Substructural Solutions to the Semantic Paradoxes: a Dialetheic Perspective

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Abstract

Over the last decade or so, a number of writers have argued for solutions to the paradoxes of semantic self-reference which proceed by dropping some of the structural rules of inference, most notably Cut or Contraction. In this paper, we will examine such accounts, with a particular eye on their relationship to more familiar dialetheic accounts.

1 Introduction

The paradoxes of semantic self-reference, such as the Liar, are profound—so profound that after over 2,000 years of investigation they are still capable of producing novel ideas. One of the most significant of these in the last decade or so has been the attempt to solve them by rejecting some of the structural rules of inference, most notably the rules of Cut or Contraction.[1] Doubtless further research in the area will produce a deeper understanding. However, the point of the present paper is to discuss how these approaches to the paradoxes relate to more usual dialetheic accounts.[2]

[1]See, e.g., Cobreros, Égré, Ripley, and van Rooij (2013a); Ripley (2012), (2013), (2015a); Zardini (2011); Mares and Paoli (2014).

[2]For example, that of Priest (1979), (1987).

In the first part of the paper, I will lay out the basics of these solutions. (The views could be formulated in various different ways. In the cause of a uniformity, my presentation of the positions involves a certain amount of regimentation, though, I think, a harmless one.) The next section gives some general considerations which provide a segue into the second half of the paper. In the two sections of this, we will look at, first, the Cut-free and, second, the Contraction-free approaches to the paradoxes. In an appendix to the paper, I will consider the approach to the paradoxes which gives up Identity, which bears a close relation to giving up Cut. As ever in philosophy, I am sure that there is more to be said about many of these issues. However, one can do only so much in one paper.

Finally by way of introduction, a brief note on some related paradoxes. Substructural solutions to the sorites paradoxes have also been suggested.[3] Though vagueness will surface from time to time in what follows, I will not attempt to discuss this possibility here, since it raises a number of quite distinct considerations. For what it is worth, I think that a Cut-free solution is much more plausible in the case of the sorites than in the case of the paradoxes of self-reference. In sorites paradoxes, it is a very natural reaction to suppose (in a way that it is not for the Liar and Curry) that things go wrong precisely because of the chaining together a sequence of individually legitimate inferences—which is exactly what Cut does. Next, there is a very close cousin of the Curry paradox, the Validity Curry, which uses a predicate supposed to express validity, rather than a conditional. The extent to which such a paradox can be handled with a substructural logic has been the subject of no little recent discussion.[4] An adequate discussion of that matter would also require a separate paper. Finally, the paradoxes of semantic self-reference are clearly closely related to the set-theoretic paradoxes of self-reference. Whether such paradoxes can be addressed with a substructural logic is a much more complex matter. One major reason for this is that the Extensionality principle poses particularly tricky problems.[5]

[3] See, e.g., Zardini (2008), Cobreros, Égré, Ripley, and van Rooij (2013b), (2015); Égré, (2019).

[4] For an introduction to the matter, see Shapiro and Beall (2018).

[5] For a detailed discussion see Weber (2021).

2 Exegesis

2.1 The Sequent Calculus *LK*

So let us turn to the matters at hand, and let us start with a standard formulation of the sequent calculus for classical logic. Following Gentzen, I will call this *LK*, though it should be noted that it differs in various ways from Gentzen's original version. (I choose the present way of setting things up to make what follows smoother.) For the time being, the language is that of the propositional calculus, with the usual connectives. Upper case Latins, A, B, etc, are schematic variables for sentences of the language; and upper case Greeks, Γ, Δ, etc., are schematic variables for finite multisets of formulas of the language. As usual, an expression of the form Γ, Δ, A represents the multiset $\Gamma \cup \Delta \cup \{A\}$. (So $\{A, B\}, B = \{A, B, B\}$.)

Sequents are things of the form $\Gamma \Rightarrow \Delta$. One might think of such an expression as saying, roughly, that from all of the members of Γ, some members of Δ follow. We will write $\emptyset \Rightarrow \Delta$ as $\Rightarrow \Delta$. (Similarly on the right hand side of $\Rightarrow$.)[6]

The calculus has only one axiom (schema), Identity (Reflexivity):

$$A \Rightarrow A$$

Each connective has a pair of rules. One introduces the connective on the left of the sequent; the other on the right. These are as follows:

$$\frac{\Gamma, A \;\Rightarrow\; \Delta}{\Gamma \;\Rightarrow\; \neg A,\ \Delta} \qquad \frac{\Gamma \;\Rightarrow\; \Delta, A}{\Gamma, \neg A \;\Rightarrow\; \Delta}$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad \frac{\Gamma_1 \Rightarrow A, \Delta_1 \quad \Gamma_2 \Rightarrow B, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow A \wedge B, \Delta_1, \Delta_2}$$

$$\frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} \qquad \frac{\Gamma_1, A \Rightarrow \Delta_1 \quad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \vee B \Rightarrow \Delta_1, \Delta_2}$$

$$\frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, A \quad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \rightarrow B \Rightarrow \Delta_1, \Delta_2}$$

[6] Gentzen's formalization of Intuitionistic Logic, *LJ*, restricts sequents to those where Δ has at most one member. A similar restriction gives "intuitionist" versions of all the substructural logics we shall meet in what follows. However, no one (as far as I am aware) has suggested using these in a solution to the semantic paradoxes. The loss in power seems to deliver no corresponding gain.

There are then three structural rules:

Weakening: $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$ $\quad$ $\dfrac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}$

Contraction: $\dfrac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$ $\quad$ $\dfrac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}$

Cut: $\dfrac{\Gamma_1 \Rightarrow \Delta_1, A \quad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$

We will write $\vdash_{LK} \Gamma \Rightarrow \Delta$ to meant that one may infer the sequent $\Gamma \Rightarrow \Delta$ from these axioms and rules.[7]

Two standard and well-known results concerning this sequent calculus are the following:

- Soundness and Completeness: $\vdash_{LK} \Gamma \Rightarrow \Delta$ iff $\Gamma \models_{CL} \Delta$, where CL is classical (multiple conclusion) logic.

- *Hauptsatz* (Cut Theorem): If $\vdash_{LK} \Gamma \Rightarrow \Delta$, there is an LK proof of the sequent without Cut.

2.2 Strict Tolerant

Let us next turn to the first of the two substructural logics that will concern us. This is called Strict/Tolerant (ST), and is obtained by simply dropping Cut from

[7]Note that the rules for $\vee$ and $\wedge$ are for what are called the *multiplicative/intensional* connectives. In the presence of Contraction and Weakening, these are equivalent to the simpler forms:

$$\frac{\Gamma, A \text{ (or } B) \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, A \text{ (or } B)}{\Gamma \Rightarrow \Delta, A \vee B} \qquad \frac{\Gamma, A \Rightarrow \Delta_1 \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}$$

(and the multisets reduce to sets). Without the two structural rules, these rules characterise a different pair of connectives called *additive/extensional*. These connectives are often added in Contraction-free logics as well. I will mention them briefly later, but for the most part they will not concern us.

It is worth noting that the conditional here is also multiplicative/intensions. There is also an additive/intentional conditional, which builds in contraction. Perhaps unsurprisingly, this allows for a version of Curry's Paradox (as noted by Hjortland (2017)).

LK. In virtue of the Cut Theorem, we have immediately that $\vdash_{ST} \Gamma \Rightarrow \Delta$ iff $\models_{CL} \Gamma \Rightarrow \Delta$.

A semantics for the language can be obtained by employing those of the 3-valued logics K_3 and LP.[8] According to these, there are three semantic values, which we may write as $1, 0.5, 0$. Given a valuation, v, let us write the value of A under v as $|A|$. Then $|A \wedge B|$ is the minimum of $|A|$ and $|B|$; $|A \vee B|$ is the maximum; $|\neg A|$ is $1 - |A|$. $A \to B$ is $\neg A \vee B$.[9]

v *strongly verifies* A if $|A| = 1$ (as in K_3). And v *weakly verifies* A if $|A| \geq 0.5$ (as in LP). And, v *verifies* $\Gamma \Rightarrow \Delta$ if:[10]

- if (materially) v strongly verifies every member of Γ, it weakly verifies some member of Δ.

As is easy to check, all the rules except Cut preserve verification. However, Cut does not. Merely consider a valuation where $|A| = 1$, $|B| = 0.5$, and $|C| = 0$. This verifies $A \Rightarrow B$ and $B \Rightarrow C$, but not $A \Rightarrow C$.

Let $\mathcal{S} \models_{ST} \Gamma \Rightarrow \Delta$ mean that every evaluation that verifies every member of $\mathcal{S}$ verifies $\Gamma \Rightarrow \Delta$. It is not difficult to show that $\vdash_{ST} \Gamma \Rightarrow \Delta$ iff $\models_{ST} \Gamma \Rightarrow \Delta$. So the ST semantics are weakly sound and complete with respect to the ST proof theory. However, let us write $\mathcal{S} \vdash_{ST} \Gamma \Rightarrow \Delta$ to mean that there is a deduction of $\Gamma \Rightarrow \Delta$ when the members of $\mathcal{S}$ are added as axioms to ST. Then since all the rules of ST preserve verification, if $\mathcal{S} \vdash_{ST} \Gamma \Rightarrow \Delta$ then $\mathcal{S} \models_{ST} \Gamma \Rightarrow \Delta$. The converse, however, does not hold, as we shall see in due course. So the semantics is strongly sound, but not strongly complete.

