How Do You Apply Mathematics?

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July 1, 2022

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Abstract

As far as disputes in the philosophy of pure mathematics goes, these are usually between classical mathematics, intuitionist mathematics, paraconsistent mathematics, and so on. My own view is that of a mathematical pluralist: all these different kinds of mathematics are equally legitimate. Applied mathematics is a different matter. In this, a piece of pure mathematics is applied in an empirical area, such as physics, biology, or economics. There can then certainly be a disputes about what the correct pure mathematics to apply is. Such disputes may be settled by the standard criteria of scientific theory selection (adequacy of empirical predications, simplicity, etc.) But what, exactly is it to apply a piece of pure mathematics? How is mathematics applied? By and large, philosophers of mathematics have cared more about pure mathematics than applied mathematics, and not a lot of thought has gone into this question. In this paper I will address the issue and some of its implications.

Key Words: mathematical pluralism, pure mathematics, applied mathematics, Quine, Field, Wigner

1 Introduction: Disagreements in Mathematics

The distinction between pure and applied mathematics is a crucial one for the philosophy of mathematics. Pure mathematics is an investigation of mathematical structures in and of themselves. Applied mathematics is the use of such structures in the investigation of other things, in physics, economics, linguistics, or whatever. Of course there is a connection between the two activities. For a start, some parts of pure mathematics (such as the original infinitesimal calculus) were developed specifically with an eye on their applications. But in principle, and even mostly in practice, the two sorts of investigations are quite distinct.

In the philosophy of mathematics, however, understanding the nature of applied mathematics has been a somewhat poor cousin of understanding pure mathematics, since philosophers of mathematics have normally been logicians, not physicists or economists. Typically, applied mathematics has been taken as of subsidiary importance, invoked to tell us something about pure mathematics—its handmaiden, as it were. In particular, the question 'What, exactly, *is it* to apply a piece of mathematics?' has not been given the attention it deserves. The aim of this paper is address that question squarely. Given the complexity of a number of the issues, a single paper can hardly address everything relevant to the question. But I hope, at least, that what follows will provide the outline of an adequate answer to that question.[1]

First, however, let me explain what this has to do with the topic of this volume. There can certainly be disputes in the area of pure mathematics. Thus, we often have some informal mathematical notion, and there is an issue about how best to formalise it. What it is for a function to be continuous, or a procedure to be algorithmic, are clear examples of this.[2] Another is the notion of what it is to be a polygon, explored by Lakatos in his *Proofs and Refutations*.[3]

[1]Talks based on this paper were given at the (online) workshop Disagreement in Mathematics, Free University of Amsterdam and the University of Hamburg, June 2021, and at Ohio State University, October 2021. I am grateful to members of the audiences for their helpful comments. I'm also very grateful to Hartry Field and Mel Fitting for their comments on earlier drafts of the paper. Finally, I'd like to thank three referees for this journal for their helpful comments.

[2]See, respectively, Bell (2013) and Copeland (2017).

[3]Lakatos (1976).

Once a notion is formalised, there can still be disagreements. Most mathematical proofs are not formalised, and there can be issues about whether all the steps can be filled in correctly. See, for example the issue over the correctness of Wiles' proof of Fermat's Last Theorem,[4] or the much more acrimonious debate over the correctness of the proof of the ABC conjecture offered by Mochizuki.[5]

However, assuming mathematical pluralism, which I will explain in a moment, what there cannot be is a dispute about whether the mathematical structure isolated in the formalisation is, as a mathematical structure, itself right or wrong. All structures are equally legitimate.

Matters are quite different with applied mathematics. Here, one has to choose a pure mathematical structure to apply; and there certainly can be a dispute over what the correct structure is—which one is right or wrong, or at least, which one is best. This is, as we shall see, an issue in scientific theory-choice.

2 (Pure) Mathematical Pluralism

Before we turn to applied mathematics itself, let me say a word about pure mathematics; for what I have to say about applied mathematics, I shall say against the background of *mathematical pluralism* in pure mathematics. I shall not defend this view here, but let me at least explain it.[6]

Start with the following fact. There are many different geometries: Euclidean, spherical, projective, hyperbolic, Riemannian, etc.[7] All are perfectly good mathematical structures (or families of structures). They have their own axiomatizations, models, theorems, and so on. There is no sense in which any of them is right and the others wrong. Perhaps some may be more mathematically interesting, powerful, have more applications, etc, than others; but that is a different matter. All this is hardly contentious.

