neg\beta(v)] \vdash \exists v(\alpha(v) \wedge \neg\beta(v))$$

This holds by existential instantiation and A1.

Let us settle, therefore, for Option 2. $Nv[\alpha(v), \beta(v)]$ is $Av[\alpha(v), \neg\beta(v)]$.

5. ALL

To summarise proceedings so far. We have identified $Sv[\alpha(v), \beta(v)]$ and $Nv[\alpha(v), \beta(v)]$ with $\exists v(\alpha(v) \wedge \beta(v))$ and $Av[\alpha(v) \wedge \neg\beta(v)]$, respectively. Doing so satisfies A4, A5, C1–C6 (assuming A1 and A2), and C8 reduces to C7. This leaves A1–A3, B1–B2, and C7, which all concern A.

We will consider the possibility that $Av[\alpha(v), \beta(v)]$ is of the form:

$$\forall v(\alpha(v) \mapsto \beta(v))$$

where $\mapsto$ is some binary connective, and specifically some sort of conditional operator. This certainly might not be the case. It seems impossible, for example, to analyse "Most αs are βs" as of the form $Mv(\alpha(v) \bullet \beta(v))$, where Mv is the quantifier "Most vs are such that" and $\bullet$ is some binary connective. But the reduction of the restricted quantifier to the absolute quantifer plus a connective is the most straightforward strategy, if some version of it can be shown to succeed.[9]

Note that, given such a reduction, Contraction and Contraposition for A reduce to Contraction and Contraposition for $\mapsto$. If $\alpha \mapsto \beta \vdash \neg\beta \mapsto \neg\alpha$, then $\forall v(\alpha \mapsto \beta) \vdash \forall v(\neg\beta \mapsto \neg\alpha)$. And if $\alpha \mapsto (\alpha \mapsto \beta) \vdash \alpha \mapsto \beta$, then $\alpha \mapsto \forall v(\alpha \mapsto \beta) \vdash \alpha \mapsto (\alpha \mapsto \beta) \vdash \alpha \mapsto \beta$.[10] So $\forall v(\alpha \mapsto \forall v(\alpha \mapsto \beta)) \vdash \forall v(\alpha \mapsto \beta)$.

There are several natural possibilities for $\mapsto$. These include:

Option 1 $\supset$, the material conditional (where $\alpha \supset \beta$ is $\neg\alpha \vee \beta$)
Option 2 $\sqsupset$, where $\alpha \sqsupset \beta$ is $*\alpha \vee \beta$, and $*$ is Boolean negation[11]
Option 3 $\rightarrow$, a standard relevant conditional
Option 4 some other conditional[12]

We will consider these options in turn: Option 1 gives C7 in straightforward fashion. It gives A2 and A3 (assuming the Law of Excluded Middle for the latter). But as we observed in the Introduction, it does not give A1. Worse, it verifies Contraposition and Contraction, since $\supset$ does. So B1 and B2 fail. Option 1 is unattractive, and we will leave it here.

Option 2 gives A1, A2 and A3. C7 becomes:

$$\flat \vdash \forall v(*\alpha(v) \vee \beta(v)) \vee \exists v(\alpha(v) \vee \neg\beta(v))$$

If the second disjunct fails, then for every a, $\alpha(a)$ fails or $\neg\beta(a)$ fails. If $\alpha(a)$ fails then $*\alpha(a)$ holds, and if $\neg\beta(a)$ fails, $\beta(a)$, holds, since $\beta(a) \vee \neg\beta(a)$. Hence, $\forall v(*\alpha(v) \vee \beta(v))$. So C7 holds.

$\sqsupset$ does not satisfy Contraposition: $*\alpha \vee \beta \nvdash *\neg\beta \vee \neg\alpha$. For suppose that α is neither true nor false and β both true and false. Then $*\alpha$ holds, as then does $*\alpha \vee \beta$. But $\neg\beta$ holds, so $*\neg\beta$ fails, as does $*\neg\beta \vee \neg\alpha$. Hence, B1 holds.

Unfortunately, $\sqsupset$ satisfies Contraction: $*\alpha \vee (*\alpha \vee \beta) \vdash *\alpha \vee \beta$. Hence, B2 is not satisfied.

For Option 3, Contraction and Contraposition may hold or fail depending on the relevant $\rightarrow$ in question. There are relevant logics of both kinds. In the stronger relevant logics, like R, $\rightarrow$ contracts; in the weaker ones, like B, it does not. In relevant logics where the semantics of negation are given using the Routley $*$, $\rightarrow$ contraposes; where a four-valued semantics is used, $\rightarrow$ does not normally contrapose.[13] So B1 and B2 can be made to hold by appropriate choice of semantics.

