Interpretations of the Third Value

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Abstract

Many-valued logic raises many interesting questions, both philosophical and technical. This essay concerns just one of them: In a three-valued logic, how should one interpret the third value? To frame the essay, I will start with the definition of a many-valued logic. I will then make some preliminary remarks on matters, which will focus us on 3-valued logics. The major interpretations of the third value fall into roughly two groups: those where it is "infectious", and those where it is not. The main part of this essay will then divide into two major sections, corresponding to that division.

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1 Introduction

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The major interpretations of the third value fall into roughly two groups: those where it is "infectious", and those where it is not. (I will explain what this means in due course.) The main part of this essay will then divide into two major sections, corresponding to that division.[1]

2 Preliminary Matters

2.1 Many-Valued Logic

So let us start by fixing some terminology and notation.

We will be concerned with propositional logics. All the logics we will meet can be extended to first (and higher) order logics in the familiar way.

[1]For comments on earlier drafts of this paper, many thanks go to Mel Fitting, Paul Égré, and David Over.

But such extensions have little relevance to the topic of this essay, so need not concern us here.

A many-valued (propositional) logic is a validity relation defined on a formal language, with an infinite set of propositional parameters, $\mathcal{P}$, and a set of propositional connectives, $\mathcal{C}$. If $c \in \mathcal{C}$ is an m-place connective, and $A_1, ..., A_m$ are formulas, then we may write the formula obtained from these using c as $c(A_1, ..., A_m)$ (though I will often use the more familiar infix notation when this is standard).

A semantic characterisation of the validity relation is given by a structure $\langle \mathcal{V}, \mathcal{D}, \{f_c : c \in \mathcal{C}\}, \nu \rangle$.[2] In the case of finitely many-valued logics, it is always possible to provide a proof-theoretic characterisation as well; but since interpretation is a matter of semantics, such matters will play little role in what follows. (In an appendix to this paper, I will, however, give natural deduction systems for the four main 3-valued logics we will meet: K_3, P_3, B_3, and H_3.)

$\mathcal{V}$ is the set of values that formulas of the language may take. $\mathcal{D} \subseteq \mathcal{V}$ is the set of *designated* values. If $c \in \mathcal{C}$ is an m-place connective, then f_c is a function from $\mathcal{V}^m$ to $\mathcal{V}$. ν is a map from $\mathcal{P}$ to $\mathcal{V}$. This is extended to a map from all formulas to $\mathcal{V}$ by recursion. Specifically, if c is an m-place connective, then:

- $\nu(c(A_1, ..., A_m)) = f_c(\nu(A_1), ..., \nu(A_m))$

It is this clause which makes the semantics truth-functional, in a general sense.

For our purposes, we need deal only with inferences with a single conclusion. So if Σ is any set of formulas, validity, $\models$, is defined in terms of preservation of designated values. That is:

- $\Sigma \models A$ iff for all ν, if $\nu(B) \in \mathcal{D}$, for all $B \in \Sigma$, then $\nu(A) \in \mathcal{D}$

2.2 Zeroing in on Our Topic

Let us now focus on our topic. First, a many-valued logic is called 'm-valued' if the cardinality of $\mathcal{V}$ is m. From now on, unless otherwise mentioned, we will take this m to be 3.

Next, the very title of this essay (given to me by the editors), suggests that we know what the interpretations of "the first two" values are. The

[2]See Priest (2008), 7.2.

natural assumption is that these are *true* (*and only true*), and *false* (*and only false*). Let me write these as 1 and 0. I will write the third value as $\mathfrak{X}$—for here lies our unknown.[3]

It is also natural to read into the title the assumption that we know how the "first two" values behave, and that this is "classically". So let c be any standard m-place classical connective in our language, and let g_c be the "classical" truth function corresponding to it. Then we will call a 3-valued logic *regular* if it satisfies the following condition. If $v_1, ..., v_m$ are 1 or 0 then:

- $f_c(v_1, ..., v_m) = g_c(v_1, ..., v_m)$

We will be concerned only with regular many-valued logics. This constraint is not toothless, since there are 3-valued logics that do not satisfy this condition. Thus, one of the oldest many-valued logics was proposed by Post.[4] In these, $\mathcal{V} = \{0, ..., m+1\}$, and negation is the cyclical function. That is:

- if $v \leq m$ then $f_\neg(v) = v + 1$
- $f_\neg(m+1) = 0$

Unless $m = 1$, so that we have a 2-valued logic, this logic is clearly not regular. Post's logics are perfectly good mathematical structures, but the behavior of negation makes it hard to give meaning of philosophical significance to its values.[5]

Next, let us turn to the connectives which will concern us. We can start with the familiar suspects: $\neg, \vee, \wedge, \rightarrow$. (Though, of course, depending on the meaning of $\mathfrak{X}$, certain other connectives may have sensible meanings.) We may assume, as usual, that $A \leftrightarrow B$ is equivalent to $(A \rightarrow B) \wedge (B \rightarrow A)$.

Much of the technical interest in 3-valued logic concerns $\rightarrow$. However, I will largely ignore the connective here. The reason is as follows. $\rightarrow$ is usually taken to represent the conditional, *if*, but no many-valued semantics does justice to the notion. One way to see this is as follows.[6] Suppose our logic satisfies the conditions:

[1] $\models A \leftrightarrow A$

[3] The various authors I will mention in what follows may have different notations. I will translate their notations into mine without comment.

[4] Post (1921).

[5] Post himself suggest an interpretation in which each sentence represents an $m+1$-tuple of sentences of the usual kind. See also Rescher (1969), p. 54 f.

[6] See Priest (2008), 7.5.4–7.5.6.

[2] $A \models A \vee B$

Then if the logic has m values, and D is $\bigvee_{1 \leq i \neq j \leq m+1} (p_i \leftrightarrow p_j)$, then $\models D$. Why? Since there is an infinite number of propositional parameters, there must be distinct p_i and p_j such that $\nu(p_i) = \nu(p_j)$. So by [1], $\nu(p_i \leftrightarrow p_j) \in \mathcal{D}$. (Since $\models A \leftrightarrow A$, then for any value v, $f_{\leftrightarrow}(v, v) \in \mathcal{D}$. So if $v(p_i) = \nu(p_j)$, $f_{\leftrightarrow}(\nu(p_i), \nu(p_j)) \in \mathcal{D}$.) The result follows by [2]. Now, let p_k be 'A molecule of water has k atoms'. Then intuitively, for distinct i and j, 'p_i iff p_j' is not true. So a disjunction of such things should not be true, let alone logically true.

There are, in fact, many-valued logics in which [1] and [2] fail. For example, as we shall see, [1] fails in K_3 and [2] fails in B_3. However, as many have noted, giving up [1] (and the things that tend to go with it) 'cripples ordinary reasoning' employing conditionals.[7] And [2] is arguably beside the point. Given [1], $f_{\leftrightarrow}(0, 0) \in \mathcal{D}$, and so if p_i and p_j are false only, $p_i \leftrightarrow p_j$ is truthlike. Now, if p_k is as before, and i and j are distinct numbers different from 3, p_i and p_j would clearly appear to be false only; yet 'p_i iff p_j' is hardly truthlike.[8]

Conditionals are essentially intensional constructions, where truth values depend on something more than the denotations of their components. What this means is that any many-valued semantics for $\rightarrow$, though it be technically useful, cannot reflect facts about meanings—and so tell us anything about the meanings of the values.