So warning: ST is sometimes defined semantically, though, in virtue of this, the semantics are not completely equivalent. I have set things up here in terms of the proof-theoretic charactersation of ST to make the relation to the Contraction-free logics clearer.

Finally, let us note a connection between ST and LP. Let $(\Gamma \Rightarrow \Delta)^{\to}$ be $\bigwedge_{\gamma \in \Gamma} \gamma \to \bigvee_{\delta \in \Delta} \delta$.[11] Then it is not hard to check that v verifies $\Gamma \Rightarrow \Delta$ iff v assigns

[8] See, e.g., Priest (2008), ch. 3.

[9] Of course, the numerical value 0.5 has no semantic significance. It is just a way of referring to a third semantic value in such a way that the truth conditions may given in a compact form. What the third value means—if anything—is a different matter, as we shall note in due course.

[10] This is to think of the sequents as expressing what is sometimes called *local validity*, that is, truth-preservation. Another possibility is to take them as expressing global validity, that is, validity-preservation. Matters are then somewhat different, though this is not the place to go into this.

[11] How to understand this when Γ or Δ is empty? A simple way is to suppose that the language

$(\Gamma \Rightarrow \Delta)^{\rightarrow}$ a value ≥ 0.5. That is, it is designated in *LP*. Hence one may think of verified sequents as designated *LP* material conditionals. From an *LP* perspective, the failure of Cut is, then, the failure of transitivity (and ultimately *modus ponens*) for the material conditional of *LP*. Moreover, let $\mathcal{S}^{\rightarrow}$ be $\{(\Gamma \Rightarrow \Delta)^{\rightarrow} : \Gamma \Rightarrow \Delta \in \mathcal{S}\}$. Then it follows that $\mathcal{S} \models_{ST} \Gamma \Rightarrow \Delta$ iff $\mathcal{S}^{\rightarrow} \models_{LP} (\Gamma \Rightarrow \Delta)^{\rightarrow}$. So semantically ST is just a fragment of *LP*.[12]

2.3 Affine and Linear Logic

Let us turn to our second substructural logic—or actually, logics, since there are two of them. The first of these is Affine Logic (*AL*). This is obtained from *LK* by dropping Contraction. The second is the somewhat better known Linear Logic (*LL*) which drops both Contraction and Weakening.[13] For present purposes, these two logics are much the same, though of course *AL* is stronger. In what follows, '*X*' may be either '*A*' or '*L*'.[14]

Both logics retain Cut; but for both there is a Cut theorem:

- if $\vdash_{XL} \Gamma \Rightarrow \Delta$ then there is an *XL* proof of $\Gamma \Rightarrow \Delta$ without Cut.[15]

As one might expect, without contraction, the conditional itself does not contract. That is:

- $\nvdash_{XL} A \rightarrow (A \rightarrow B) \Rightarrow (A \rightarrow B)$

contains two logical constants, $\top$ and $\bot$, governed by the axioms $\Rightarrow \top$ and $\bot \Rightarrow$. $\bigwedge \emptyset$ and $\bigvee \emptyset$ are then $\top$ and $\bot$, respectively.

[12] As noted, in effect, by Barrio, Rosenblatt, and Tajer, (2015), and Cobreros, Égré, Ripley, and van Rooij (2020). In particular, then $\{\emptyset \Rightarrow A_1, ..., \emptyset \Rightarrow A_n\} \models_{ST} \emptyset \Rightarrow B$ iff $\{A_1, ..., A_n\} \models_{LP} B$. So *LP* is the "external" consequence relation of ST.

[13] *LL* first appeared in Girard (1987). *AL* first appeared (as far as I know) in Grišin (1981). These logics are often formulated with both the extensional and intensional connectives. It is also standard to include certain "exponential" connectives. For the most part, these will not concern us, though they will play an important role towards the end of this essay.

[14] A solution to the paradoxes which endorses the propositional logic *LL* is given by Mares and Paoli (2014); one that endorses *AL* is given by Zardini (2011).

[15] There is a sense, then, in which *LL* has no structural rules. The Lambek Calculus goes even further. This gives up all structural rules including Associativity, which is implicit in taking premises and conclusions to be multisets. In the Lambek Calculus, these are taken to be sequences. No one, as far as I know, has suggested using the Lambek Calculus to solve the semantic paradoxes. The very weakness of the logic makes its natural home in the domain of grammatical parsing. See Moortgat (2010).

What might not be expected, is that connectives other than $\rightarrow$ are also affected. For example, distribution fails:

- $\nvdash_{XL} A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$

(To get from $A \wedge (B \vee C)$ to $(A \wedge B) \vee (A \wedge C)$, the premise has to be used twice. Once to get the $B \vee C$; then again to get the As to add to each disjunct.) Moreover:

- $\nvdash_{XL} A \Rightarrow A \wedge A$

Indeed, the last of these is equivalent to Contraction. For suppose that $\Gamma, A, A \Rightarrow \Delta$. Then $\Gamma, A \wedge A \Rightarrow \Delta$. Hence, given $A \Rightarrow A \wedge A$, we can cut to get $\Gamma, A \Rightarrow \Delta$.

Both AL and LL can be given various semantics, including world-semantics.[16] However, they are more complicated than those for ST, and we need not go into them here.

2.4 Adding Truth

Quantifiers can be added to our substructural logics, though the failure of Contraction gives rise to some tricky issues in XL. Again we need not go into this matter here. Since the semantic paradoxes are our topic, what we do need to consider is the addition of a truth predicate. So now we take it that our atomic formulas may be of the form $T\langle A\rangle$, where T is a monadic predicate, and for any formula of the language, A, $\langle A\rangle$ is a constant (thought of as the name of A). We also assume that the language contains some form of self-reference, so that there are sentences that can contain their own name. For the predicate T, we add the rules:

$$T\text{-Rules}: \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, T\langle A\rangle} \qquad \frac{A, \Gamma \Rightarrow \Delta}{T\langle A\rangle, \Gamma \Rightarrow \Delta}$$

Let us call our substructural logics, thus augmented, ST_T and XL_T. It can be shown that the Cut Theorem holds for XL_T.

Moreover, it can be shown that truth behaves transparently in these logics. That is, if A is any formula and ζ is a sequent, let $\zeta_{A/T\langle A\rangle}$ be ζ with occurrences of A replaced by $T\langle A\rangle$. Then for any of our logics:

- $\vdash \zeta$ iff $\vdash \zeta_{A/T\langle A\rangle}$

[16] See, e.g., Avron (1988), Troelstra (1992), ch. 8, and Allwein and Dunn (1993).

We can now see that standard paradoxical arguments, notably those of the Liar and Curry, fail in these logics. Here are the standard proofs of each. We note where Cut and Contraction (and for good measure, Identity) are used.

For the Liar, let L be a sentence of the form $\neg T\langle L\rangle$. Then the argument is:

$$
\begin{array}{ccc}
L \Rightarrow L & L \Rightarrow L & \text{Identity} \\
T\langle L\rangle \Rightarrow L & L \Rightarrow T\langle L\rangle & \\
T\langle L\rangle \Rightarrow \neg T\langle L\rangle & \neg T\langle L\rangle \Rightarrow T\langle L\rangle & \\
\Rightarrow \neg T\langle L\rangle, \neg T\langle L\rangle & \neg T\langle L\rangle, \neg T\langle L\rangle \Rightarrow & \\
\Rightarrow L, L & L, L \Rightarrow & \\
\Rightarrow L & L \Rightarrow & \text{Contraction} \\
\hline
\multicolumn{2}{c}{\Rightarrow} & \text{Cut} \\
\multicolumn{2}{c}{\Rightarrow A} &
\end{array}
$$

(The last step also fails in LL_T, since it applies Weakening.)

For Curry, let A be any sentence one wishes, and let C be a sentence of the form $T\langle C\rangle \to A$. Then the argument is as follows:

$$
\begin{array}{ccc}
\vdots & \dfrac{T\langle C\rangle \Rightarrow T\langle C\rangle \quad A \Rightarrow A}{T\langle C\rangle, T\langle C\rangle \to A \Rightarrow A} & \text{Identity} \\
T\langle C\rangle \Rightarrow A & & \\
\Rightarrow T\langle C\rangle \to A & T\langle C\rangle, T\langle C\rangle \Rightarrow A & \\
\Rightarrow T\langle C\rangle & T\langle C\rangle \Rightarrow A & \text{Contraction} \\
\hline
\multicolumn{2}{c}{\Rightarrow A} & \text{Cut}
\end{array}
$$

The ellipsis on the left hand proof represents the first three lines of the right hand proof.

Indeed, using Cut-freedom, it is not difficult to see that none of our T-augmented logics is trivial. That is:

- $\nvdash \Rightarrow$

Hence, we are assured that similar paradoxes are not lurking in the wings.

3 Approach to an Evaluation

Before we turn to a discussion of our two approaches, let us note two matters of orientation.