It might be suggested that, though we have a plurality here, there is an underlying unity, in that all of the geometries can be defined within a single mathematical structure: Zermelo-Fraenkel set theory (with the Axiom of Choice), ZFC. Such reductionism is still, perhaps the standard view in

[4] See Singh (1997).

[5] See, e.g., Klarreich (2018).

[6] Mathematical pluralism is defended at length in Priest (2013), (2019).

[7] For some discussion, see Gray (2017).

the philosophy of mathematics, though it has always been contentious. From the perspective of category theory, for example, things look quite different. ZFC is, itself, just one framework amongst many.

But in any case, the view is no longer sustainable. It is not just that it is hard to find a framework based on classical logic to which all other parts of mathematics can be reduced; we now know that there are mathematical structures based on non-classical logics which cannot be formulated in terms of classical logic, on pain of triviality. These are things like the intuitionist theory of smooth infinitesimals, various branches of inconsistent mathematics, and so on.[8] All of these have their axiomatisations, models, etc; and the structures have clear mathematical interest. *Qua* pure mathematical structures, they are all equally good; there is no sense in which one is right and the others wrong. It may be the case that some are more rich, interesting, have more applications, etc, than others. But again, that is a different matter.

3 The Application of Mathematics

Matters of equal correctness are entirely different when we turn to applied mathematics. In this, a branch of mathematics or a mathematical structure is chosen and applied in an analysis of some natural or social phenomenon. That is, applied mathematics is the use of a pure mathematical structure in the investigation of other things, in physics, economics, linguistics, or whatever. There is then a question of which structure is the right one for the job—or at least which ones are better than others. What exactly this means, we will come to in due course. At any rate, pure mathematical structures are *not* all equally good in this regard.

Thus, geometry has what one might call a canonical application: charting the structure of space (or maybe nowadays, space/time). For most of the history of mathematics it was assumed that Euclidean geometry was the correct geometry for that application; Euclidean geometry was developed for just that purpose. Indeed, in the history or mathematics until the 19th Century it is hard to disentangle this geometry from its application. But of course, Kant notwithstanding, we now know better. The spatial structure of the cosmos is not Euclidean; it is not even one of constant curvature. Euclidean geometry was not the right geometry for the purpose. Naturally, a geometry can have many applications, and this by no means implies that

[8]See, e.g., Shapiro (2014), Priest (2019), Bell (1998), Mortensen (2017).

Euclidean geometry is not the right geometry for some other application. What can be right for one application can be wrong for another, and vice versa.

Thus, in Quantum Mechanics there are two well known kinds of statistics, Bose-Einstein and Fermi-Dirac, each of which implements a different kind of probability distribution for the entities involved.[9] And it turns out that the different kinds of statistics are appropriate for different kinds of particles. Bose-Einstein statistics works for Bosons; Fermi-Dirac statistics works for Fermions.

Hence, given any potential application, there is the question of which mathematical structure of a certain kind is best for the job. And conversely, given different mathematical structures of the same kind, there is the question of which application or applications they are appropriate for.

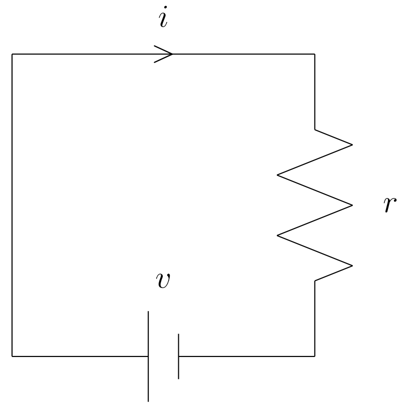
Indeed, there is a question of whether they have any applications at all. Some do not. There is, for example, as far as I know, no known application of the theory of large cardinals. And some branches of pure mathematics were found to have an application only well after their development: for example, the theory of electricity for complex numbers, and Special Relativity for group theory.

4 Two Examples

Against this background, we can now turn to our central issue. How, exactly, does one apply a pure mathematical structure? Let us start with a couple of simple examples.

The first uses Ohm's law to determine the current in a circuit. Suppose we have a simple electrical circuit with a battery and a resistor.

[9] See, e.g., French (2019).