For similar reasons, other intensional operators, such as modal operators, are not on the list of connectives with which we will be concerned. It is well known that no standard modal logic has a finitely many-valued semantics, as proved by Dugundji (1940).[9] (Indeed, the proof I have just given is a simple variation on his argument.)

In the present context, this fact is not toothless either. Indeed, the earliest many-valued logic was given by Łukasiewicz in his paper in Polish, 'Philosophical Remarks on Many-Valued Systems of Propositional Logic', of 1920.[10] This was a regular 3-valued logic, $Ł_3$. Motivated by Aristotle's argument about future contingents in ch. 9 of *De Interpretatione*, he interpreted

[7] Field (2008), p. 73.

[8] See, further, Santorio and Wellwood (2023).

[9] Based on a technique used by Gödel (1933) to prove the same result about intuitionist logic.

[10] Translated into English in Borkowski (1970), pp. 153–178.

$\mathfrak{X}$ as 'is possible' (though he does gloss this at one point as *neither true nor false*[11]). Things that are true are possible, of course, and this motivates the truth function for $\Diamond$:

- $f_\Diamond(v) = 1$ if $v = 1$ or $v = \mathfrak{X}$
- $f_\Diamond(v) = 0$ if $v = 0$

The rub, of course, is that things that are false can be possible too. So this interpretation gives strange results. For example, in Ł$_3$, $\mathcal{D} = \{1\}$ and if $f_\wedge(v_1, v_2) = 0$ then $v_1 = 0$ or $v_2 = 0$. It follows, as is easy to check, that, $\Diamond A, \Diamond B \models \Diamond(A \wedge B)$. This is clearly not right. For both A and $\neg A$ may be possible, whilst $A \wedge \neg A$ is not. Moreover, this inference would be valid even if one took $\mathcal{D} = \{1, \mathfrak{X}\}$. These things are, again, little more than the result of inappropriately forcing an infinite variety into a finite set of pigeon holes, which is at the basis of Dugunji's theorem.

Hence, to explore plausible interpretations of $\mathfrak{X}$ in what follows, we focus on our old friends, $\neg$, $\vee$, and $\wedge$.

3 Non-Infectious $\mathfrak{X}$s

Let us now turn to our subject proper. Given that we have three values, and two of them are already pinned down, there are 3 ways of completing the specification of values for $\neg$, and 5^3 ways of doing it for each of $\wedge$ and $\vee$. This gives us $3 \times 5^3 \times 5^3$ possible combination of functions for out three connectives.[12] The reader will be pleased to know that we will not consider all of these here... Fortunately, there are two major philosophical strategies for handling $\mathfrak{X}$, so we may consider just these.

3.1 First Degree Entailment

The first strategy is to make the natural generalisation of the classical truth/falsity conditions for the connectives. To see how, let us make a detour through a well known 4-valued logic.[13]

[11]Borkowski (1970), p. 153.

[12]In the logics we will consider, De Morgan's laws and Double Negation hold. If one insists on these, conjunction may be defined in terms of disjunction (or vice verse), so the number drops to 3×5^3.

[13]On the following, see Priest (2008), ch. 8.

The natural truth/falsity conditions for our connectives in classical logic are as follows. Let us write t for truth (in an interpretation) and f for falsity (in an interpretation) then:

- $\neg A$ is t iff A is f
- $\neg A$ is f iff A is t
- $A \wedge B$ is t iff A is t and B is t
- $A \wedge B$ is f iff A is f or B is f
- $A \vee B$ is t iff A is t of B is t
- $A \wedge B$ is f iff A is f and B is f

Of course, if one assumes that truth and falsity are exclusive and exhaustive, as is the case in classical logic, the second of each pair is redundant. But if one drops this assumption, this is no longer the case. And that is exactly what we are going to do now.

For any formula, we then have four possibilities: that it is *true and only true*, *false and only false*, *both*, and *neither*. So the possible sets of values are $\{t\}$, $\{f\}$, $\{t, f\}$, and $\emptyset$. Let us write these as 1, 0, b, and n, respectively (Recall that 1 and 0 meant *true only* and *false only*.) We may then have a 4-valued logic, where $\mathcal{V} = \{1, 0, b, n\}$.

An easy way to specify the values of $f_\neg$, $f_\wedge$, and $f_\vee$ can be obtained if we represent our four values in a Hasse diagram, the Diamond Lattice:

$$\begin{array}{ccccc} & & 1 & & \\ & \nearrow & & \nwarrow & \\ b & & & & n \\ & \nwarrow & & \nearrow & \\ & & 0 & & \end{array}$$

Let us write the lattice-value of A as $|A|$. Then applying the truth conditions above, it is simple to show that $|A \wedge B|$ is the greatest lower bound of $|A|$ and $|B|$. For example, suppose that $|A| = 1$ and $|B| = b$. Then both A and B are true, so $A \wedge B$ is true. But B is false, so $A \wedge B$ is false. That is, $|A \wedge B| = b$. Or suppose that $|A| = b$ and $|B| = n$. Then it is not the case that both A and B are true, so $A \wedge B$ is not true. But A is false, so $A \wedge B$ is false. That is, $|A \wedge B| = 0$. The other cases are left as exercises. Similarly, one can check that $|A \vee B|$ is the least upper bound of $|A|$ and $|A \wedge B|$.

For negation: if $|A| = 1$ then $|\neg A| = 0$. (If A is true but not false; then $\neg A$ is false but not true.) Similarly if $|A| = 0$ then $|\neg A| = 1$. But if $|A| = b$ then $|\neg A| = b$. (If A is true and false, then $\neg A$ is false and true, which is the same thing—order does not matter.) Similarly, if $|A| = n$, $|\neg A| = n$. In other words, $\neg$ toggles between 1 and 0, and b and n are fixed points.

Finally, validity is defined as in classical logic, as truth preservation. In other words, an inference is valid if whenever all the premises are 1 or b, so is the conclusion. (Recall that truth can now come in two flavours: *true and not false*, and *true and false*.) In other words, $\mathcal{D} = \{1, b\}$.