3.1 Arguments For

Various arguments are offered for the two substructural approaches to the paradoxes, and this is not the place to go into the ins and outs of these. But let me discuss briefly what I think is the most persuasive motivation for such an approach. This is the fact that it allows essentially the same solution to the Liar and the Curry paradoxes. By contrast, the usual dialetheic solutions to the Liar reject Explosion; whilst they solve Curry by appealing to some feature of the conditional.[17]

Now, one cannot deny that the unification to solutions this would provide is a virtue, as are all forms of simplicity. However, it is but one virtue amongst many, and as an argument for such a solution it is a weak one. Thus, a dialetheic solution can be given for the Liar paradox and Zeno's arrow paradox.[18] But this fact, on its own, hardly speaks in favour of a dialetheic solution to either of them.

Of course, if these paradoxes are manifestations of essentially the same phenomenon then one *should* expect the same solution. This may be called the Principle of Uniform Solution: same kind of paradox, same kind of solution.[19] Those who endorse substructural solutions tend to assume, without argument, that they are of the same kind. This is not at all obvious, however. Both the Liar and Curry involve self-reference; but there are paradoxes of self-reference that are clearly of a quite different kind; for example, some versions of the surprise exam paradox.[20] True, these do not involve truth, but they do involve knowledge; and there are paradoxes concerning knowledge which pretty clearly belong in the same class as the Liar, such as the Knower paradox.[21]

Indeed, an appeal to the Inclosure Schema shows that the Liar and Curry are not the same kind of paradox. That the Schema captures the essence of a certain kind of self-referential paradox is certainly contentious, and this is not the place to go into the matter in detail here.[22] But some important facts are easy

[17] And the usual gap-theoretic solutions to the Liar give up Excluded Middle, whilst they solve Curry by appealing, again, to some feature of the conditional. See Field (2008). Égré and some of the others who endorse substructural solutions also take these to apply to sorites paradoxes, and so to deliver a further unified approach. In fact, I agree that the sorites paradoxes are inclosure paradoxes, and so endorse the same kind of solution to these that I endorse for the Liar. (See Priest (2019).) However, as I have already said, paradoxes of vagueness are not on the agenda in this essay.

[18] See Priest (1987), chs. 1, 12.

[19] See Priest (2005a), 11.5, 17.6.

[20] Halpern and Moses (1986).

[21] See Priest (2005a), 10.2.

[22] For references and discussion, see Priest (2017), §15.

to grasp. First, paradoxes are not statements. They are *arguments*. The Liar argument allows us to establish something which one may be inclined to reject, namely a contradiction. But one can use the Curry argument to establish any conclusion—even one that is obviously true, such as that 1+1=2, that London is the capital of England, and so on. Clearly, one ought not to be able to establish such things by this kind of argument, even though one is (hopefully!) already inclined to accept them. The conclusion, acceptable or otherwise, is playing no role in what is problematic about the argument. This is enough to suggest that there is something quite different going on in the two arguments—the Liar and Curry.[23]

3.2 Arguments Against

Let us now turn to the other side of the matter: arguments against the substructural approaches to the paradoxes. Those to be considered in what follows may be broken up into four separate but connected issues. Let me say something general about each of these.

1. *Why should one reject the substructural principles in question?* Breaking paradoxes is easy. Simply write out the argument; choose any premise or inference employed. Deny its truth/validity. Of course, such a procedure is entirely unsatisfactory. Without a rationale for the move, it is completely *ad hoc* and uninformative. Nor does the fact that the premise/inference is involved in the supposedly unacceptable conclusion provide any suitable ground: the same can be said of any possible choice.

2. *Can one live without them?* Rejecting the substructural principle blocks the conclusion. But of course, it blocks much more. As those who endorse these accounts of the paradoxes are well aware, we use these principles, and apparently legitimately, on many occasions. How so? This is a crucial question. Blocking triviality shows that rejecting a form of inference does enough; but the question is whether it does too much. Unless one is to be a radical revisionist (as in intuitionist philosophy), one must give an

[23]Of course, one can infer anything using the liar argument if one appeals to Explosion at the end. But this is extrinsic to the argument, as is witnessed by the fact that it played no role in discussion of the argument for over 2,000 years. Relatedly, a referee demurred at this point since, they said, the two deductions of 2.4 are 'isomorphs'. This strikes me as false: the arguments are quite different. For example, in the case of the Liar, the argument to an arbitrary conclusion uses Weakening; not in the case of Curry.

account of how it is that the rejected principle is used correctly on many an occasion.[24]

3. *Are there paradoxes of the kind in question which proceed without them?* Rejecting the principles in question breaks the paradoxical arguments in question, but are there others of the same kind? Of course, as we noted, one cannot prove $\Rightarrow A$, for arbitrary A, with the machinery to hand. But could there be other legitimate machinery which can be added to allow one to do so? For example (though we will not pursue the matter here), the paradoxes of denotation, such as Berry's, use definite description terms essentially. An appropriate sequent-calculus treatment of these is not at all obvious; neither, then, is the fact that they can be avoided by getting rid of substructural rules.[25]

4. *Is this simply a form of dialetheism?* And when all is said and done, notwithstanding all these matters, contradictory conclusions of certain kinds have not been avoided. Do we have just a new logic for dialetheism—or even an old one in new guise?

Let us address each of these questions with respect to our two substructural accounts, starting with ST.

4 ST

4.1 Why Reject Cut?

Why should one reject Cut? Semantically, as we have seen, the rationale is that we use different designated values for the premises of an inference (the antecedents of a sequent) and the conclusions (the succedents of the sequent). What could motivate the change? A rationale is sorely needed. You can get any kind of argument to break down if you allow some relevant things to change, such as the denotations of terms employed. (Aristotle was a philosopher; Jackie Kennedy married Aristotle; so Jackie Kennedy married a philosopher.) But the standard understanding

[24] This has been a clearly understood feature of dialetheism about the paradoxes right from the start, and much attention has been paid to the matter. See, e.g., Priest (1979), §4, Priest (1987), chs. 8 and 16 (2nd edn).

[25] Indeed, many discussions of the semantic paradoxes miss the fact that it is not clear how to apply the mechanism of the proffered solution when descriptive terms are involved. See Priest (2006).

of logical consequence is that the relevant parameters are not allowed to move around under one's feet. In particular, terms in the premises and conclusion are not allowed to change their senses/referents; the truth of premises and conclusions must be understood in the same sense; if context is involved, it must be held fixed. There is a standard name for inferences which violate this constraint: they are called *fallacies of equivocation*.

In the present case, what this would seem to entail is that what it means for the premises and conclusions to hold should be interpreted in the same way. If not, things start to look very arbitrary. Why not a notion of validity according to which, if the premises are true in an infinite model, the conclusion is true in a finite model; or if the premises are true in a model of size 36, the conclusion is true in a model of size 37? For the premises to hold in *some* sense, and the conclusion to hold in *some other* sense, just seems to be changing the subject.

It would seem, then, that we should stick to sequents that preserve the value 1, giving the logic K_3, or the values ≥ 0.5, giving LP. These logics, like any many-valued logic in which validity is defined in terms of the preservation of one of a set of designated values, verifies Cut.

In an attempt to justify the switch, Égré argues as follows:[26]

1. Truth is a vague predicate like 'drunk', 'adult'.

2. Truth for statements with vague predicates come by degrees, say in [1,0].

3. These may be partitioned into three sections, $[0,\alpha]$,(α,β),$[\beta, 1]$—where α and β are contextually determined—meaning *true* (enough), *borderline true*/false, *false* (enough).

4. The three ST values can be interpreted as falling within the three parts of the partition.

5. A valid inference may allow one to move from a formula to one with a lesser degree of truth.

6. So may move from the ST value 1 to 0.5.

That sentences with vague predicates have a degree of truth in $[0, 1]$ is certainly contestable, though plausible. I see no reason to believe that the truth predicate is vague—at least when all the other predicates in the language are precise, as they

[26]See, e.g., Égré (2019).

are if they are those of arithmetic (which is all you need to get self-reference).[27] Shoehorning the continuum of values into a trichotomy seems arbitrarily procrustean, and in any case may not produce values that behave like ST. Thus, depending on the context, A and $\neg A$ may both be in $[\beta, 1]$.[28]

But the main problem is with 5. Once the acceptable degree of truth is fixed, we still want for a reason as to why we should endorse a notion of validity which allows for it to be decreased.

Discussing the sorites paradox, Zardini (not wearing his Contraction-free hat) suggests that when reasoning with vague sentences, we have a practice of reasoning, and so an understanding of a perfectly legitimate notion of validity—which takes us from good (or very good) premises to conclusions that are 'good enough'.[29] Exactly what 'good enough' means is not spelled out. But whatever it means, if we are allowed to use inferences which *decrease* the degree of truth or acceptability, then they may well take us *below* the level of what is good enough in one application.[30]

Indeed, on the degree of truth approach, it is precisely the fact that we are wont use degree-of-truth-decreasing inferences which generates the unacceptable conclusion of the sorites. That is exactly why we should *not* use them. Such a practice, if we have one, is, then, a bad practice, and the corresponding notion of validity is an incorrect one.

In other words, once the set of acceptable degrees of truth (or acceptability) is fixed by the context, we should be interested in inferences that preserve membership of that set.[31] Nor is it relevant to point out that the degree of acceptability is

[27] And even if there are clearly vague predicates in the language, truth could still be a crisp predicate. It could be the case that $T\langle A\rangle$ is completely true if A is completely true, and completely false otherwise. One might object that $T\langle A\rangle$ should have the same degree of truth as A. So if A has a non-extreme value, so does $T\langle A\rangle$. But if L is the Liar sentence, $\neg T\langle L\rangle$, this obviously gives us no non-question-begging reason to suppose that L does not have an extreme value.