If the voltage, v, produced by the battery is 6 volts, and the resistance, r, of the resistor is 2 ohms. What current, i (in amps), flows? Ohm's law, $v = ri$, tells us that it is 3. What is going on here?

First, we start with a state of affairs in the physical world. This will involve some wires and other bits of electrical paraphernalia. Understanding what is going on, and making predictions about it, will involve the following steps. (The sequence here is not a temporal one.) First, we need to invoke three physical quantities: the current flowing, I, the resistance, R, and the voltage in the circuit, V. Call the set of physical quantities, $\mathbb{P}$. Next, these have to be assigned some mathematical values. Hence, there are three functional expressions, μ_i, μ_r, μ_v, such that μ_i means 'the value in amps of', μ_r means 'the value in ohms of', and μ_v means 'the value in volts of'. (I will often omit the subscripts on 'μ' when they are clear from the context.) In our case, the mathematical values are real numbers, members of $\mathbb{R}$. So the denotation of each μ is a map from $\mathbb{P}$ to $\mathbb{R}$. We can now enunciate Ohm's Law:

- $\forall V, R, I(\mathcal{F}(V, R, I) \to \mu(V) = \mu(R) \times \mu(I))$

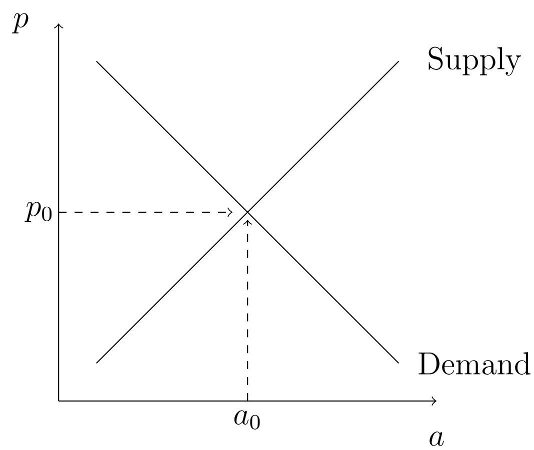
where $\mathcal{F}(V, R, I)$ states that V, R, and I are the quantities in an electrical circuit. Finally, we have to determine exactly how the mathematical entities and the operations on them work. In the case at hand, this is provided by the mathematical structure of the classical reals, $\mathfrak{R}$, $\langle \mathbb{R}, +, \times, 0, 1, < \rangle$.

Now, in our present example, we have three particular quantities, V_0, R_0, and I_0, such that $\mathcal{F}(V_0, R_0, I_0)$. Hence applying Ohm's Law, we have $\mu(V_0) = \mu(R_0) \times \mu(I_0)$. We also have $\mu(V_0) = 6$ and $\mu(R_0) = 2$. If we now chose new terms, v, r, and i, for $\mu(V_0)$, $\mu(R_0)$, and $\mu(I_0)$, respectively, we have the equations:

- $v = r \times i$
- $v = 6$
- $r = 2$

Moving to the pure mathematical level, if these statements hold in $\mathfrak{R}$ then so does $i = 3$. That is, moving back to the empirical level again, $\mu(I_0) = 3$. That is, the current in the circuit is 3 ohms.

Next, an example from economics.[10] According to standard microeconomics, other things being equal, in an equilibrium state, the price of a commodity is the point where supply and demand balance.



So, given some good, let p be the price per unit, and a be the amount produced. Suppose that for supply, $p = a$; and for demand, $p = 6 - a$. Then the equilibrium price per unit is given by the point, $\langle p_0, a_0 \rangle$, where these lines cross; that is, p_0 is 3. What is going on here?

The situation we are dealing with concerns people who produce, and who buy and sell things. To understand the situation, we need to invoke two quantities, A, the amount produced, and P, the price. $\mu(A) \in \mathbb{R}$ will be the amount of A in, say, kilograms; $\mu(P) \in \mathbb{R}$ will be the price in, say,

[10] There may well, of course, be important differences between the natural and the social sciences. As far as I can see, however, any such differences do no affect the points at issue here. Indeed, I choose an example from the social sciences precisely to show that the account of applied mathematics I am giving applies just as much to the social sciences as to the natural sciences.

dollars. But here we have two other things to consider, the causal relations of supply and demand between these quantities. Let these be S and D, respectively. Note that the appropriate mathematical structure is now the second-order reals, and $\mu(S)$ and $\mu(D)$ are relations between reals. That is, $\mu(S), \mu(D) \subseteq \mathbb{R}^2$. The Law of Supply and Demand tell us that:

- $\forall A, P, S, D(\mathcal{F}(A, P, S, D) \rightarrow \forall x, y(\langle x, y\rangle \in \mu(S) \cap \mu(D) \rightarrow \mu(A) = x \wedge \mu(P) = y))$

where A and P are the equilibrium values, and $\mathcal{F}(A, P, S, D)$ states that the quantities concerned are in the appropriate relation (and we assume, for the sake of simplicity that there is a unique point common to the two relations).