This 4-valued logic is usually called First Degree Entailment (FDE).[14]

3.2 K_3 and P_3

Given FDE it is easy to see that if no propositional parameter has the value b then no formula does; and if no propositional parameter has the value n then no formula does. Hence the left and right hand sides of the lattice give perfectly well defined 3-valued logics. Though b and n have intuitively different meanings, the lattices for the two cases are isomorphic. We may diagram this in the following tables:

$\neg$	
1	0
$\mathfrak{X}$	$\mathfrak{X}$
0	1

$\wedge$	1	$\mathfrak{X}$	0
1	1	$\mathfrak{X}$	0
$\mathfrak{X}$	$\mathfrak{X}$	$\mathfrak{X}$	0
0	0	0	0

$\vee$	1	$\mathfrak{X}$	0
1	1	1	1
$\mathfrak{X}$	1	$\mathfrak{X}$	$\mathfrak{X}$
0	1	$\mathfrak{X}$	0

Here, $\mathfrak{X}$ could be either b or n. The only formal difference between the logics is in the designated values. b is designated and n is not. So if $\mathfrak{X} = n$, $\mathcal{D} = \{1\}$; and if $\mathfrak{X} = b$, $\mathcal{D} = \{1, b\}$. In the first case we have Strong Kleene logic, K_3. In the second case, we have the Logic of Paradox, LP. For reasons of uniformity, I will write this as P_3.[15] Note that in $Ł_3$ the tables for these

[14] FDE was invented/discovered by Anderson and Belnap. (See Anderson and Belnap (1975), ch. 3.) The 4-valued semantics is due to Dunn. (See esp. §18.)

[15] On the two logics, see Priest (2008), ch. 7. For K_3, see Kleene (1938), with a much fuller discussion in Kleene (1952), §64. For P_3, see Priest (1979). The tables appear, without mention of designated values in Asenjo (1966). The tables also appear in Asenjo and Tamburino, where $\mathfrak{X}$ is designated.

connectives and the designated values are exactly the same as those for K_3. So the $\neg$, $\wedge$, $\vee$ fragment of Ł$_3$ is the same as that of K_3.[16]

3.3 Varieties of Undesignated $\mathfrak{X}$

Let us consider the case where $\mathfrak{X} = n$. There are, in fact, many reasons why one might suppose that certain sentences are neither true nor false. This is not the place to enter into a philosophical discussion of how plausible such claims are, but let us note these reasons. (I make no claim that this list is exhaustive.)

Perhaps the oldest claim that there are such statements is due to Aristotle, who, in the somewhat notorious ch. 9 of *De Interpretatione*, argued that there are certain ("contingent") statements about the future whose truth values are not yet determined, and so which are (as yet) neither true nor false—such as 'there will be a sea battle tomorrow'.[17] Aristotle did not, of course, have the modern notion of a many-valued logic; but as we noted, Łukasiewicz, who was motivated by Aristotle, sometimes calls his third value *neither true nor false*.

Kleene took sentences whose value is classically undefined as neither true nor false. Thus, a partial recursive function, f, may be defined by a computation which, for certain inputs, i, does not terminate. Any statement of the form $f(i) = j$ is then such a sentence. Actually, Kleene insists that $\mathfrak{X}$ is not on a par with 1 and 0: it is not a value, but the *absenc*e of a value. But if one recalls that n represents $\emptyset$, and so can be thought of as having a value in the empty set, it is hard to see real philosophical substance in Kleene's insistence.[18]

[16]The tables for $\rightarrow$ are different, however. The table for K_3 is the first of those below. The table for Ł$_3$ is the second:

$\rightarrow$	1	$\mathfrak{X}$	0
1	1	$\mathfrak{X}$	0
$\mathfrak{X}$	1	$\mathfrak{X}$	$\mathfrak{X}$
0	1	1	1

$\rightarrow$	1	$\mathfrak{X}$	0
1	1	$\mathfrak{X}$	0
$\mathfrak{X}$	1	1	$\mathfrak{X}$
0	1	1	1

$\rightarrow$	1	$\mathfrak{X}$	0
1	1	$\mathfrak{X}$	0
$\mathfrak{X}$	1	$\mathfrak{X}$	0
0	1	1	1

The table for $\rightarrow$ in P_3 is the same as that for K_3. The table for Ansenjo and Tamburino (1975) is the third.

[17]See, further, Priest (2008), 7.9 and 11a.7.

[18]Recall also that there is no formal difference between a function being undefined for some input and its having a "dummy value", say $*$—or n—for those inputs.

Another reason for supposing that there are valueless sentences was given by Frege. In 'Sense and Reference' he argued that if a sentence contains a non-denoting name, it is neither true nor false.[19] Of course, Frege did not have the notion of a modern many-valued logic either, but Kleene's approach could be seen as implementing the claim, since it is natural to suppose that if $f(i)$ is undefined, '$f(i)$' fails to refer. Frege's rationale, however, delivers, not K_3, but B_3, as we will see in due course.

Reference failure might be thought of as some kind of presupposition failure. There are other kinds. For example, 'I have stopped beating my dog', might well be taken to presuppose that I used to beat it, and so to fail in some way, if I did not. Some, such as Strawson, have held that cases of presupposition failure are neither true nor false.[20] However, perhaps presupposition-failure motivates B_3 more naturally than K_3, as well.

Next, there is a long history of taking paradoxical sentences of semantic self-reference, such as the liar sentence, to be neither true nor false. However, perhaps the now most celebrated advocate of this view was Kripke.[21] Kripke also insisted that n should be regarded as *lacking* a value, since the evaluation of paradoxical sentences he envisaged never results in 1 or 0.

Another kind of paradox where the notion of being neither true nor false has a possible application is the sorites paradox. In this, a vague predicate delivers borderline cases. Thus, a person, p, who is on the cusp of puberty seems to be as much adult as child, and as little adult as child. It is a natural thought that 'p is a child (or adult)' is neither true nor false. Some have endorsed K_3 as an appropriate logic for such borderline cases.[22] The sorites argument then breaks down. However, a 3-valued approach of this kind is often coupled with some kind of supervaluation technique,[23] or generalised to a continuum-valued logic in an attempt to handle "higher order vagueness".[24]

Finally, if one identifies truth with verification, then, since there may well be statements, A, such that neither A nor $\neg A$ is verified, such an A

[19]See Geach and Black (1970), pp. 56–98.

[20]Strawson (1952), pp. 174–179. See also Beaver, Guerts, and Denlinger (2021).

[21]Kripke (1975).

[22]E.g., Tye (1994).

[23]Given a K_3 evaluation, a *resolution* is a classical evaluation which is the same, except that any value n is replaced by either t or f. A formula is then true/false on a *supervalulu-ation* if it is true/false on all resolutions (and neither otherwise). Validity is then defined in terms, not of evaluations, but of supervaluations. Such a definition preserves classical (single-conclusion) validty. See Priest (2008), 7.10.3–7.10.5a.