[28] Of course, the desired results can be obtained by putting appropriate constraints on α and β. In particular, for negation to be a fixed point of the trichotomy, it must be the case that $\alpha \leq 0.5 \leq \beta$. But imposing constraints of this kind appears to be incompatible with the fact that acceptability is a contextual matter, and not guaranteed to obey any such constraints.

[29] Zardini (2008), pp. 347-9.

[30] Priest (1998) defines a notion of local validity which allows for a drop in truth value. But as is pointed out there (p. 335), that an inference is locally valid does *not* justify using it.

[31] In Łukasiewicz' continuum-valued semantics, one sometimes sees a valid inference characterised as one which is degree-of-truth-non-decreasing. That is, an inference $A \vdash B$ (one premise, one conclusion, for simplicity), is valid if for every valuation, $|A| \leq |B|$. This is not the same as the ST strategy. It is just a way of characterising inferences that preserve designated values (which are upward closed) whatever they are. (See Priest (2008), 11.4.10, 11.4.11.) And it satisfies Cut.

context-dependent. Context-change is a standard fallacy. (Consider merely the effect of a geographical context change on the inference: it's midnight so it's dark.)

Moving from the semantic characterisation of validity to a proof theoretic one, Ripley has suggested that one should understand a sequent $\Gamma \Rightarrow \Delta$ as saying that it is incorrect ("out of bounds") to assert every member of Γ and deny every member of Δ.[32] Cut then fails. To see why, consider again a one premise/conclusion Cut:

$$\frac{A \Rightarrow B \quad B \Rightarrow C}{A \Rightarrow C}$$

Suppose one asserts A; then one cannot deny B. But it does not follow that one must assert B, and so must deny C. Maybe B is neither deniable nor assertible.

However, such an understanding of inference seems quite inadequate. Suppose that I endorse A (maybe as an axiom of some system), and I infer B. This tells me more than that I simply can't deny B. The inference shows that I am committed to it. That is, I must assert it. ('Yea, I know that B follows from A, and that I endorse A, but I refuse to endorse B'!)[33]

I think it fair to assume that by 'assert' here Ripley means what it is standardly taken to mean. (There is no comment otherwise.) That is, to assert A is to utter it with the intention that hearers, thereby, come to believe A, or at least believe that the utter believes A.[34] Later in the same paper (p. 153) Ripley rechristens this *strict assertion*, and introduces a different notion of assertion, which he calls *tolerant assertion*. One may then suggest that if one strictly asserts all member of Γ, one may at least tolerantly assert some member of Δ.

I confess that don't really understand the notion of tolerant assertion. Exactly what linguistic act constitutes it, and what is its effect supposed to be?[35] Outwith some clearer understanding of the notion, it is hard to see whether this move is of any help. But at least the following seems clear.

Either the tolerant assertion of A is meant to deliver some sort of commitment to A or not. Suppose it does not; then we still have no commitment to the logical

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[32]Ripley (2013).

[33]See Rumfitt (2008), p. 80. Another way of bringing the point home was suggested to me by Nils Kürbis. For any A, it certainly seems to be incoherent to assert A and deny 'I assert A'. So on this account of validity, 'I assert A' follows from A, which it clearly does not.

[34]For a more careful formulation, see Priest (1987), 4.6.

[35]In the paper (p. 153), Ripley says that one is entitled to make a tolerant assertion when one is not entitled to make strict denial; but he tells me in correspondence that he now thinks this to be false. A referee suggested that one might understand it as the appropriate way to assert a borderline case. If so, this just takes us back to the matter of the fallacies involved in soritical reasoning above.

consequences of the things to which we are committed. Suppose it does. Then we still want for a reason as to why a valid inference should allow us to settle for a lesser kind of commitment than the one to which we are committed to the premises. Anything less just does not seem to be the kind of thing we want.[36] To illustrate, suppose that a tolerant assertion at least rules out a strict denial. A person who takes the truth or otherwise of a certain claim to be beyond human cognition will certainly rule out denying it; but this delivers no endorsement of it in any interesting sense.

Finally, Cobreros, Égré, Ripley, and van Rooij point out that without Cut one can endorse other things that one might like: the validity of the tolerance principle for vague predicates—they might equally have said the unrestricted T-Schema—and the usual left and right rules for the conditional ('the deduction theorem, and *modus ponens*') whilst avoiding paradox.[37] It is clear that such a justification for rejecting Cut provides no rationale independent of its ability to avoid supposedly unacceptable conclusions. Moreover, one might point out, equally, that if one rejects the usual rules for the conditional, one can have the other things one might like: an unrestricted T-schema and Cut. And given that the material conditional—which the usual conditional rules deliver—has always been held problematic, unlike Cut (aka the transitivity of deduction), one might well prefer the latter choice.[38]

[36] Incurvati and Schlöter (2018) argue for a notion of weak assertion. According to them, to weakly assert A is similar to asserting 'Perhaps A', and prevents $\neg A$ being added to the common ground of the conversation (p. 755). I do not know whether they take this notion to be the same as Ripley's tolerant assertion; but if it is, it just makes my point. If it is agreed that one may infer a theorem, T, from the axioms of a mathematical theory, 'perhaps T' is not the appropriate response.

[37] Cobreros, Égré , Ripley, and van Rooij (2013b), p. 373.

[38] Zardini (2015) has a sophisticated defence of the possibility if the non-transitivity of logical consequence in general. This is not the place to discuss the many considerations he mounts. However, in §2 he gives three reasons one might reject transitivity. 1: Validity should hold in virtue of some commonality of content between premises and conclusion. 2: Transitivity can naturally be thought of as failing in sorites arguments. 3: Transitivity naturally fails in inductive (aka non-monotonic inferences). Let me make three very brief comments about these. *Re 1*: that validity (as opposed to, say, the truth of a conditional) should require commonality of content is contentious. And in any case, such relations can be (though they need not be) transitive. *Re 2*: I have set aside issues to do with the sorites here, but the discussion of Égré above makes a number of relevant comments. *Re 3*: ST is a monotoic (deductive) logic, not an inductive one.

4.2 Can Logic Function without Cut?

But can one reasonably reject Cut anyway? We seem to use it all the time. Thus, given some axioms, Σ, we deduce lemma A, $\Sigma \Rightarrow A$, and then use this to infer theorem T, $\Sigma, A \Rightarrow T$. We then apply Cut to infer the theorem, $\Sigma \Rightarrow T$. At the very least, we are owed a cogent story of how and why.

It may be pointed out that if $\Sigma \Rightarrow A$ and $\Sigma, A \Rightarrow T$ are classically valid, then so is $\Sigma \Rightarrow T$. The trouble, though, is that we seem to apply Cut legitimately when this is not the case. For example, suppose we assume that $p \Rightarrow q \wedge r$. Despite the fact that $\vdash q \wedge r \Rightarrow q$, we cannot cut to infer that $p \Rightarrow q$. Nor can this be inferred from $p \Rightarrow q \wedge r$ without it. (There is no rule for eliminating a conjunction.) As we will note in due course, the various elimination rules can be added to the sequent calculus without adding Cut. But the general point remains: reasoning from assumptions often involves Cut, and legitimately so. Patching up the sequent calculus in various ways can only be a partial fix.[39]

One way to see this is to note the following. Since $\Sigma, A \Rightarrow B$ is equivalent to $\Sigma \Rightarrow A \rightarrow B$, the failure of Cut is equivalent to the failure of *modus ponens*.[40] In particular, suppose that we have, either by assumption or by proof, that $\Sigma \Rightarrow A$ and $\Sigma \Rightarrow A \rightarrow B$. Then we cannot infer $\Sigma \Rightarrow B$ without Cut. Thus, *modus ponens* fails, even though $\rightarrow$ has its standard inferential meaning, as specified by its left and right sequent rules. *Modus ponens* is ubiquitous in reasoning from assumptions/axioms.

It might be pointed out that in dialetheic *LP*, *modus ponens* fails anyway. But of course, Cut holds in *LP*. Moreover, it is the fact that *modus ponens* fails for the material conditional which motivates the standard suggestion that the language of *LP* be augmented with a new conditional which does satisfy *modus ponens*. Perhaps the substructuralist could also add a new conditional to the language, but this would appear to undercut the ST strategy to the paradoxes.

In fact, the failure of Cut is more problematic than any of these considerations show. It means that valid inference cannot function as a closure operator on a set of beliefs. It is standard to define a theory as a set of sentences, Σ, closed under deducibility. That is, if $\Sigma \vdash A$ then $A \in \Sigma$. Now, as far as real-life theories, like the

[39] Or can deliver disaster. Murzi and Rossi (2020a) give rules for an unparadoxicality predicate which allow one to perform a cut on A on the assumption that it is not paradoxical. The rub is that closing ST plus a truth predicate under these rules gives triviality.