In the present situation, there are particular values, A_0, P_0, S_0, D_0, such that $\mathcal{F}(A_0, P_0, S_0, D_0)$. Hence, we know that $\forall x, y(\langle x, y\rangle \in \mu(S_0) \cap \mu(D_0) \rightarrow \mu(A_0) = x \wedge \mu(P_0) = y)$. We also know that $\mu(S_0) = \{\langle x, y\rangle : y = x\}$ and $\mu(D_0) = \{\langle y, x\rangle : y = 6 - x\}$. Let us now choose new terms a, p, s, d such that $s = \mu(S_0), d = \mu(D_0), a = \mu(A_0), p = \mu(P_0)$. Then we have the pure mathematical statement:

- $\forall x, y(\langle x, y\rangle \in \{\langle x, y\rangle : y = x\} \cap \{\langle y, x\rangle : y = 6 - x\} \rightarrow a = x \wedge p = y)$

Ascending to this level, if this statements holds in $\mathfrak{R}$, then so does $p = 3$. That is, descending to the empirical level again, $\mu(P_0) = 3$. That is, the equilibrium price is 3 dollars per kilogram.

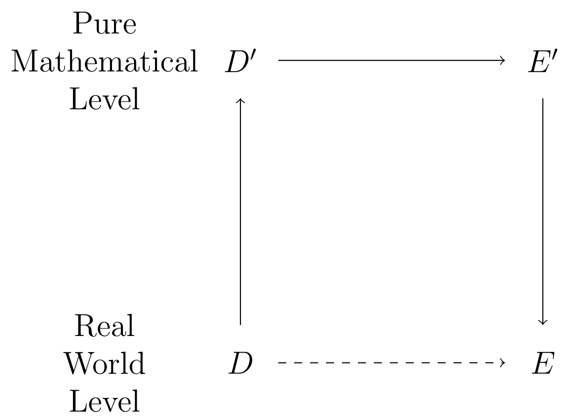
5 The General Picture and its Ramifications

Bearing our examples in mind, I can now sketch the general situation involved in applying mathematics.[11]

Let us call the topic to which we are applying mathematics, for want of a better phrase, the "real world". The real-world state of affairs will concern various real-world quantities. The situation describing these and the laws of nature can expressed by a set of statements, D, using the μ functional terms. Pure mathematical statements, D', concerning some structure, $\mathfrak{S}$, are abstracted from these statement, ignoring the physical interpretation of the mathematical quantities. Using what we know about $\mathfrak{S}$, we can infer

[11] The basic idea of this is to be found in Priest (2005), 7.8.

some other statements, E', that hold in the structure. These can be "de-abstracted", bringing the physical interpretation—that is, μ—back into the picture, to deliver some descriptions of the real-world situation, E. We may depict this as follows:



Hence it is that we can use the pure mathematical structure to infer facts about the "real world".[12]

What takes us from D' to E' in this procedure is pure mathematics: proofs concerning $\mathfrak{S}$. The rest of the picture is a matter of empirical discovery. That this is so in the case of the scientific laws involved is, of course, clear. But the finding of an appropriate mathematical structure, $\mathfrak{S}$, is, in principle, equally an empirical discovery. This is exactly what the situation concerning the replacement of Euclidean geometry by Riemannian geometry showed us.

How does one determine the correct (or best) mathematical structure to be employed? μ-sentences may be subject to empirical testing by an appropriate measuring procedure.[13] For example, in the Ohm's Law example, we can test the claim that $\mu(I_0) = 3$ by using an ammeter. Thus if the truth of this statement is already known, it may be explained. And if not, the procedure may be used to confirm or refute the machinery employed to get

[12] This account bears notable similarities to the "inferential conception" of applied mathematics, described and advocated by Bueno and Colyvan (2011). (See esp. their diagram on p. 353.) A main difference is that they take the transitions involved to be between empirical phenomena and mathematical structures, whereas the account I give here takes them to be between statements about these things. In the last analysis, however, this difference may not be particulary significant.