[24]See Priest (2008), ch. 11, and Keefe and Smith (1996b).

will be neither true nor false. Mathematical intuitionists identify truth with verifiability. However, intuitionist logic is not a finitely many-valued logic.[25] Reichenbach suggested using Ł$_3$ (augmented by some further connectives) as a logic for quantum mechanics. $\mathfrak{X}$ is the value of unverifiable statements concerning quantum states.[26] He interprets $\mathfrak{X}$ as a certain kind of meaninglessness. As one might then expect, $A \vee \neg A$ is not a logical truth. However, in K_3/Ł$_3$ $\neg(A \vee \neg A)$ is not a logical truth either. This is more problematic, since quantum theory does not seem to be in the business of denying the law of non-contradiction.[27] More problematic is that what the observation of quantum states seems to require is the failure of distribution: $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$. (It would appear that a particle can hit a screen having gone through one or other of two slits, but not having gone through either of them.) But K_3/Ł$_3$ verifies distribution. More recent quantum logics reject this principle, and are not finitely many-valued.[28]

3.4 Probability and Conditional Assertion

A somewhat different interpretation of K_3 was proposed by de Finetti in his '*La Logique de la Probabilité*' (1935).[29] 1 is interpreted as *having subjective probability 1*; 0 is interpreted as *having subjective probability 0*; and $\mathfrak{X}$ is interpreted as *uncertain*.[30] De Finetti tends to identify having probabilty 1 with truth (and dually for 0). That is, of course, dubious, even for a highly idealised agent. Many things that are true and do not have probability 1. (E.g., either 'there is intelligent life in our galaxy which is not on Earth' or its negation.) But even passing this over, we hit an obvious problem. If A has value $\mathfrak{X}$, so does $\neg A$, and so does $A \vee \neg A$, which it should not have on this understanding. Probabilities just do not work in this way. Indeed, probability is not a truth function of any kind. If a is the value of a die rolled at random, then 'a is even' and 'a is ≤ 3' both have probability 1/2, and, we

[25] As proved by Gödel (1933). See Priest (2008), 7.11.

[26] Reichenbach (1944), §§30, 32. See also Rescher (1969), p. 341.

[27] The situation is complicated by the fact that Reichenbach has three negations, only one of which is $\neg$; and for one of others ('complete negation'), $\overline{A}$, one does have $\overline{A \wedge \overline{A}}$ (though why this is some kind of negation is unclear, since if $|A| = 1$, $|\overline{A}| = \mathfrak{X}$).

[28] See, e.g., Mittelstaedt (1978).

[29] English translation, Angell (1995). De Finetti does not say which values are to be understood as designated, but it is pretty clear that it is just 1.

[30] Sometimes he also speaks of these as *known to be true*, *known to be false*, and *doubtful*.

may suppose, are both true. But 'a is 1 and a is even' has probably 0, while 'a is 1 and $a \leq 3$' has probability 1/6.

De Finetti augments the usual connectives with a new conditional. Let us write this as $A \rightarrowtail B$. He gives it the following truth table:[31]

$\rightarrowtail$	1	$\mathfrak{X}$	0
1	1	$\mathfrak{X}$	0
$\mathfrak{X}$	$\mathfrak{X}$	$\mathfrak{X}$	$\mathfrak{X}$
0	$\mathfrak{X}$	$\mathfrak{X}$	$\mathfrak{X}$

The idea is that if the antecedent is true the conditional states what the consequent states. If not, it is 'null' as he puts it, or 'defective' as it is sometimes put in the literature.[32] That is, "all bets are off".[33] Following some thoughts of Jeffrey (1963), and apparently with no knowledge of de Finetti, the idea is worked out at greater length by Belnap (1973), though (with reservations) he prefers the table:

$\rightarrowtail$	1	$\mathfrak{X}$	0
1	1	$\mathfrak{X}$	0
$\mathfrak{X}$	1	$\mathfrak{X}$	0
0	$\mathfrak{X}$	$\mathfrak{X}$	$\mathfrak{X}$

Belnap calls $\mathfrak{X}$ 'non-assertive', a species of being neither true nor false. A different articulation of the same idea is to be found in Cooper (1968), where he also terms the value $\mathfrak{X}$ *neither true nor false*.

As one would expect from the discussion of conditionality and many-valuedness in 2.2, as matrices for a conditional, these deliver implausible properties. For example, if 1 is the only designated value, the de Finetti matrix verifies:

- $A \rightarrowtail B \vdash B \rightarrowtail A$ [if I shoot you, you will die; so if you die I will shoot (shot?) you]

And both matrices verify:

[31] He appears to identify this with conditional probability though, since probability is not a truth function, the matrices do not capture the way that conditional probability works. (Thus, if $Pr(A) = 0.5$, $Pr(A/A) = 1$.) There are, of course well-known connections between conditionals and conditional probability. See Edgington (2020) and, in the present context, Cruz and Oliver (202+).

[32] E.g., Over and Baratgin (2017).

[33] Indeed, he motivates the table in terms of conditional bets.

- $A \rightarrowtail B \vdash B$ [if I shoot you, you will die; so you will die]
- $\neg(A \rightarrowtail B) \vdash \neg B$ [it is not the case that if I shoot you Donald Trump will die; so Donald Trump will not die

whilst neither matrix verifies:

- $\vdash A \rightarrowtail A \vee B$ [if it's red, it's either red or blue]

Of course, one can wring the changes by defining validity in other ways, but all have problematic consequences.[34]

3.5 Varieties of Designated $\mathfrak{X}$

Let us now turn to the same matter where $\mathfrak{X}$ is b—that is, where one is thinking of $\mathfrak{X}$ as the value of sentences that are both true and false (dialetheias). Some possibilities here are as follows. (Again, I make no claim to the exhaustiveness of the list, or the plausibility of its members.)

It's probably fair to say that the candidate that has received the most air-time over the last 40 years concerns the paradoxes of self-reference. In these,[35] we have an apparently cogent argument—concerning notions that are semantic, set theoretic, intensional, and maybe of some other kinds—which ends in a contradiction, $A \wedge \neg A$.[36] The orthodox view is that there is something wrong with the arguments; a different view is that there is nothing wrong with them: they just establish that something is both true and false. This is a view that was mooted by Priest and Routley,[37] and was behind the construction of P_3.

Further candidates for dialetheias are contradictions related to other familiar paradoxes. Sorites paradoxes are one such kind. Since statements in the borderline area of sorites progressions seem to be symmetrically poised between truth and falsity, one may take them to be, not *neither* true nor false, but *both* true and false.[38] The sorites arguments then break down, but for different reasons.

Another relevant kind of paradox comprises the paradoxes of motion—and more generally, change.[39] For example, Zeno's arrow paradox notes that

[34] See Égré, Rossi, and Sprenger (2021). See, further, Sprenger (202+).
[35] Or at least most of them: Curry's paradox is not of this kind.
[36] See Bolander (2017).
[37] Priest (1979) and Routley (1977).
[38] See, e.g., Hyde (1997) and Priest (2019a).
[39] See Priest (1987), chs. 11, 12.

at any instant of its journey, since it *is* an instant, an arrow makes zero progress. It then concludes that the arrow can make no progress at all, since the progress made on the journey is the sum of the progresses made at each instant. One solution to the paradox is to hold that the arrow *does* make progress at an instant. Just *because* it is in motion, it has already gone a little bit further than where it is. So it is both there and not there.[40]

Another example of a dialetheia that has been offered—which one might or might not think of as paradoxical—arises in Christian thought. It is standard theology to hold that Christ was both fully divine and fully human, where these are contradictory predicates. Orthodox Christian thinking tries to defuse this contradiction in some way or other, but a more recent approach is simply to claim that Christ was dialetheic in this regard.[41] Of course, an example of this kind is hostage to certain religious views; but Christianity is not the only religion that produces apparently paradoxical claims.