[40] It should be noted that the notion *modus ponens* can be articulated in a number of different ways. (See, e.g., Ripley (2015a), §4.2.) Claiming that one of these is the real *modus ponens*, strikes me a senseless. What I have in mind by the term is clear enough in what follows. On the fact that *modus ponens* delivers Cut, see Negri and von Plato (2001), p. 19

Theory of Evolution, are concerned, this is a logicians' fiction. No scientist ever thinks of them in this way. The point of the fiction of closure is simply that if we are committed to some things, we are committed to whatever they entail. Now, when we have a set of beliefs or of information, we are not normally interested in the inferential relations within the set. Probably, the set could be axiomatized in many different ways, for example. We are just interested in what information can be extracted from it. Thinking of a theory as a closed set of sentences is a way of conceptualising this fact.[41]

Now, suppose that Cut fails for some set of premises, Σ. That is, for some A and B, Σ entails A, and Σ, A entails B, but Σ does not entail B. If Σ were closed, we would have $A \in \Sigma$. But then Σ entails B, so $B \in \Sigma$, and Σ entails B, contrary to hypothesis. So Σ cannot be closed.

In other words, logic cannot perform its function (or this one of its functions, if it has more than one) if Cut fails.

4.3 Higher Level Paradoxes

Let us turn to the question of whether one can obtain paradoxes of self-reference with additional machinery. One way to do this is by ascending a level and considering meta-inference relations.[42]

Specifically, let us now write $\Rightarrow$ as $\Rightarrow_0$, and consider the propositional language which contains this as an additional connective. Call this language $\mathcal{L}_0$. We may then consider sequents of the form $\eta_1, ..., \eta_n \Rightarrow_1 \xi_1, ..., \xi_m$, where the ηs and the ξs are formulas of $\mathcal{L}_0$. We add names for sequents of the lower level to the ground language, and take as rules:

$$\frac{\eta \Rightarrow_1 \xi}{(\emptyset \Rightarrow_0 T\langle\eta\rangle) \Rightarrow_1 \xi} \qquad \frac{\eta \Rightarrow_1 \xi}{\eta \Rightarrow_1 (\emptyset \Rightarrow_0 T\langle\xi\rangle)}$$

$\emptyset \Rightarrow_0 T\langle\eta\rangle$ says $T\langle\eta\rangle$ holds, in whatever sense it is in which the consequent of a sequent operator holds.[43] So these rules are simply a version of the transparency

[41] It might be suggested that it is more appropriate to think of the transitive closure in this connection. But if this account is correct, the transitive closure can produce exactly things that are *not* acceptable.

[42] These observations were triggered by some as yet unpublished work of Brian Porter on the Validity Curry. Brian observed that even if jettisoning some substructural rules voids the arguments at one level, they appear at the next.

[43] There is, note, no problem about applying the truth predicate to sequents. These are, after all, simply metatheoretic conditionals, with their own verification conditions.

of truth in the antecedent and the consequent, respectively. If we now assume the *LK* rules for level $\Rightarrow_1$-sequents, we have the following versions of the Liar and Curry paradoxes. Let λ be a formula of the form: $\neg(\emptyset \Rightarrow_0 T\langle\lambda\rangle)$. Then we have:

$$
\begin{array}{c}
\dfrac{\begin{array}{c}
\lambda \Rightarrow_1 \lambda \\
(\emptyset \Rightarrow_0 T\langle\lambda\rangle) \Rightarrow_1 \lambda \\
(\emptyset \Rightarrow_0 T\langle\lambda\rangle) \Rightarrow_1 \neg(\emptyset \Rightarrow_0 T\langle\lambda\rangle) \\
\Rightarrow \neg(\emptyset \Rightarrow_0 T\langle\lambda\rangle), \neg(\emptyset \Rightarrow_0 T\langle\lambda\rangle) \\
\Rightarrow_1 \lambda, \lambda \\
\Rightarrow_1 \lambda
\end{array}
\qquad
\begin{array}{c}
\lambda \Rightarrow_1 \lambda \\
\lambda \Rightarrow_1 (\emptyset \Rightarrow_0 T\langle\lambda\rangle) \\
\neg(\emptyset \Rightarrow_0 T\langle\lambda\rangle) \Rightarrow_1 (\emptyset \Rightarrow_0 T\langle\lambda\rangle) \\
\neg(\emptyset \Rightarrow_0 T\langle\lambda\rangle), \neg(\emptyset \Rightarrow_0 T\langle\lambda\rangle) \Rightarrow_1 \\
\lambda, \lambda \Rightarrow_1 \\
\lambda \Rightarrow_1
\end{array}}{\Rightarrow_1} \\
\Rightarrow_1 \zeta
\end{array}
\quad
\begin{array}{l}
\text{Iden} \\ \\ \\ \\ \\ \text{Con} \\ \text{Cut} \\ \\
\end{array}
$$

Similarly, let α be any sequent of $\mathcal{L}_0$, and let γ be a sentence of the form $(\emptyset \Rightarrow_0 T\langle\gamma\rangle) \to \alpha$. Then the argument is as follows:

$$
\dfrac{\begin{array}{c}
\vdots \\
(\emptyset \Rightarrow_0 T\langle\gamma\rangle) \to \alpha \Rightarrow_1 \alpha \\
\gamma \Rightarrow_1 \alpha \\
(\emptyset \Rightarrow_0 T\langle\gamma\rangle) \Rightarrow_1 \alpha \\
\Rightarrow_1 (\emptyset \Rightarrow_0 T\langle\gamma\rangle) \to \alpha
\end{array}
\qquad
\begin{array}{c}
\dfrac{\dfrac{(\emptyset \Rightarrow_0 T\langle\gamma\rangle) \Rightarrow_1 (\emptyset \Rightarrow_0 T\langle\gamma\rangle) \qquad \alpha \Rightarrow_1 \alpha}{(\emptyset \Rightarrow_0 T\langle\gamma\rangle) \to \alpha, (\emptyset \Rightarrow_0 T\langle\gamma\rangle) \Rightarrow_1 \alpha} \qquad \alpha \Rightarrow_1 \alpha}{(\emptyset \Rightarrow_0 T\langle\gamma\rangle) \to \alpha, (\emptyset \Rightarrow_0 T\langle\gamma\rangle) \to \alpha \Rightarrow_1 \alpha, \alpha} \\
(\emptyset \Rightarrow_0 T\langle\gamma\rangle) \to \alpha \Rightarrow_1 \alpha
\end{array}}{\Rightarrow_1 \alpha}
\quad
\begin{array}{l}
\text{Con} \\ \text{Cut}
\end{array}
$$

Of course, a natural move at this point would be to deny Cut at this level. But paradox then arises at the next (as one can see simply by increasing subscripts). Hence, one can avoid the higher level paradoxes only by rejecting Cut at all levels. Indeed, it would seem that this move is forced on the sub-structuralist by considerations of coherence. Whatever logic is used at the object level should be used at the metalevel.[44]

The problem with this is that we then have what Chris Scambler calls 'Tortoise Logic'.[45] In his well known paper 'What the Tortoise Said to Achilles',[46] Lewis

[44]Indeed, just as the insistence that the truth predicate for a language can occur only in a different language is an artifice, so is the claim that the connective $\Rightarrow$ for a language can only occur in a different language. What one really needs to employ, then, is a type-free language containing $\Rightarrow$. Such a language can be found in von Kutschera (1968). (An account of the calculus in English can be found in Wansing and Priest (2015), §4.) Von Kutschera's logic is not a sub-structural one. Sub-structural variations on this, what can and what cannot be done in them, require further investigation.

[45]Scambler (2020).

[46]Carroll (1895).

Carroll asks us to consider someone, a tortoise, who endorses $A \wedge (A \rightarrow Z)$ (call this B) but not Z. We may add as a premise $B \rightarrow Z$ (call this C), but the tortoise may then endorse this, and still refuse to endorse Z. We may then add the premise $C \rightarrow Z$. The tortoise may agree to this, but still not to Z. And so it goes on. Once *modus ponens* is rejected, one can have all the infinite number of premises of this form, but not accept Z.

The situation of rejecting Cut at every level is exactly the same. Suppose we have as premises A and $A \rightarrow Z$. Because of the failure of *modus ponens*, aka Cut, we need not have Z. We can add [I insert brackets to aid readability]:

- $[\Rightarrow_0 A, \Rightarrow_0 A \rightarrow Z] \Rightarrow_1 [\Rightarrow_0 Z]$

But for the same reason, given $\Rightarrow_0 A$, we may not have $\Rightarrow_0 Z$. We can then add:

- $[\Rightarrow_1 \Rightarrow_0 A, \Rightarrow_1 \Rightarrow_0 (A \rightarrow Z)] \Rightarrow_2 [\Rightarrow_1 \Rightarrow_0 Z]$

but may not have $\Rightarrow_1 \Rightarrow_0 Z$. And so on all the way up the natural numbers. In other words, we can have our premises, and a statement of the validity of Cut/*modus ponens* at all levels, without having anything that amounts to the conclusion of our premises, because it *actually* fails at all levels. Scambler (2020) nicely shows how this argument can be formulated in a precise model-theoretic way.

4.4 ST_T as a Version of Dialetheism

Let us turn to the final point. Notwithstanding any of the above, ST_T is a dialetheist view[47]—maybe even an old form in a new guise.

If L is the liar sentence, then $\vdash_{ST_T} \Rightarrow L$ and $\vdash_{ST_T} \Rightarrow \neg L$ (and so $\vdash_{ST_T} \Rightarrow L \wedge \neg L$). We can prove the Liar contradiction. True, even though $\vdash_{ST_T} L \wedge \neg L \Rightarrow A$, for arbitrary A, we cannot prove $\vdash_{ST_T} \Rightarrow A$ because of the failure of Cut. The failure of Cut is, then, exactly what we need to stop the contradiction exploding.[48]

It might be suggested that this does not mean that dialetheism is true, at least in a full-blooded sense, since this means that L has the value 0.5 in any model. That is, L is only "half true". So this is dialetheism-lite. Of course, one should not be carried away by numerological considerations. The question is what the value

[47]Ripley himself makes the point (2013), p. 155, as does Égré, in a more guarded form (2019), §4.5.2.