[13] For a discussion of measurement in science, see Tal (2020).

there. The correct mathematical structure for any kind of application is the one that gives the best results in this way.

If one is some kind of non-realist about the domain of application, nothing more matters. Of course, there is a question about why this piece of mathematics gives the right (or good) results. The standard realist answer[14]—to which I am sympathetic—is that the mathematical structure tracks the real-world structure.[15] That is, we can read off facts about the real-world from mathematics since the two configurations have the same structure.[16] That is, in our examples, 'μ' refers to a homomorphism. (It cannot be an isomorphism, since different physical quantities can have the same mathematical value.)

Well, matters have to be a little more complicated than that. Often, the appropriate mathematical structure is much more than an instrumentalist black box, but the mathematical structure and the physical structure are not exactly the same. For example, the mathematics can be continuous while the physical system is discrete. Thus, in our example of supply and demand, the real numbers are continuous, but the quantities produced will normally be discrete. Or again: in the mathematics of fluid flow, continuous mathematics is used, but we know that the fluids are composed of molecules, and so are discrete. The mathematics may also ignore factors whose effect is below the level of empirical significance (such as, maybe, the gravitation effects of planets on the Sun).

In such cases, though the real-world and mathematical structures are not the same, the one approximates the other, at least to the order of magnitude with which we are concerned. How, exactly, to understand this matter is a somewhat tricky issue; but fortunately, one that we do not need to go into here. Note, however, that we do not have to give a mathematical proof that the approximation is a good one. The fact that the application gives the right results to an appropriate level of accuracy provides an *a posteriori* demonstration that it does so.

Whatever one makes of these matters is not relevant to the main point

[14]See, e.g., Chakravartty (2017), 2.1.

[15]Essentially this account is defended in Pinnock (2004), where he calls it a *structuralist account*. The account is discussed by Bueno and Colyvan (2011), where they call it a *mapping account*. They argue there that it should be subsumed under what they call an *inferential conception* of applied mathematics.

[16]In just the way that the *Tractatus* reads off the structure of the world from the structure of the classical predicate calculus.

here, though, which is simply that one chooses the pure mathematical structure to apply which gives the best empirical results—however one theorises that matter.

6 *A Posteriori* Mathematics

The example of geometry has already taught us that the pure mathematical structure to be employed in the step from D' to E' may not be the one we would have expected, might have to revised in favour of something unexpected, and that what this is is an *a posteriori* matter. In this section, I want to hammer that message home.

In the two examples of §4, the mathematical structure deployed was that of the standard reals. However, there is no *a priori* reason why this has to be the case. We might discover (or have discovered) that better predictions for electrical quantities are made using operations on the real numbers where multiplication is non-commutative. For example, this could be a structure $\mathfrak{R}' = \langle \mathbb{R}, +, \times', 0, 1, < \rangle$, $\times'$ being defined as follows:

- if $x > y$ then $x \times' y$ is $x \times y$
- if $y \geq x$ then $x \times' y$ is $x \times y - \frac{y-x}{r}$

where r is a large real number, representing, perhaps, some physical constant.

Nor does the underlying logic of a structure applied have to be classical logic. There is no *a priori* reason why using the intuitionist reals might not give better predictions than using the classical reals. Let me make the point about using a mathematical structure with a non-classical logic with a more detailed, entirely hypothetical, example. This concerns arithmetic.[17]

There are arithmetics (even axiomatic ones) which contain all the truths in the (classical) standard model of arithmetic, and then more. These are inconsistent, of course, but non-trivial. Their structure is now relatively well understood.[18] Take a simple one of these. In this, there there is a tail, $\{0, 1, ...n - 1\}$, for some $n > 0$. The numbers in this behave consistently. Then, for $i \geq n$, the numbers cycle, so that for some period, $p > 0$, $i + p = i$—and of course, $i + p \neq i$ too. We might depict the sturucture thus ($\rightarrow$ is

[17] I take this from Priest (2003).