A3 holds trivially, as does A1. But A2 fails, since $\forall v\beta(v) \nvdash \forall v(\alpha(v) \rightarrow \beta(v))$. C7 also fails, since this becomes:

$$\flat \vdash \forall v(\alpha(v) \rightarrow \beta(v)) \vee \exists v(\alpha(v) \wedge \neg\beta(v))$$

which is invalid in virtually all relevant logics. (The quantifier-free version: $\flat \vdash (\alpha \rightarrow \beta) \vee (\alpha \wedge \neg\beta)$ is invalid in the relevant logic R, for example.) From a semantic perspective, $\alpha(a) \rightarrow \beta(a)$ may fail because of what happens to a at 'worlds' other than the actual, thereby leaving the actual status of $\exists v(\alpha(v) \wedge \neg\beta(v))$ open.

6. Some Other Conditional

This leaves Option 4. Assume that the relevant logic has a ternary-relation semantics: interpretations are furnished with a ternary relation, R, which is employed to give the truth conditions of $\rightarrow$, thus:[14]

$$\alpha \rightarrow \beta \text{ is true at } x \text{ iff for all } y, z, \text{ such that } Rxyz,$$

$$\text{if } \alpha \text{ is true at } y, \beta \text{ is true at } z.$$

Assume, also, that interpretations are furnished with a binary relation on worlds, $\sqsubseteq$, such that:

$$\text{If } x \sqsubseteq y \text{ then anything true at } x \text{ is true at } y$$

This will be of great use when it comes to validating A2.[15]

For *relevant* conditionals, the inference $\beta \vdash \alpha \rightarrow \beta$ is going to fail. It holds in classical logic, but, as we have seen, classical conditionals are inappropriate for our task. The inference is also valid in *intuitionist* logic. An intuitionist conditional $\Rightarrow$ can be modelled by requiring $\alpha \Rightarrow \beta$ to be true at a 'world', x, just when for every y where $x \sqsubseteq y$, if α is true at y then β is true at y. This validates $\beta \vdash \alpha \Rightarrow \beta$ trivially. If β is true somewhere (say, x), then so is $\alpha \Rightarrow \beta$, since to evaluate this we only check worlds y where $x \sqsubseteq y$, and so β is true there, giving $\alpha \Rightarrow \beta$ at x. We will use this insight to construct a kind of conditional appropriate for our use in restricted quantification. Of course, we cannot use a conditional with the intuitionistic clause (that would validate Contraction), but we can connect our new conditional to the relevant conditional (the truth conditions of which are given by the ternary relation). It is pleasing that the solution to this second issue (connecting this conditional to the relevant conditional) brings with it the solution to the first issue (invalidating Contraction).

To model this new conditional we introduce a new ternary relation using the material at hand. $R'xyz$ obtains if and only if $Rxyz$ and $x \sqsubseteq z$. We may now give the truth conditions for $\mapsto$ as follows:

$$\overset{T}{\mapsto} : \alpha \mapsto \beta \text{ is true at } x \text{ iff for all } y, z, \text{ such that } R'xyz,$$

$$\text{if } \alpha \text{ is true at } y, \beta \text{ is true at } z$$

So, to evaluate $\mapsto$ at x, we look at the same sorts of pairs, y, z, of worlds related by the R involved in the relevant conditional $\rightarrow$, but we do not consider all of them: we consider only those where $x \sqsubseteq z$. This makes $\mapsto$ a kind of enthymematic conditional: we evaluate the consequent of the conditional only at those worlds where all the information at x is preserved.[16]

Assuming (as is required for *modus ponens* for $\rightarrow$) that $R@@@$ (where @ is the base world of the interpretation), we have *modus ponens* for $\mapsto$ (since $@ \sqsubseteq @$). This makes A1 true. A3 also holds, since $R'xyz$ entails $Rxyz$, and so, $\alpha \rightarrow \beta \vdash \alpha \mapsto \beta$. But now, A2 holds as well. For suppose that $\forall v\beta(v)$ is true at @. Then for any a, $\beta(a)$ is true at @. But if $R@yz$ and $@ \sqsubseteq z$, $\beta(a)$ is true at z, so $\alpha(a) \mapsto \beta(a)$ is true at @. Hence, $\forall v(\alpha(v) \mapsto \beta(v))$. In fact, we have a stronger version of A2: $\vdash \forall v\beta(v) \rightarrow \forall v(\alpha(v) \mapsto \beta(v))$, since for *any* worlds x, y and z, if $R'xyz$ then $Rxyz$.