Candidates for dialetheias that have nothing much to do with paradox concern laws and other systems of norms. Thus, for example, one might have laws of the form:

- Anyone in category X may do so and so
- Anyone in category Y may not do so and so

Things are perfectly consistent as long as there is no one in both categories. But if at some stage there is a person in categories X and Y, that person both may and may not do so and so—at least until the law is changed.[42]

A number of the examples above concern boundaries in one way or another. Now, boundaries are contradictory entities almost by definition, since they both join and separate their two sides. (Interestingly, the English word 'cleave' has both these meanings.) A final example of dialetheias may therefore be delivered by a general theory of boundaries, according to which a point on a boundary is on one side of it, and so not on the other, for both of the sides![43]

[40]This, incidentally, was Hegel's solution to the paradox, and is part of a much larger story about the role of contradictions in Hegel's philosophy. See Priest (1990).

[41]See Beall (2021).

[42]See, further, Priest (1987), ch. 13, and Priest (2022).

[43]See Weber and Cotnoir (2015) and Weber (2021), esp. 1.3.

3.6 Information-Theoretic Interpretations

Some philosophers sympathetic to many-valued logic have balked at a reading of $\mathfrak{X}$ as alethic, but have endorsed an information-theoretic understanding of the value. That is, 'A is b' is understood as having been *told* both A and $\neg A$; one has both in one's data base, as it were. 'A is n' is understood as having been *told* neither A nor $\neg A$; one has neither A nor $\neg A$ in one's data base. The suggestion is made concerning FDE, for example, by Belnap.[44]

A problem with this interpretation is the same as that with de Finetti's: it does not sit well with the behavior of the connectives. For a start, one might have been told A by one source, and $\neg A$ by another. But both sources might reject $A \wedge \neg A$. So A has the value b, but $A \wedge \neg A$ does not even get the value 1. Even more obviously, one can have been told that $A \vee B$ whilst having been told neither A nor B. So $A \vee B$ can get the value 1 whilst neither A nor B does.[45]

Kleene also suggests an information-theoretic interpretation of n, not as *undefined*, but as *unknown*.[46] But we have the same problem. Thus, one can know that $A \vee B$ without knowing either A or B. (Just let B be $\neg A$.) Similarly, Brady suggests that A taking the value b means that both A and $\neg A$ are provable; and A taking the value n means that neither A nor $\neg A$ is provable. But at least classically, one can prove $A \vee B$ without proving A or B—in which case, $A \vee B$ can have value 1 when neither A nor B does. Dually, one can prove $\neg(A \wedge B)$ when one can prove neither $\neg A$ nor $\neg B$. So $A \wedge B$ can have the value 0, though neither A nor B does.

Information (what one is told, what one knows) is just not truth-functional—even if there are more than two truth values. If one has to pool information from different sources, the most sensible thing to do, it seems to me, is use a discursive logic, which is not truth-functional.[47]

Of course, we may restrict our data base to atomic sentences.[48] We then have to decide what to believe about compound sentences. Thus, suppose—to illustrate a 3-valued procedure—that we have produced our data base from the information provided by a number of (classically minded) "ex-

[44] In 'How a Computer Should Think', and 'A Useful Four-Valued Logic', pp. 35–54 and 55–76 of Omori and Wansing (2019).

[45] Perhaps one might attempt to overcome the first issue using supervaluations, but this does nothing to help the second.

[46] Kleene (1952), p. 335.

[47] See Priest (2002), 4.2.

[48] As suggested by Fitting (1994).

perts". Without loss of generality, we may suppose that there are just two. Each is asked whether A and answers either *true* or *false*. If both say *true*, we mark the sentence 1; if both say *false*, we mark it 0. And if they disagree, we mark it $\mathfrak{X}$. If we adopt a conservative policy, we may interpret $\mathfrak{X}$ as n, since we have an under-supply of information. If we adopt a liberal policy, we may interpret $\mathfrak{X}$ as b, since we have an over-supply of information. In either case we use the K_3/P_3 matrices to compute the values of compound sentences. In both cases, however, we may end up with things that are different from what our two experts think. Thus, despite their differences, both experts will think that $A \vee \neg A$ is true, though the conservative policy may value it as $\mathfrak{X}$, neither true nor false. And both will think that $A \wedge \neg A$ is false, though the liberal policy may mark it $\mathfrak{X}$, both true and false. And if we are not prepared to accept the experts' views about these things, why should we accept their views about atomic sentences?

3.7 Two Final Comments

Before we move on to the second general strategy for handling $\mathfrak{X}$, let me make two final comments.

[I] The intersection of K_3 and P_3 is itself a natural logic. Let us call this K_3P_3.[49] So:

- $\Sigma \models_{K_3P_3} A$ iff $\Sigma \models_{K_3} A$ and $\Sigma \models_{P_3} A$

One may think of this as a logic where $\mathfrak{X}$ is ambiguous. Specifically, it is what one would get if one took it that $\mathfrak{X}$ could be either n or b, the valid inferences being the ones that hold in both possibilities.

[II] There is a generalisation of many-valued logic in which there are two sets of designated values, $\mathcal{D}_\pi, \mathcal{D}_\kappa$, one for the premises and one for the conclusion, so that:

- $\Sigma \models A$ iff for all ν, if $\nu(B) \in \mathcal{D}_\pi$, for all $B \in \Sigma$, then $\nu(A) \in \mathcal{D}_\kappa$

In a series of papers, Cobreros, Égré, Ripley, and van Rooij (hereafter CERR) apply this technique with the values and matrices of K_3/P_3, and take $\mathcal{D}_\pi =$

[49]Dunn (2000) calls it RM_{fde}. Proof theoretically, it is obtained from FDE by adding the rule of inference $A \wedge \neg A \vdash B \vee \neg B$.

$\{1\}$ and $\mathcal{D}_\kappa = \{1, \mathfrak{X}\}$.[50] The logic has come to be known as ST (Strict-Tolerant) and one may show that an inference is valid in ST iff it is valid in classical logic. However valid classical metainferences, notably Cut, fail.

The construction itself tells us nothing about how to interpret $\mathfrak{X}$. We know that some As are such that both A and $\neg A$ can have that value. CERR's examples of such As are paradoxical sentences of self-reference, and borderline cases of vague predicates. We have already met these in connection with K_3 and P_3, and so this adds nothing new to the present issue.[51]

Perhaps the most interesting philosophical question specific to this construction is why one might change designated values from premises to conclusion in the way required. It is not clear to me that there is a good answer to this question. However, the matter is not one that is relevant here.[52]

4 Infectious $\mathfrak{X}$s

4.1 B_3 and H_3

Let us now turn to the second general strategy for handling $\mathfrak{X}$. This is to take $\mathfrak{X}$ to be infectious, in the sense that if it is the value of any sub-formula of A, it is the value of A. The truth tables for our connectives then become as follows:

$\neg$	
1	0
$\mathfrak{X}$	$\mathfrak{X}$
0	1

$\wedge$	1	$\mathfrak{X}$	0
1	1	$\mathfrak{X}$	0
$\mathfrak{X}$	$\mathfrak{X}$	$\mathfrak{X}$	$\mathfrak{X}$
0	0	$\mathfrak{X}$	0

$\vee$	1	$\mathfrak{X}$	0
1	1	$\mathfrak{X}$	1
$\mathfrak{X}$	$\mathfrak{X}$	$\mathfrak{X}$	$\mathfrak{X}$
0	1	$\mathfrak{X}$	0

The crucial matter now is whether one takes $\mathfrak{X}$ to be undesignated or designated. In the first case we have the logic B_3; in the second, we have the logic H_3.