[18] See, for example, Priest (1997) and (2000).

the successor function):

$$\begin{array}{ccccccccc} 0 & \to & 1 & \to & \dots & \to & n & \to & n+1 \\ & & & & & & \uparrow & & \downarrow \\ & & & & & & n+p-1 & \leftarrow & \dots \end{array}$$

How might such an arithmetic come to be applied? Let us suppose that we predict a collision between a star and a huge planet. Using a standard technique, we compute their masses as x_1 and y_1. Since masses of this kind are, to within experimental error, the sum of the masses of the baryons (protons and neutrons) in them, it will be convenient to take a unit of measurement according to which a baryon has mass 1. In effect, therefore, these figures measure the numbers of baryons in the masses. After the collision, we measure the mass of the resulting (fused) body, and obtain the figure z, where z is much less than $x_1 + y_1$. Naturally, our results are subject to experimental error. But the difference is so large that it cannot possibly be explained by this. We check our instruments, suspecting a fault, but cannot find one; we check our computations for an error, but cannot find one.

Some days later, we have the chance to record another collision. We record the masses before the collision. This time they are x_2 and y_2. Again, after the collision, the mass appears to be z (the same as before), less than $x_2 + y_2$. The first result was no aberration.

We investigate various ways of solving the anomaly. We might revise the theories on which our measuring devices depend, but there may be no obvious way of doing this. We could say that some baryons disappeared in the collision; alternatively, we could suppose that under certain conditions the mass of a baryon decreases. But either of these options seems to amount to a rejection of the law of conservation of mass(-energy), which would seem to be a rather unattractive course of action.

Then we realise that the results can be accommodated by supposing that when we count baryons we have to use a non-classical arithmetic. (As noted, we already know that different sorts of fundamental particles obey different statistics. Baryons are certain kinds of fermion.) The empirical results can be accommodated by using an inconsistent arithmetic of the kind just described, where z is the the least inconsistent number, n, and $p = 1$. For in such an arithmetic $x_1 + y_1 = x_2 + y_2 = z$, and our observations are explained without having to assume that the mass of baryons has changed, or that any are lost in the collisions.

The thought experiment can be continued in ways which make the application of an inconsistent arithmetic even more apt—indeed, even accommodating the fact that if $z' > z$ then $z' = z$—but we do not need to go into the details here.[19] Of course, these facts can be accommodated in a consistent—though still highly non-standard—arithmetic. What you cannot have in such an arithmetic is the rest of standard arithmetic; or even the fragment axiomatised in Peano Arithmetic. And this rest may well be important in applying the structure. At any rate, the point is made. There is nothing in principle against applying such a paraconsistent arithmetic.

Truth itself—at least some part of it—may then be governed by a non-classical logic. So even the logic we take to apply to the real world may end up being revised for empirical reasons.

Let me summarise the picture of applied mathematics that has emerged in the preceding discussion. In applying mathematics, one uses a pure mathematical structure as depicted in the diagram of Section 5. The structure to be used is the one which gives the right empirical results (whatever that means). Sometimes, the pertinent pure mathematical structure will have arisen out of some kind of real-world practice, making the distinction between the pure structure and a certain application almost invisible.[20] (Euclidean geometry and natural-number arithmetic are cases in point.) That, however, provides no guarantee that a different structure will not do that job better. (Geometry again illustrates.) Sometimes, a certain sort of application will occasion the development of a whole new kind of pure mathematical structure which seems to be right for doing the job in question. (The infinitesimal calculus illustrates.[21]) Many pure mathematical structures were, however, produced and investigated with no thought of application in mind. (The investigation of higher infinitudes is a case in point.) Though sometimes it will turn out, later, that such pure structures are just what seems to be required for a certain application. (Group theory and the Special Theory of Relativity provide a case in point.) Historically, then, the connection between pure and applied mathematics can be a somewhat tangled one. All the more reason to keep the fundamentals of the relation between them straight.

[19] They can be found in Priest (2003), §7.

[20] On the genealogy of some mathematical concepts in certain human practices, see Lakoff and Núñez (2000); and for further discussion, see Kant and Sarikaya (2021).

[21] For an illustration from contemporary mathematical biology, see Montévil (2018) and Pérez-Escobar (2020).

7 Comments on Three Philosophers: Quine, Field, and Wigner

To round out the discussion, let me end with some comments on three notable philosophers of mathematic, pointing out where, in the light of the preceding account, they are right, and where they are wrong.