As in standard relevant logics, $\mapsto$ will not satisfy Contraction and Contraposition, at least without further constraints on R'. The standard condition for Contraction is that $R'xxx$. If the condition for Contraction fails for R (which must be the case for the applications we have in mind), so that for some x it is not the case that $Rxxx$, then it *must* fail also for R'. The failure of Contraction is not only *allowed* on this position; it is *mandated* by the failure of Contraction for the relevant conditional.

If negation is handled by the Routley *, the condition that verifies (the rule form of) Contraposition is: if $R@yz$ then $R@z^*y^*$. In fact, this is a condition that is satisfied automatically in the standard semantics.[17] However, even if Contraposition holds for the relevant conditional governed by R, it does not follow that it holds for R'. For if $R'@yz$, then we have $R@yz$ (and so $R@z^*y^*$) and $@ \sqsubseteq z$. To conclude that $R'@z^*y^*$ we would need $@ \sqsubseteq y^*$. But we have *nothing* that would lead us to this conclusion. The introduction of the $\sqsubseteq$ clause blocks the inference from Contraposition for $\rightarrow$ to Contraposition for $\mapsto$. If negation is handled in a four-valued fashion[18] then Contraposition fails even for $\rightarrow$, and the additional clause concerning $\sqsubseteq$ does nothing to help matters.[19] Thus, however negation is handled, we cannot only arrange for conditions B1 and B2 to be satisfied, they come for free, given the definition of $\mapsto$.

This leaves only C7, which becomes:

$$\flat \vdash \forall v(\alpha(v) \mapsto \beta(v)) \vee \exists v(\alpha(v) \wedge \neg\beta(v))$$

For all we have said so far, there is no reason to suppose this to hold; and maybe it should just be given up. Of all the constraints, it appears one of the softest. But we can arrange for this to obtain too if we wish.

If we take the semantics of the relevant logic to be a four-valued one, we have to give $\mapsto$ not only truth conditions but falsity conditions as well. Given the truth conditions, perhaps the natural falsity conditions are:

$$\alpha \mapsto \beta \text{ is false at } x \text{ iff for some } y, z, \text{ such that } Rxyz,$$

$$\alpha \text{ is true at } y \text{ and } \beta \text{ is false at } z$$

However, suppose we tweak these a little, ignoring the world-shift, to obtain:

$$\overset{F}{\mapsto} : \alpha \mapsto \beta \text{ is false at } x \text{ iff } \alpha \text{ is true at } x \text{ and } \beta \text{ is false at } x$$

Now C7 holds. For we have $\forall v(\alpha(v) \mapsto \beta(v)) \vee \neg\forall v(\alpha(v) \mapsto \beta(v))$. In the second case, we have $\exists v \neg(\alpha(v) \mapsto \beta(v))$, so $\exists v(\alpha(v) \wedge \neg\beta(v))$.

One should note that, given these falsity conditions for $\mapsto$, there is no reason to suppose that the Law of Excluded Middle will hold for $\mapsto$-formulas: $(\alpha \mapsto \beta) \vee \neg(\alpha \mapsto \beta)$ (LEM$\mapsto$). If $\alpha \mapsto \beta$ fails to be false at world x, this tells us nothing about how α and β behave at other worlds.[20]

7. Conclusion

To summarise: we defined a connective, $\mapsto$, with truth and falsity conditions given by $\overset{T}{\mapsto}$ and $\overset{F}{\mapsto}$. Restricted quantifiers were defined as:

- All αs are βs: $\forall v(\alpha(v) \mapsto \beta(v))$
- No αs are βs: $\forall v(\alpha(v) \mapsto \neg\beta(v))$
- Some αs are βs: $\exists v(\alpha(v) \wedge \beta(v))$

Given that the relevant conditional underlying $\mapsto$ has appropriate properties, these definitions satisfy all the constraints mooted: A1–A5, B1–B2, C1–C8. Hence we have a plausible account of relevant restricted quantification.

Notes

[1] See, e.g., Priest (1987), ch. 6, and Beall (2004).

[2] In the parlance of one of the authors, we wish the resulting logic of restricted quantification to be *robustly contraction-free.* (See Restall (1993).)

[3] It should be noted that this account works just as well for relevant logics with Contraction, though these are not suitable for the applications we have in mind.

[4] Note that A2 may send a jarring note down the spine of any relevant logician. Yet it certainly seems intuitively valid. (Part of the problem is how to reconcile this fact with the failure of $\alpha \rightarrow (\beta \rightarrow \alpha)$ in relevant logic.) Consider, for comparison, the principle dual to A2, $\neg\exists v\alpha(v) \vdash Av[\alpha(v), \beta(v)]$. This does not have the same intuitive appeal. If everything is a β, then any α is a β (even if α is totally irrelevant to β). It is a very different thing to infer from the absence of any α that any α is a β. So, we do not take this dual principle as important to preserve in an account of quantification. If it 'falls out' as valid then well and good. If it is invalidated, that seems to be no cost.