[48]Technically speaking, ST is not a paraconsistent logic, since it validates Explosion; but it is effectively so, just because of the failure of Cut.

0.5 means—if it means anything, and is not just a piece of instrumental model-theoretic machinery. As we shall see in a moment, a natural interpretation is *both true and false*—in which case, we certainly have full-blooded dialetheism.

As we noted, Égré suggests understanding 0.5 as something like *true but not completely true*. Truth is a vague predicate. Having the value 0.5 means being borderline true, but not fully true.[49] Now, as I said, I see no reason to believe that the truth predicate is vague; but if it is, it is not having the maximal degree of truth which is relevant. It is having a high enough degree of truth to be acceptable. (After all, only a pedant would deny that I have a new car when I have just picked it up from the showroom, though I have driven it home.) Assuming that there are degrees of truth, the degree of truth required to make something acceptable is context-dependent. What, in the present case, is the degree of truth of the Liar sentence, and what it the degree of truth required to make it acceptable in the present context? I don't have any sense of how one might go about answering these questions sensibly. But in a sense, they are irrelevant anyway. Whatever the answers, they had better be such as to make the Liar sentence acceptable. Given the ST_T machinery, you can *prove* it. I assume that substructuralists do not want to end up in the same somewhat desperate camp as those[50] who cannot accept the consequences of their own theory!

As we saw, Ripley suggest that the Liar sentence is not strictly assertible, only tolerantly assertible. However one understands the latter notion, it remains the case that $L \wedge \neg L$ follows from things all of which we accept (namely the members of the empty set), so it would seem to be a failure of nerve not to endorse it. True, $L \wedge \neg L \Rightarrow$ as well, which cannot happen with things that are strictly assertible (non-paradoxical), but so what? That just shows that the Liar is an odd true sentence. But we knew that.

But is ST_T really a *new* form of dialetheism? Looking at it in the appropriate way, ST is just *LP*, or a fragment thereof. As we noted, ST is weakly sound and complete with respect to the ST semantics. As we also noted, it is strongly sound. However, as Dicher and Paoli point out,[51] it is not strongly complete. $p \Rightarrow q \wedge r \models_{ST} p \Rightarrow q$, but as is easy to see, it is not the case that $p \Rightarrow q \wedge r \vdash_{ST} p \Rightarrow q$. (There are no rules for eliminating a conjunction.) Quite generally, Cut is required to deliver all the standard elimination rules. What Dicher and Pauli prove is that when these (but not Cut) are added—call the augmented system ST^+—we obtain

[49] He sometimes glosses this as *true in some respects, but not all*. But I really can't see how to apply the terminology of respects in this case.

[50] Such as, in effect, Maudlin. See Priest (2005b) for references and discussion.

[51] Dicher and Paoli (2019).

a proof theory which is strongly sound and complete. And as they then note:[52]

- $\mathcal{S} \vdash_{ST^+} \Gamma \Rightarrow \Delta$ iff $\mathcal{S} \models_{ST} \Gamma \Rightarrow \Delta$ iff $\mathcal{S}^{\rightarrow} \models_{LP} (\Gamma \Rightarrow \Delta)^{\rightarrow}$

ST^+ is then simply a fragment of *LP*. One may understand the value 0.5 as just meaning *both true and false*.

Note that $0.5 \geq |(\Gamma \Rightarrow \Delta)^{\rightarrow}|$ iff $0.5 \leq |\neg(\Gamma \Rightarrow \Delta)^{\rightarrow}|$. It follows that $(\Gamma \Rightarrow \Delta)^{\rightarrow} \models_{ST} (\Sigma \Rightarrow \Pi)^{\rightarrow}$ iff $(\Gamma \Rightarrow \Delta)^{\rightarrow} \models_{LP} (\Sigma \Rightarrow \Pi)^{\rightarrow}$ iff $\neg(\Sigma \Rightarrow \Pi)^{\rightarrow} \models_{K_3} \neg(\Gamma \Rightarrow \Delta)^{\rightarrow}$. Cobreros, Égré, Ripley, and van Rooij (2020), p. 1076, claim, on this ground, that ST is no more tightly related to *LP* than to K_3. Obviously, that claim is somewhat stretched. The connection in the case of K_3 goes via significant logical tampering. The real connection with K_3 is not with ST, but with TS, as we shall note in 7.1.

It remains the case that the significance of an embedding theorem is a philosophically sensitive issue. Ripley (in correspondence) suggests that one may view the embedding of ST in *LP* as analogous to Glivenko's theorem in intuitionism, which shows that A is valid in classical logic iff $\neg\neg A$ is valid in intuitionist logic. To infer that classical logic is simply a fragment of intuitionist logic would be a bridge too far, however.

However, the situation here is somewhat different. One reason is that, relatively uncontentiously, a classical logician and an intuitionist logician mean different things by the logical connectives. However, this does not apply to ST and *LP*, since the truth conditions (and values) of the two logics are identical.

At the very least, there appears to be no bar to understanding the machinery of ST in terms of the *LP* embedding. One who endorses *LP* can simply, then, embrace the ST machinery, and apply it where it is useful to do so.

5 *AL* and *LL*

Let us now turn to the second substructural solution—that which rejects Contraction. And again let us consider our four questions. Our discussion here may be briefer, since some of the ground has already been laid.

5.1 Why Reject Contraction?

First, why should one reject Contraction? Usually, Linear and related logics are thought of as logics which keep track of resource use. Thus, standard presenta-

[52] Drawing on the work of Pynko (2010).

tions of Contraction-free logic interpret $A, A \Rightarrow B$ as something like 'we use two data (resources) of type A to obtain one datum of type B'.[53] Of course, there is no reason why this should tell us anything about logical consequence. Logic is, in some undeniable sense, about truth-preservation. One may think of premises as bits of data, but if a premise is true, this is not affected by how many times it is used. Even if, for reasons of data storage, it ceases to become available after some number of uses, it is still true.

Zardini gives a very different account of the failure of Contraction.[54] He starts with a discussion of states of affairs. There is a relation of causation between these. This is a generic notion of causation, however, not causation in the physical sense (though that is a special case of it). Zardini sees it as more analogous to what is sometimes called grounding—though, as we are about to see, it has a crucial feature, instability, which usual accounts of grounding do not have.

We may write that state a causes state b as $a \curvearrowright b$. The relation $\curvearrowright$ induces an ordering on the class of states of affairs called *stages*. Some states of affairs are unstable, in the sense that they hold at one stage, but not the next. For physical causation, this is clear—being alive is a (partial) cause of dying; but if someone dies, they are no longer alive—though why this should be the case for non-physical causation is less clear. A state for which this is the case is *unstable*. And since there are such states, it is not the case, in general, that, for example, $a \curvearrowright a$, or that if $a \curvearrowright b$ then $a \curvearrowright a \wedge b$.[55] So if A describes a state of affairs that holds at a stage, you may lose it at the next stage. In that sense, we may lose information.

Zardini argues that the state of affairs described by A causes the state of affairs described by $T\langle A \rangle$. Moreover, the state of affairs described by the liar sentence is a paradigm unstable state, since it causes its negation. I confess that this example strikes me as dubious. It seems to me that causation here is simply a case of entailment, and entailment simply adds more information; it does not take away anything.[56]

However, let this pass. The theory of states of affairs is not a theory of logical

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[53] Troelstra (1992), p. 1.

[54] This is explained at greatest length in Zardini (2019), to which page numbers in what follows refer.

[55] Here, $\wedge$ is whatever it is for states that corresponds to conjunction for statements.

[56] It is clear that the behavior of stages is modeled on that of Gupta and Herzberger's revision-theoretic semantics. I have never been able to make much sense of what this is and why it is supposed to behave as it does. See Priest (1987), 1.6, for references and discussion. However, Zardini's stages are clearly ontological whereas, I think, Gupta and Herzberger's stages are meant to be more semantical.

consequence, which is a 'truth related implication' (p. 176). (Following the notation I have used, I will write this as $\Rightarrow$.) Nonetheless, it gives rise to one. Exactly how, is only gestured at.[57] One possibility would be to take $\Rightarrow$ to track $\curvearrowright$. This will not work, at least to motivate AL, since one does not then have $A \Rightarrow A$. Another possibility is to take it to be the logical closure of information that holds at a stage. But that will not work either, for then we do not have $A \Rightarrow T\langle A\rangle$.

Zardini's solution is to allow it to do both, 'Implication does have the power, given a condition, either to develop it interstage or to keep it intrastage, but it does not have the power to do both' (p. 178). Such an insistence seems rather arbitrary. If implication can do this, why can't it track other cross-stage relations too? (And if the reason for this is that if we mix stages we will end up inconsistency, a dialetheist can only say 'quite so'!)

However, in any case it is not clear that this will do what is required. For a start, this understanding will not give you $T\langle A\rangle \Rightarrow A$, since crossing the arrow takes you to be previous stage. However, more importantly, it seems to verify Contraction. We have that $A \Rightarrow A$. Now, whichever stage the right hand A is at, so is $A \wedge A$. So we have $A \Rightarrow A \wedge A$.