The first is Quine. Famously, according to Quine,[22] claims of pure mathematics are verified (established as true) holistically, together with our empirical scientific claims. And since they are true, and quantify over abstract mathematical objects, these exist.[23] The ontological claim, depending as it does on the view that anything quantified over in a true statement exists—at least if the language is an appropriately regimented one—is highly debatable.[24] But set that matter aside here and concentrate just on the epistemic claim.

It has many problems. The people who are best qualified to judge whether a claim of pure mathematics is true are pure mathematicians; and they care not at all about applications. What is important to them is proof. Next, as noted, there are important parts of pure mathematics that have (as yet) no empirical applications, such as the theory of large transfinite numbers, infinitary combinatorics, and the theory of surreal numbers.[25] Next, some theories can be applied in different areas. In some they are verified; in some, they would not be. Thus, the pure mathematical theory of the Lambek Calculus is verified when applied to grammatical parsing, but not when applied to simplifying Boolean electrical circuits. Does this mean that the pure mathematical statements are both true and false? Presumably not. As far as truth goes, one application would have to be privileged. But any such privileging is arbitrary.

But what underlies all these issues, as we may now see, is that the account

[22]Perhaps most notably in Quine (1951). See Colyvan (2001), esp. 2.5. For more on Quine's philosophy of mathematics see Hylton and Kemp (2019) and Priest (2010).

[23]This is sometimes known as 'Quine's indispensability argument': abstract entities exist because they are indispensable for science. See, e.g., Colyvan (2001). Note that the plausibility of the ontological conclusion goes via the thought that we are justified in taking the statements to be true. The existence of a God is indispensable for Christian theology; but this provides no argument for the existence of God if the theological statements are not true.

[24]See Priest (2005).

[25]On the last of these, see Knuth (1974).

simply has the wrong take on applied mathematics. What gets confirmed or otherwise by an application are the statements which describe the real world—the likes of our D and E above. Application has no relevance to pure mathematical statements, like D' and E'. All that is confirmed or not in their application is whether they are the right bits of mathematics for the job at hand. The criterion of truth for statements like D' and E' is proof. And indeed, assuming the correctness of mathematical pluralism, it is not even truth *simpliciter* which is at issue here, but truth-in-a-structure.[26]

Let us now turn to Field. His *Science without Numbers*[27] is an essay on applied mathematics. The major explicit driver of Field's project is what he calls nominalism—perhaps better (for reasons we will come to in a moment), anti-Platonism: the view that no abstract entities, notably mathematical entities, exist. Field shows how an important example, Newtonian gravitational theory, may be formulated quantifying over only physical entities. In scientific practice, pure mathematics, which quantifies over abstract entities can be (and is) used, but this is a conservative extension of the physical machinery, and so is not involved essentially.[28] We are free, then, to adopt a fictionalist understanding of the pure mathematics. The pure mathematical statements are not really true. They are just "true in the fiction" which is, say, $\mathfrak{R}$.

Field's view has some notable similarities and notable differences with the view described above. To start with, I agree with Field's anti-Platonism. However, I prefer a noneist approach to the matter. By all means, quantify over abstract entities. These are just non-existent objects.[29]

Next, according to the two accounts of applied mathematics, the world (or an aspect of it) is described in empirical terms. We then interpret some of what is going on in pure mathematical terms, and use the results of this to infer an empirical situation. There are three important differences between our approaches.

First, in my case, though not in Field's, the empirical statements refer to mathematical objects, such as numbers. However, I stress, and as I have already noted, these statements may be empirically verifiable by means of familiar measuring devices. (Nor do I regard the truth of such statements as

[26] On the difference between the two, see Priest (2021).

[27] Field (1980).

[28] In the way that non-finitary statements are used, according to Hilbert, in arithmetic. See Zach (2019).

[29] See Priest (2005), ch. 7.

committing to the existence of abstract entities, for reasons I have already noted.[30])

Secondly, as the introduction to the second edition of Field's book makes clear (see esp. p. P-4 ff.) there is a second thought which drives his approach. Descriptions at the empirical level should make use only of intrinsic notions. How, exactly to understand the notion of intrinsicality here is not a straightforward matter. But certainly the use of measuring scales and coordinate systems are not intrinsic. As is clear, my empirical level makes use of such notions. Now, I can understand the pull of intrinsicality from a certain theoretical perspective; but I think it fair to say that it is of virtually no importance for practicing (applied) mathematicians. If, in the end, science makes essential use of things such as measuring scales,[31] and this introduces an ineliminable conventionality into actual science, so be it.[32]

Third, it is important for Field that the application of the pure mathematics is conservative over the empirical level.[33] Conservativity plays no role in my account. Indeed, it is important that the result is not conservative. It is precisely this fact with allows for novel empirical predications, which can be used to test the machinery deployed. However, as just observed, Field and I understand different things by the empirical level.