[5] It is worth noting that there are other reasons why one might suppose contraposition to fail for restricted universal quantification. Thus, Sylvan and Nola (1991) argue for this as a solution to the Ravens Paradox.

[6] The argument is given for truth-theory. A similar argument could be given for set-theory. See Beall (2004) and references therein.

[7] In Brady (2003), pp. 321–31, the notion of restricted quantification is taken as syntactically primitive, and an axiomatisation is given. The axiomatization was produced with a different desideratum in mind, namely that whatever is a theorem for unrestricted quantification is also a theorem for the corresponding restricted quantification, and *vice versa*. The axiomatisation can be tweaked to deliver relativised quantifiers that satisfy our desiderata provided that one may help oneself to existential assumptions of the form $\exists v\alpha(v)$ freely.

[8] See, e.g., Restall (2000), pp. 28–30.

[9] An analysis of restricted quantification which is not of this kind is the "conditional assertion" account of Belnap (1973). We will not discuss this here, except to note that both A5 and B2 fail in Belnap's account. Belnap's account is taken up and developed in Cohen (1992), with the same consequences.

[10] We assume here that if $\vdash \alpha \mapsto \gamma$ and γ entails δ then $\vdash \alpha \mapsto \delta$.

[11] By which we mean a connective such that $*\alpha$ holds iff α fails. For an appropriate semantics, see Meyer and Routley (1973). Option 2 can be looked at another way. To say that $\forall x(*\alpha(x) \vee \beta(x))$ is true is to say that for every a such that $\alpha(a)$ holds $\beta(a)$ holds. Thus, it is to say that the set of things of which α is true is a subset of the things of which β is true, and so, effectively, to define restricted quantification in terms of subsethood.

[12] We will consider only one in what follows. It is not the only possibility. Another is the conditional of the logic given in Cooper (1968). This is closely related to the conditional employed in Belnap (1973). (See fn. 9.) It fares quite well according to our criteria. But it fails B2, and Cooper's logic is not closed under uniform substitution.

[13] See Priest (2001), chs. 9, 10.

[14] See, e.g., Priest (2001), ch. 10.

[15] The relation $\sqsubseteq$ is standard in the semantics of a number of relevant logics. See, e.g., Restall (2000), ch. 11. If it is not already present, it can always be added.

[16] A simlar effect can be obtained by employing the constant t, thought of as the conjunction of all (actual) truths, and defining $\alpha \mapsto \beta$ as $(\alpha \wedge t) \rightarrow \beta$. Given that $\vdash t$, A1 holds; given that $\beta \vdash t \rightarrow \beta$, A2 holds; A3 is straightforward. $\mapsto$ contracts only if $\rightarrow$ does. Contraposition fails, even if it holds for $\rightarrow$. From $(\alpha \wedge t) \rightarrow \beta$, we can infer $\neg\beta \rightarrow (\neg\alpha \vee \neg t)$, and hence $\neg\beta \mapsto (\neg\alpha \vee \neg t)$. But even given t, we cannot infer $\neg\beta \mapsto \neg\alpha$.

[17] Thus, in the simplified semantics, it drops out of the condition that $R@yz$ iff $y = z$.

[18] As in Priest (2001), ch. 9.

[19] The easiest way to get Contraposition for $\rightarrow$ in a four-valued context is to modify the truth conditions, building in not only truth-preservation forwards, but also falsity-preservation backwards. Contraposition *can* be obtained without modifying the truth conditions, but not simply by imposing extra constraints on R. (See Routley (1984) and Restall (1995).)

[20] Indeed, the desiderata actually rule out LEM$\mapsto$. For consider the instance of the T-schema involved in the Curry paradox: $T\langle\gamma\rangle \leftrightarrow (T\langle\gamma\rangle \mapsto \bot)$. Suppose that we have $(T\langle\gamma\rangle \mapsto \bot) \vee \neg(T\langle\gamma\rangle \mapsto \bot)$. Given the first disjunct, an application of *modus ponens* gives us $T\langle\gamma\rangle$, and a couple more give us $\bot$. Given the second disjunct, the falsity conditions for $\mapsto$ give us $T\langle\gamma\rangle$, and the result is the same.

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JC BEALL
Department of Philosophy,
University of Connecticut,
Storrs, CT, USA
e-mail: jc.beall@uconn.edu

ROSS T. BRADY
Department of Philosophy,
La Trobe University,
Melbourne, Australia

A. P. HAZEN, GRAHAM PRIEST AND GREG RESTALL
Department of Philosophy,
University of Melbourne,
Melbourne, Australia