5.2 Can Logic Function without Contraction?

Let us turn to the second issue: that logic must be able to function without contraction. If we have a set of axioms, it would appear be silly to insist that each be used only once. Similarly, if a lawyer is making a case for the innocence of their client, it would be absurd to suppose that they could not appeal to a piece of evidence a second time.[58] To a certain extent, this fact can be accommodated. We may say that, given a set of axioms or a body of evidence, Σ, A may be held to follow from this if $\vdash_{XL} \Sigma^* \Rightarrow A$, where Σ^* is a multiset such that its members are members of Σ repeated some finite number of times.

But problems do not end there. It is not just how many times a piece of information is used, but what happens to it after it is used. To illustrate: Suppose that we are reasoning about sets, and we wish to establish that $\{x : A(x)\} \subseteq \{x : B(x)\}$,

[57] 'Nonnatural causation constitutes the basic structure on which implication develops, so that all the core cases of non-natural causation are cases of implication: a theory of non-natural causation thus provides at least the beginnings from which to extract a theory of implication' (p. 177).

[58] Zardini tells me in conversation that he thinks it may indeed be correct that one can use some axioms only once: for example, if the Liar sentence is an axiom. This strikes me as quite contrary to the way that axioms are used in practice; but it certainly raises the tricky question: given an axiom, how many times can you use it, and how do you know?

where $A(x)$ is some perfectly ordinary set theoretic condition (that is, of a no-funny-business kind). If Γ is our set of axioms, we need to show that $\Gamma^* \Rightarrow A(x) \rightarrow B(x)$. But to do so we would standardly show that that $\Gamma^*, A(x), ..., A(x) \Rightarrow B(x)$, for some finite number, n, of repetitions of $A(x)$. The trouble now is that though we can infer $A(x) \rightarrow B(x)$, if $n > 1$ we will have left behind $n - 1$ unwanted occurrences of $A(x)$.

Nor should one underestimate the severity of the challenge. If one looks at a standard axiomatization of *LL*,[59] it is clear that it is a subsystem of the intensional fragment of the relevant $R - W$. (*AL* adds the *K* axiom, $A \rightarrow (B \rightarrow A)$.) And one thing we have learned from investigations of relevant logic is how hard it is to reconstruct many standard mathematical arguments using one.[60] This is not to offer an argument that this challenge cannot be met. But it has to be met if this approach to the paradoxes is even to be competitive.[61]

It is worth noting that there are translations of classical logic into *LL*. One such is due to Grišin.[62] Grišin provides a pair of maps, + and −, from the language of classical logic to that of the (full) language of *LL*—which contains both the extensional (additive) connectives and the intensional (multiplicative) connectives such that:

- $\vdash_{LK} \Gamma \Rightarrow \Delta$ iff $\vdash_{LL} \Gamma^- \Rightarrow \Delta^+$

where $\Gamma^- = \{A^- : A \in \Gamma\}$ and similarly for Δ^+. Clearly, the left to right direction follows also for *AL*. It might be thought that this provides the required recapture. It does not. The reason is that the Grišin translation treats $\wedge$ (and $\vee$) differently under the two maps. In one, the connective is intensional; in the other it is extensional. Hence, the inference does not preserve an unambiguous translation of the axioms/theorems in question, as is surely intended in a classical deduction. The classical inferences appear valid only because they trade on an ambiguity. Indeed, Mares and Paoli (2014) argue that the classical paradoxical arguments are fallacious precisely because they trade on this ambiguity.

[59]E.g., Avron (1988), §2, Troelstra (1992), ch. 7.

[60]To give just two examples: the problem of recapturing classical Peano Arithmetic in $R^\#$ (see Friedman and Meyer (1992)); and the difficulty of recapturing standard results concerning cardinals and ordinals in relevant set theory, even with unrestricted set-theoretic comprehension (see Weber (2010) and (2012))—though here the problem is exacerbated by the fact that the set theoretic principles force the use of a relatively weak relevant logic.

[61]I note that there is always a brute-force way of obtaining what one wants. Whenever one wants to contract on a particular A, one just adds axioms of the form $A \Rightarrow A \wedge A$ and $A \vee A \Rightarrow A$. Such an *ad hoc* procedure, bereft of systematic and theoretical insight, has little to recommend it.

[62]See, e.g., Troelstra (1992), 5.5.

One can also obtain a certain amount of contraction, and so of standard reasoning, using the "exponential" operators;[63] but these must be eschewed if "extended" paradoxes are to be avoided, as we shall see in the next subsection.

Finally, I note that Ripley (2015b) argues persuasively that Contraction-free logics cannot function as closure operators, any more that Cut-free logics. Given this, problems about the failure of closure apply to Contraction-free logics as well.[64]

5.3 Higher Level and Exponential Paradoxes

Let us turn to the next issue: paradoxes which use other machinery. And let us start with higher level paradoxes again. As we saw in 4.3, the solution to the paradoxes which proceeds by jettisoning Cut is subject to paradoxes of the same kind at a higher level. As the deductions given there show, exactly the same is true of the solution which jettisons Contraction.

Exactly as with the ST case, then, one has to give up Contraction at all levels—with all the ramifications thereof. In particular, it leads to a tortoise situation. Suppose that we get the tortoise to agree that A and that $A, A \Rightarrow_0 Z$. The Contraction-savvy tortoise refuses to accept Z, since we need A twice. So we add the information that $\Rightarrow_0 A$, and point out that we can now reach $\Rightarrow_0 Z$, simply with Cut, but with no Contraction. The tortoise points out, correctly, that you need to Cut twice, so you need two occurrences of $\Rightarrow_0 A$. So we add a statement of the validity of this inference: $[\Rightarrow_0 A, \Rightarrow A_0] \Rightarrow_1 [\Rightarrow_0 Z]$. Now, given $\Rightarrow_1 \Rightarrow_0 A$, we can get to $\Rightarrow_1 \Rightarrow_0 Z$ with Cut and no Contraction. But the tortoise points out, correctly, that we need to Cut twice here too. Obviously, we are off on an infinite regress, and we never get anything delivering Z. Whether this argument can be made into a precise model-theoretic argument of the kind that Scambler gives for tortoise logic in the Cut case requires further investigation.[65]

Another way in which versions of the Liar and Curry arise is by the use of the "exponential" connectives. These are standard in Contraction-free logics. Their very purpose is to allow for a certain amount of contraction. And hardly surpris-

[63] In particular, interpreting the intuitionist conditional as $!A \to B$ allows Intuitionist Logic to be embedded in Linear Logic. (For !, see the next subsection.) See Girard (1995), p. 34 ff.

[64] For a possible reply, see Cintula and Paoli (202+). This is not the place to go into it, but I confess that I find the reply given somewhat tortured.

[65] A way of avoiding the regress is simply to add $A \Rightarrow_0 A \wedge A$ at the second stage. But this does not show that the present regress is not vicious.

ingly, then, they bring back paradox.[66]

Exactly how one formulates the rules for these operators depends on other matters. But it is standard that there is an operator, !, which satisfies the following rules:[67]

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A! \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A}{!\Gamma \Rightarrow !A} \qquad \frac{\Gamma, !A, !A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta}$$

Given these, we have a version of the liar paradox. Let L be $!\neg T\langle L\rangle$.Then:

$$\dfrac{\begin{array}{c} \vdots \\ !\neg T\langle L\rangle \Rightarrow \\ L \Rightarrow \\ T\langle L\rangle \Rightarrow \\ \Rightarrow \neg T\langle L\rangle \\ \Rightarrow !\neg T\langle L\rangle \end{array} \qquad \begin{array}{c} L \Rightarrow L \\ L \Rightarrow T\langle L\rangle \\ L, \neg T\langle L\rangle \Rightarrow \\ !\neg T\langle L\rangle, \neg T\langle L\rangle \Rightarrow \\ !\neg T\langle L\rangle, !\neg T\langle L\rangle \Rightarrow \\ !\neg T\langle L\rangle \Rightarrow \end{array}}{\Rightarrow}$$

The ellipsis on the left just reproduces the first five lines of the deduction on the right.

Similarly, we have a version of Curry. Let A be arbitrary, and let C be $!(!T\langle C\rangle \to A)$. Then:

$$\dfrac{\begin{array}{c} \vdots \\ !T\langle C\rangle \Rightarrow A \\ \Rightarrow !T\langle C\rangle \to A \\ \Rightarrow !(!T\langle C\rangle \to A) \\ \Rightarrow C \\ \Rightarrow T\langle C\rangle \\ \Rightarrow !T\langle C\rangle \end{array} \qquad \begin{array}{c} \dfrac{!T\langle C\rangle \Rightarrow !T\langle C\rangle \qquad A \Rightarrow A}{!T\langle C\rangle, !T\langle C\rangle \to A \Rightarrow A} \\ !T\langle C\rangle, !(!T\langle C\rangle \to A) \Rightarrow A \\ !T\langle C\rangle, C \Rightarrow A \\ !T\langle C\rangle, T\langle C\rangle \Rightarrow A \\ !T\langle C\rangle, !T\langle C\rangle \Rightarrow A \\ !T\langle C\rangle \Rightarrow A \end{array}}{\Rightarrow A}$$

Hence, XL_T is beset with these "exponential" paradoxes. One might, of course, write off the ! operator as meaningless in some sense. However, the unity and simplicity of a Contraction-free solution to the paradoxes then starts to disappear.[68]

[66] What follows is taken essentially from Ripley (202+).