Of course, making novel predications does play an important role in science, and Field is well aware of this. He is happy with the fact that in practice mathematical machinery is used to make novel predications. In principle, however, these could be obtained simply from the empirical base, from which the appropriate mathematics may be thought of as abstracted via the appropriate representation theorem. His approach is therefore a sort of "rational reconstruction", which mine is not.

Finally, Field's fictionalism and my version of mathematical pluralism are not exactly the same; but there are, at the very least, strong similarities between the prefixes 'In the fiction F it is the case that' and 'In the structure

[30] In the examples of §4, the empirical languages contained terms that refer to mathematical objects, though not quantification over them, e.g., with things such as $\exists r\, r = \mu(I)$. The procedure I sketched carries over straightforwardly to such a syntax. The quantifiers are simply preserved in the abstracted pure mathematical statements (and back).

[31] So that they cannot be "factored out" with invariance under the appropriate transformations.

[32] See the discussion of conventionalism in Tal (2020).

[33] One has to be a bit careful as to how to spell this out, though. See the discussion of conservativity in §0.4 of the second edition of the book.

$\mathfrak{S}$ it is the case that'. Both are closed under some kind of logic, both are non-veridical (that is, for both operators, Θ, ΘA does not imply A); and in neither case does the truth of ΘA entail the existence of the objects referred to in A—though perhaps for different reasons. Indeed, one may hold that a mathematical structure, characterised in a certain way, may not even have its characterising properties at the actual world, but at some other worlds.[34] The connection between the operator 'In fiction F' and world semantics is well known.[35]

Finally, let us turn to Wigner's essay 'The Unreasonable Effectiveness of Mathematics in the Natural Sciences'.[36] As the title suggests, in this paper Wigner avers that we have no right to expect that mathematics can be effective in our engagement with the world. In a sense he's right; in a sense, he's wrong.

First, there is absolutely no *a priori* reason why the world should be ordered or have structure. It is entirely logically possible that the world should be as random as can be. And if it were, no mathematics would help to explain or predict events.

But of course, the world is not like this. We know that it has order—at least, pockets of it—because we are part of it, and we are ordered beings, as is our immediate environment. There is, then, structure in the world. Mathematics is a science of structure (or structures); hence we can expect mathematics to get some grip on a least some aspects of the world.

Of course, there is no *a priori* reason why any particular mathematical structure should get a grip on it. But it is hardly surprising that some of the mathematics we have does so, since it evolved out of relevant practices, or was developed specifically for that purpose. That it does so is, then, no more surprising than that a telescope allows us to see at a distance, or that the 'flu vaccine protects against 'flu.

Naturally, it may turn out that the mathematics we have developed is the wrong mathematics for the project. As we know, it has been so sometimes in the past; and maybe it will be so again in the future. But if that turns out the be the case, people (or maybe, now, computers!) will at least attempt to design mathematics that works better.

Such development is certainly not the point of pure mathematics, which

[34] Priest (2005), 7.8.
[35] See, e.g., Kroon and Voltolini (2019), §2.
[36] Wigner (1960).

is to investigate interesting abstract structures in their own right. But designing pure mathematical structures for intended applications is obviously a legitimate project as well.

8 Conclusion

The main point of the paper was to address the question ‘What exactly is applied mathematics?’. That is, ‘what goes on when a piece of pure mathematics is applied to an empirical realm?’. The answer I have given is as follows. Given statements about some empirical situation, statements concerning a pure mathematical structure are abstracted. We can then use what we know about that structure to establish other pure mathematical claims, which are then “de-abstracted” to deliver statements about the empirical situation. Which pure mathematical structure should be used is an *a posteriori* and fallible matter.

I'm sure that there is much more to be said about the philosophy of applied mathematics. This paper is no more than an attempt to spell out a basic—and I hope correct—understanding of the topic, as a distinctive part of the philosophy of mathematics. If it serves to put that topic on the agenda for further reflection, the paper will have served it purpose.[37]

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