[50]See, e.g., Cobreros, Égré, Ripley, and van Rooij (2013), (2015).

[51]In some papers, $\mathfrak{X}$ is glossed as *tolerantly assertible*, as contrasted with 1, which is glossed as *strictly assertible*. This suggests a connection with speech act theory; but what exactly tolerant assertibility might be, beyond being able to endorse both A and $\neg A$, is unclear. For some discussion, see Cobreros, Égré, Ripley, and van Rooij (2015), 21.2.3.

[52]For a full discussion of matters, see Priest (202+).

4.2 Varieties of Undesignated $\mathfrak{x}$

The first person to suggest the logic B_3 was Bochvar in a paper in Russian of 1937.[53]

What are the candidate interpretations for $\mathfrak{X}$ in B_3? Bochvar himself interprets $\mathfrak{X}$ as *meaningless* (though he sometimes glosses this as *neither true nor false*). This makes it plausible to see the value as infectious, since it is natural to suppose that a sentence with meaningless parts is itself meaningless. Bochvar gives as examples of meaningless sentences those self-referential sentences that arise in the paradoxes.

B_3 and various modifications are also endorsed by Goddard and Routley, who take $\mathfrak{X}$ to indicate a species of meaninglessness, and call it *non-significant*.[54] They give as examples things like category mistakes (e.g., 'The number 3 is red'), and Wittgenstein's "hidden nonsense" (e.g., 'It is 18.00h on the Sun').[55] It might be thought that these things should not be allowed to occur in a well-formed language, but the fact that the nonsense may be "hidden", means that there may be no effective way of ruling such things out; and for this reason or others, we may find ourselves reasoning with them.[56]

Smiley (1960) endorses B_3 and reads the infectious value as *neither true nor false*. He does this by implementing a Fregean understanding of the notion. According to Frege, the referent of a name is its bearer, and the referent of a sentence is its truth value. Moreover, the referent of any linguistic phrase is a function of the referents of its components (except when the referent occurs in what we would now call an intensional context). It follows that any sentence that contains a non-denoting name has no truth value; and if a subsentence of a sentence has no truth value, neither does the sentence. So $\mathfrak{X}$ behaves infectiously. Assuming that presupposition-failure is infectious, it may also be taken to motivate B_3.[57]

[53] Not translated into English till Bergmann (1981), though the basic ideas of the paper were known to English-speaking logicians due to a review by Church (1939). See also Rescher (1969), pp. 29–33.

[54] See Goddard and Routley (1973), ch. 5. This book is now hard to find. A brief summary of it can be found in Szmuc and Omori (2018), who show how $\mathfrak{X}$, as a species of lacking a value, can be interpreted using the notion of plurivalent logic (where sentences may take any number of values—including none). A brief discussion of the controversy which the book generated can be found in Szmuc and Ferguson (2021).

[55] Goddard and Routley (1975), §§1.4, 1.5.

[56] See, further, Routley (1969).

[57] On presupposition-failure, see also Spector (202+).

Kleene (1952), p. 334 also briefly specifies the logic B_3. (So the logic is sometimes known as 'weak Kleene', as opposed to K_3, which is 'strong Kleene'.) Again, computational issues are driving the interpretation. To see how, suppose that we have a compound sentence, say, for example, $A \vee B$. To compute its value we proceed by, first, computing the values of A and of B, and then, second, using classical truth tables. But if one or other of these computations never terminates, so that the value of A or of B is undefined, we never get around to the second stage, so the value of the whole is undefined. (If we applied truth tables as soon as we found the value of A (or B) to be 1, we would get, instead, the truth table of K_3 for $\vee$. Dually for $\wedge$.)

A slight variation on this idea is that in determining the value of $A \vee B$ (or $A \wedge B$) we compute the value of the left component first.[58] If we obtain a value that is sufficient to determine the classical truth value of the compound, we assign that value and do no more. If the value obtained is not thus sufficient, we then compute the value of the second component.[59] In either case, a computation that never terminates will result in an undefined value. As is easy to see, the tables for conjunction and disjunction are now as follows:

$\wedge_L$	1	$\mathfrak{X}$	0
1	1	$\mathfrak{X}$	0
$\mathfrak{X}$	$\mathfrak{X}$	$\mathfrak{X}$	$\mathfrak{X}$
0	0	0	0

$\vee_L$	1	$\mathfrak{X}$	0
1	1	1	1
$\mathfrak{X}$	$\mathfrak{X}$	$\mathfrak{X}$	$\mathfrak{X}$
0	1	$\mathfrak{X}$	0

The connectives are clearly asymmetric (in the sense that $A \models A \vee B$ but $B \not\models A \vee B$—and dually for conjunction). And if we had deployed a procedure that computed the second component first, we would have obtained connectives with the reversed asymmetry, $\vee_R$ and $\vee_R$. These matrices are also to be found as part of a theory of presupposition failure in Peters (1979).[60] The asymmetry naturally arises if we think of processing compound sentences, component by component, left-first.[61] This is perhaps more natural in the context of linguistics than logic.

Fitting formalises the computational procedure behind B_3 as follows. We

[58] As suggested by McCarty (1963), and formalised in the following way in Avron and Konikowska (2009).

[59] In computer science terminology, this is often called 'lazy evaluation', since we do no more computation than is necessary.

[60] Krahmer (1998), §4.2.2, calls the logic 'Middle Kleene'.

[61] See George (2014). See also Beaver and Krahmer (2001).

can define a monadic guard operator, $\mathcal{G}$, which tests to see whether the computation of the value of a formula terminates. Essentially, we just run the computation, and see whether it stops. $\mathcal{G}$ will then have the table:

$\mathcal{G}$	
1	1
$\mathfrak{X}$	$\mathfrak{X}$
0	1

One may then think of conjunction in B_3 as $\mathcal{G}A \wedge_L (\mathcal{G}B \wedge_L (A \wedge B))$—the the final conjunction being that of K_3—or in fact, any regular conjunction. Similarly, one will obtain the disjunction of B_3 with $\mathcal{G}A \wedge_L (\mathcal{G}B \wedge_L (A \vee B))$ (given a similar understanding of the disjunction).[62]

But why might one want to insist that the computations of the values of A and B severally terminate? Ferguson suggests the following idea.[63] The computation of the value of one of the conjuncts (or disjuncts), might not just fail to terminate, but might actually crash. For example, in computation, before a computation starts, the variables need to be "declared". That is, a memory location in the computer needs to be assigned. If this is not done, then when the variable is called during the computation, the computation will crash.