[67] See Troelstra(1992), p. 18. $!\Gamma$ is $\{!A : A \in \Gamma\}$.

[68] Moreover, there must be *some* legitimate way of expressing the thought that you can use A as many times as you like; and that is always going to threaten to bring back enough contraction to generate paradox. See, further, Murzi and Rossi (2020b).

5.4 XL_T as a Version of Dialetheism

Finally, let us turn to the question of whether this is simply a new form of dialetheism. As we saw, without Contraction, one cannot infer $\Rightarrow L$ and $L \Rightarrow$, but one can infer $\Rightarrow L, L$ and $L, L \Rightarrow$, that is $\Rightarrow \neg L, \neg L$. I'm not exactly sure what this means, since I'm not exactly sure what $\Rightarrow A, B$ means. But this simply looks like dialetheism with a stutter. In particular, if we are to take a proof of A from Σ to be of the form $\vdash \Sigma^* \Rightarrow A$, then symmetry requires it equally to be of the form $\vdash \Sigma^* \Rightarrow A^*$. So the repetition is doing no work.

Paoli (in conversation), noted that $\Gamma \Rightarrow A, B$ is equivalent in XL to $\Gamma \Rightarrow A \vee B$, and suggested that the righthand comma can be therefore be interpreted as an intensional disjunction of some kind. (One might think of this as meaning $\neg A \rightarrow B$. In that case, $\Rightarrow L, L$ means $\Rightarrow \neg L \rightarrow L$.) But by a duality of reasoning, $A, B \Rightarrow \Delta$ is equivalent to $A \wedge B \Rightarrow \Delta$. So the lefthand comma should be interpreted as an intensional conjunction of some kind. But this seems wrong. If we are working in an axiom system, the things on the left of the $\Rightarrow$ are the axioms. These are standardly given simply in the form of a list. There need be no intensional relation between them. For example, the axioms of Peano Arithmetic are, in principle, quite independent of each other.[69]

Zardini (2019), p. 180 f. suggests that one may take $A \vee B$ to express the fact that the the situation described is such that one or other of the disjuncts holds, but it is ontologically (as opposed to epistemologically) undetermined which. That strikes me as a perfectly coherent idea in general. But when A and B are the same, my grasp of its sense disappears. What would it be like for a situation to be such that either A holds or A holds, but it is ontologically under-determined which?

However, whatever one says about these matters, we have proved two disjunctions, $\Rightarrow L \vee L$ and $\Rightarrow \neg L \vee \neg L$. Now, the essence of any disjunction, $A \vee B$, worth the name, is to give you a choice: you can have either A or B—or maybe both. Now, in the case of L, I choose the first disjunct; and in the case of $\neg L$, I choose the second...

In sum, then, contradiction with a stutter hardly seems any more consistent that contradiction without one.

[69]And in truth, a right-hand comma does not look much like an intensional disjunction, at least in AL, since we have $A \Rightarrow A \vee B$ (by Identity and Weakening).

6 Conclusion

This concludes our discussion of the two substructural putative solutions to the semantic paradoxes. What we have seen is that giving up the structural rules in question poses real problems. But even if these can be surmounted, they still deliver what amounts to a dialetheic account of the paradoxes. Far better, it would seem, is to accept a simple dialetheic account of the paradoxes, and keep the structural rules.[70]

7 Appendix

The sequent calculus rule of Identity is not standardly characterised as a structural rule. However, this is largely a terminological matter. There is a clear sense in which Identity is dual to Cut. Solving the paradoxes by jettisoning Identity has been given much less attention that solving them by jettisoning Cut or Contraction—perhaps for good reason, as we shall see. However, it has at least been considered by some.[71] In this appendix I will consider this strategy. Again, first I will give an exegesis; then an evaluation.

7.1 Tolerant Strict

The substructural logic obtained by dropping Identity from LK has been come to be called Tolerant/Strict (TS). As a little thought suffices to show, for no A and B do we have $\vdash_{TS} A \Rightarrow B$.

Semantics for the language can be obtained by reversing those for ST. For TS, v verifies $\Gamma \Rightarrow \Delta$ if:

- if (materially) v weakly verifies every member of Γ, it strongly verifies some member of Δ.

As is easy to check, all the rules except Identity preserve verification. However, Identity does not. Merely consider a valuation where $|A| = 0.5$.

[70]A conclusion also reached by Barrio, Rosenblatt and Tager (2015). Noting the fact that ST can be embedded in LP, they argue that LP is preferable because, essentially, it keeps the structural rules.

[71]Most notably, French (2016). A possible solution that drops the final structural rule, Weakening, has been explored by Da Ré (202+), though here one has to modify the rules for $\rightarrow$ as well, to avoid Curry paradoxes.

Note, here, a connection between ST and K_3. It is not hard to check that v verifies $\Gamma \Rightarrow \Delta$ iff v assigns $(\Gamma \Rightarrow \Delta)^{\rightarrow}$ the value 1. That is, it is designated in K_3. Hence one may think of verified sequences as valid K_3 material conditionals. From a K_3 perspective, the failure of Identity is just, then, the failure of the material conditional, $A \rightarrow A$.

It follows that $\mathcal{S} \models_{TS} \Gamma \Rightarrow \Delta$ iff $\mathcal{S}^{\rightarrow} \models_{K_3} (\Gamma \Rightarrow \Delta)^{\rightarrow}$, and hence that if $\mathcal{S} \vdash_{TS} \Gamma \Rightarrow \Delta$, $\mathcal{S}^{\rightarrow} \models_{K_3} (\Gamma \Rightarrow \Delta)^{\rightarrow}$. As for ST, the converse is not true, since establishing the connective elimination rules requires Identity and Cut. Plausibly, again as for ST, adding these rules delivers completeness, though I know of no proof of this.[72] However, since TS has no theorems, and K_3 has no logical truths, $\vdash_{TS} \Gamma \Rightarrow \Delta$ iff $\models_{K_3} (\Gamma \Rightarrow \Delta)^{\rightarrow}$ iff $\models_{TS} \Gamma \Rightarrow \Delta$, vacuously. So the converse holds in the zero-premise case; and, for good measure, the TS semantics are weakly complete with respect to the TS proof theory.

Truth can be added to the TS sequent calculus, as for the other substructural logics. Let us call the result TS_T. Truth then behaves transparently, as for the others. Moreover, the standard arguments for the Liar and Curry of 2.4 fail, since they use Identity. Indeed, the theory cannot prove any paradox, since it cannot prove anything! There is nothing of the form $\vdash_{TS_T} \Gamma \Rightarrow \Delta$.

7.2 *TS*

Let us now turn to an evaluation of the TS solution. We will do this by addressing our four questions. Actually, the answer to the last question is *no*. There is no way in which a contradiction—or anything else—can be deduced; and there is nothing much more to be said about this. The other questions are more interesting.

The first question is why one should reject Identity. *Prima facie*, this seems a most innocuous of principles.[73] And recall that merely pointing out that it is part of an argument for a contradiction will not provide the independent ground required. As we noted, in TS the validity of a sequent is essentially the truth of a material conditional in K_3. So it might be suggested that Identity fails for exactly the same reason that it fails for the conditional in K_3, which is that A may be neither true nor false. But it is precisely a standard criticism of K_3 that one needs Identity (and all the other conditionals which it delivers) to do any significant reasoning. It is because of this that those who have been sympathetic to the K_3

[72]The result is stated without proof in Da Ré, Szmuc, and Teijeiro (202+).

[73]In particular, it it comes to a choice between Identity and the T-Schema, the latter seems much less secure.

machinery, such as Harty Field, have augmented the machinery with a conditional that does verify Identity.[74] Clearly, this undercuts the whole substructural strategy in question here.

French suggests that we may reject Identity, at least for those A which lead to paradox, because they are logically defective, meaning that we cannot use them in reasoning.[75] I'm not clear what sense of 'cannot' is at issue here. But why should we not use L in reasoning? *Prima facie*, if L is the Liar sentence, there is nothing problematic about reasoning from $L \wedge L$ to L. As I have already stressed, we need an independent argument for such failure.

Turning to the second issue: it is clear that TS is useless as a canon of inference, simply because, according to it, no inferences are valid. I suppose that it might be suggested that we can help ourselves to an instance of Identity whenever it is legitimate to do so. But this simply raises the question of which instances are legitimate; and, as *per* the previous objection, why those generating paradox are not.

French (§4) suggests that the fact that there are no valid inferences is not a significant problem, since there are valid meta-inferences. This seems to me to miss the point. It remains the case that you cannot infer anything from information Σ. Every Σ is vacuously closed!

Finally, the third issue. As the arguments of 4.3 show, there are higher order paradoxes of self-reference. And now, if one rejects Identity at the higher level, this undercuts the reply to the last objection. Moreover, the rejection needs to go all the way up, or the paradox would occur at higher levels. In other worlds there are no valid inferences, meta-inferences, meta-meta-inferences, ... at all! Logic is impotent.[76]

All up, then, the TS putative solution seems to have much less to recommend it than the other two putative solutions.[77]

[74] See Field (2008).

[75] French (2016), §6.3.

[76] In this context note also that Murzi and Rossi in their (202+) extend their result about triviality following from certain ways of expressing non-paradoxicality from Cut-free and Contraction-free logics to Identity-free logics.

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