For comparison, think of a sentence of natural language with an indexical. For example, 'She is happy'. This has no meaning until a denotation is fixed for 'she'. In a certain sense, the sentence is meaningless until this is done. One might then argue that the same applies to any grammatical compound of this, e.g., 'London is in the UK and she is happy'. It could be said that such sentences expresses no proposition.[64]

In the same way, when the computation of one of the components of a formula crashes, the very notion of the program having an output has no well-defined computational meaning. Clearly, we are back with the thought that $\mathfrak{X}$ expresses a species of meaninglessness.

[62] Fitting introduced his guard operator as a primitive in (1994), §5. In (2006), §4.2, he shows how it may be defined in the context of the theory of bilattices.

[63] Ferguson (2014). This and all the other sole-authored papers by Ferguson referred to in what follows appear, in sometimes extended form, in Ferguson (2017), to which §4 of this essay is heavily indebted.

[64] Ferguson (2016) also considers the case where, because of storage issues, a program inadvertently overwrites a variable. The indexical analogue of this is when the denotation of the indexical is changed mid-reasoning.

There is also a close connection between an infectious $\mathfrak{X}$ and logics of "analytic implication" proposed by Parry, and later by Angell.[65] These are logics in which the inference $A \vdash A \vee B$ fails, the rationale being that the conclusion of an inference should not introduce concepts that are not in the premise.[66]

Deutsch[67] uses B_3 to invalidate the inference $A \vdash A \vee B$. Later in the thesis (p. 48), he considers the logic obtained by adding b to the values, and proves that the logic then satisfies the strong variable-sharing condition: if $A \vdash B$ then every propositional parameter in B occurs in A.[68] This suggests another interpretation of $\mathfrak{X}$. One might suppose that some of the sentences in the language have concepts familiar to a reasoning agent, and others are alien (in the way that the notion of a black hole or the internet are alien to a Medieval monk). $\mathfrak{X}$ can then be interpreted as the value of a sentence with alien concepts. True, some such sentences might still be true, but the consequence relation can now be thought of as meaning 'if all the premises are true and expressed in non-alien concepts, so is the conclusion'.

In a similar spirit, Oller (1999), who uses the same 4-valued logic, interprets *A having the value* $\mathfrak{X}$ as having no information about A. Since he is also working in the context of Parry-style logics, one might interpret the information in question as of a conceptually inaccessible kind. The same idea can obviously be applied to the $1/\mathfrak{X}\ /0$ part of the logic, B_3.

A similar idea is to be found in Beall (2016). A discourse is about something or things. Call this its topic. Not all the sentences in which a discourse is conducted will be about that topic. For example, if we are discussing the capital city of France, then statements about Beijing are off topic. Beall suggests that we interpret $\mathfrak{X}$ as the value of statements that are off topic. (So 1 is being true and on topic; and 0 is being false and on topic.) Given that a compound sentence is about whatever its components are about, it is natural to suppose that $\mathfrak{X}$ is infections. The consequence relation is now not one of preserving truth, but of preserving truth that is on topic.[69]

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[65] E.g., Parry (1932), (1989); Angell (1977), (1989).

[66] For a detailed investigation of these logics and the connection with an infectious third value, see Ferguson (2015a).

[67] Deutsch (1981), p. 9.

[68] Parry calls this the *Proscriptive Principle*. On the connection between Fitting's guard operator in the context of bilattices, and this logic, see Ferguson (2015a). If one adds n to the values as well, one obtains another analytic implication. See Ferguson (2016).

[69] Beall's idea is generalised to a natural 4-valued logic in Song, Omori, Arenhart, and

A final suggestion for an interpretation of $\mathfrak{X}$ is 'ineffable'. It is natural to take this to be infectious, since if something is ineffable, so is anything which contains it. If one follows this path, it makes no sense to interpret the bearers of semantic values as sentences: sentences wear their effability on their face. They have to be interpreted as propositions or states of affairs.[70] Again, in this case one might suggest that on this understanding, it does not make sense for $\mathfrak{X}$ to be undesignated: ineffable propositions might still be true—or ineffable states of affairs might be the ontological equivalent: existent.[71] But the consequence relation can now be thought of as meaning: 'if all the premises are true and effable, so is the conclusion'.[72]

4.3 Varieties of Designated $\mathfrak{X}$

The other possibility is that $\mathfrak{X}$ is both infectious and designated, which gives the logic H_3. This was first formulated by Halldén. Halldén's work is perhaps not as well known as it should be, and since the logic is paraconsistent, it is more often called 'Paraconsistent Weak Kleene'.[73]

How is such an $\mathfrak{X}$ to be interpreted? As the title of his work suggests, Halldén suggests that it should be interpreted as the value of things that are nonsense in some sense. He gives as examples of such things the paradoxical statements of self-reference, and borderline cases of vague predicates. Perhaps it is reasonable to take such statements to have an infectious value (though there is always the question of why the tables of K_3/P_3 might not be more appropriate). What is not clear is why $\mathfrak{X}$ should be designated. It would seem bizarre for a valid inference with a true premise to have a meaningless conclusion.[74] Halldén, it must be said, provides no very good reason.[75]

Tojo (202+), by splitting Beall's third value into two: *true and* $\mathfrak{T}$ and *false and* $\mathfrak{T}$, where $\mathfrak{T}$ is *off topic*. Some other possible interpretations of $\mathfrak{T}$ (e.g., known) are suggested in Omori (202+).

[70] Further, see Priest (2018), ch. 5. Actually, the logic there is 5-valued. The values are the four FDE values, and an infectious $\mathfrak{X}$. However, again, B_3 is its $1/\mathfrak{X}/0$ fragment.

[71] Kapsner (2020).

[72] See Priest (2020), 3.4.

[73] See, e.g., Ciuni and Carrara (2016).

[74] As noted by Goddard and Routley (1975), p. 274.

[75] As noted by Goddard and Routley (1975), §5.5, who, however, consider both logics in which $\mathfrak{X}$ is designated ("C logics") and those where it is not ("S logics") for reasons of technical comparison (p. 276).

Of course, given that $\mathfrak{X}$ is likely to be interpreted as some species of non-falsity, the definition of validity amounts to preservation of non-falsity. But the question is what $\mathfrak{X}$ could mean such that it makes this a sensible notion of validity. None of the candidates for $\mathfrak{X}$ which we looked at in the last subsection seem to provide this.

In Chapter 5 of *Time and Modality*,[76] Prior introduces a system of tense/modal logic he calls Q. The values of his logic are ω-sequences of the values 1, 0, and $\mathfrak{X}$. The points of the ω-sequence are thought of as points in time, and the values are the values of statements at each time. At each point, the values work in a B_3/H_3 fashion. And $\mathfrak{X}$ behaves in a designated fashion. Prior thinks that there are some things about the future that cannot be said. (For example, one cannot refer to a future object, though one can do so when it comes into existence.) He reads the value $\mathfrak{X}$ as *true but inexpressible*—and so we have a species of temporal ineffability. That explains why it should be both designated and infectious. The oddity of this interpretation is that one would expect a fourth value, $\mathfrak{Y}$, meaning *false but inexpressible*, where negation toggles between $\mathfrak{X}$ and $\mathfrak{Y}$. The negation of a true inexpressible statement is hardly true and inexpressible.

But note that there is no 4-valued logic that corresponds to B_3 and H_3 in the way that FDE corresponds to K_3 and P_3. It cannot be the case that both $\mathfrak{X}$ and $\mathfrak{Y}$ are infectious.[77]

4.4 Information-Theoretic Interpretations

Just as in the case of K_3/P_3, one might suggest reading the semantic values of B_3/H_3 in an information-theoretic way. However, there would still be the same problems. Information values are just not truth-functional.

As in 3.6, we could elect to restrict our information to atomic sentences only, and then decide what it is rational to believe on the basis of this. However, instead of using the K_3/P_3 matrices, we could, for one of the reasons considered above, decide to use the B_3/H_3 matrices.

For example, suppose we abstract the entries of our data base from two "experts", as described in 3.6—recall that our experts are classical. The conservative policy will then justify taking $\mathfrak{X}$ to be undesignated (B_3); and

[76] Prior (1957).

[77] Some 4-valued logics where one of $\mathfrak{X}$ and $\mathfrak{Y}$ is infections and the other is "almost infections" are discussed in Ciuni, Ferguson, and Szmuc (2019), and Da Ré, Pailos, and Szmuc (2020).

the liberal policy will justify taking it to be designated (H_3). On this interpretation $\mathfrak{X}$ is a warning label, infecting everything it touches. With the conservative policy, it means something like 'Alert: information underdetermined'. With the liberal policy, it means something like 'Alert: information overdetermined'.[78]

But in any case, we still face exactly the same problem we had with K_3/P_3. Namely, the result may conflict what our agents actually hold about compound sentences. For example, it could be the case that one agent thinks that A is true and that B is false; the other has it the other way round. Then despite their differences, both will think that $A \vee B$ is simply true, whilst the policy makes it $\mathfrak{X}$. And both will think that $A \wedge B$ is simply false, though the policy makes it $\mathfrak{X}$.

4.5 Two Final Comments

Let me end with two final comments, corresponding to those made in 3.6.

[I] The intersection of B_3 and H_3 is itself a natural logic, though as far as I am aware, it is not to be found in the literature. Let us call this B_3H_3. So:

- $\Sigma \models_{B_3H_3} A$ iff $\Sigma \models_{B_3} A$ and $\Sigma \models_{H_3} A$

One may think of this as a logic where $\mathfrak{X}$ is ambiguous between designation and non-designation—whatever that means.

[II] One may also use the techniques of different designated values for premises and conclusions with B_3 and H_3, as well as with K_3 and P_3. Szmuc and Ferguson[79] show that this logic has exactly the same consequence relation as classical logic—though metainferences, notably Cut, still fails for the same reason.

However, the problem of why one should define consequence in this way remains. In fact it is worse, since there is even less reason to allow an only true premise to have a meaningless conclusion than there is for allowing it to have true and false conclusion.

[78] This is a simplified version of the interpretation suggested by Szmuc (2019). Implementing a more general approach, he reads off his 3 values in a more complicated way, on the basis of judgements of "experts" that are themselves 3-valued. $\mathfrak{X}$ may be thought of as either *neither true nor false* or *both true and false*. Whichever the case, A will take the value $\mathfrak{X}$ if one of our two agents says 1 and the other says 0 *or* if either of them says $\mathfrak{X}$. However, the extra complexity does not add much to the present matter. It is the duality between under- and over-determination that is really doing the work.

[79] Szmuc and Ferguson (2021).

5 Conclusion

We have now met four well known 3-valued logics for our three connectives: K_3, P_3, B_3, and H_3 (or six if one counts the two logics with asymmetric conjunction and disjunction). We have looked at the ways in which it is natural to interpret the "third" value in each case—indeed, ways which motivated the construction of these logics in some cases.

Philosophically, the most problematic of the four these logics is H_3. In my opinion, we still want for a plausible philosophical interpretation of $\mathfrak{X}$ in this case.

Of course, there are also the other 46,871 ($= 3 \times 5^3 \times 5^3 - 4$) combinations of matrices. Doubtless, for many (maybe most) of these, $\mathfrak{X}$ has no natural philosophical interpretation. But it would be surprising if for none of them did it have one. Time will tell if such are to be found.

6 Appendix: Proof Systems for Four 3-Valued Logics

In this appendix I will give Prawitz-style natural deduction systems for the four main 3-valued logics we have met. A bar over an assumption means that the rule discharges it, and a double line in an inference means that it goes both ways. $A^\dagger$ is any formula containing all the propositional parameters in A. Proofs of soundness and completeness are given in Priest (2019b).

The rules in play are the following:

- $\wedge I$:

$$\frac{A \quad B}{A \wedge B}$$

- $\wedge E$:

$$\frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

- $\vee$I:

$$\frac{A}{A \vee B} \qquad \frac{B}{A \vee B}$$

- $\vee E$:

$$\dfrac{A \vee B \quad \begin{array}{c}\overline{A}\\ \vdots \\ C\end{array} \quad \begin{array}{c}\overline{B}\\ \vdots \\ C\end{array}}{C}$$

- DeM:

$$\dfrac{\neg(A \vee B)}{\neg A \wedge \neg B} \qquad \dfrac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

- DN:

$$\dfrac{A}{\neg\neg A}$$

- Excluded Middle (EM):

$$\overline{B \vee \neg B}$$

- Explosion:

$$\dfrac{A \wedge \neg A}{B}$$

- Weak $\vee I$:

$$\dfrac{A \quad B^{\dagger}}{A \vee B} \qquad \dfrac{A^{\dagger} \quad B}{A \vee B}$$

- Weak $\wedge$E:

$$\dfrac{A \wedge B}{A \vee B^{\dagger}} \qquad \dfrac{A \wedge B}{A^{\dagger} \vee B}$$

- Weak EM:

$$\dfrac{A^{\dagger}}{A \vee \neg A}$$

- Weak Explosion:

$$\dfrac{A \quad \neg A}{A^{\dagger}}$$

The rules for the various systems are as follows:

K_3: $\wedge I$, $\wedge E$, $\vee I$, $\vee E$, DeM, DN, and Explosion.

P_3: $\wedge I$, $\wedge E$, $\vee I$, $\vee E$, DeM, DN, and EM.

B_3: $\wedge I$, $\wedge E$, Weak $\vee I$, $\vee E$, DeM, DN, Explosion, and Weak EM.

H_3: $\wedge I$, Weak $\wedge E$, $\vee I$, $\vee E$, DeM, DN, Weak Explosion, and EM.

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