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INCONSISTENT MODELS OF ARITHMETIC
PART II: THE GENERAL CASE

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Abstract. The paper establishes the general structure of the inconsistent models of arithmetic of [7]. It is shown that such models are constituted by a sequence of nuclei. The nuclei fall into three segments: the first contains improper nuclei; the second contains proper nuclei with linear chromosomes; the third contains proper nuclei with cyclical chromosomes. The nuclei have periods which are inherited up the ordering. It is also shown that the improper nuclei can have the order type of any ordinal, of the rationals, or of any other order type that can be embedded in the rationals in a certain way.

§1. Introduction. Paraconsistent logics have been invoked for many different purposes, such as the solution of the paradoxes of self-reference.[1] Such applications are, of course, philosophically contentious, and one cannot subscribe to them unless one takes a paraconsistent logic to be, in an appropriate sense, the correct logic. But paraconsistent logics also make possible many mathematical structures interesting in their own right;[2] and one may explore these, whether or not one is a card-carrying paraconsistent logician—just as a classical logician may explore the nature of intuitionist structures without subscribing to the correctness of intuitionist logic.

Some of the most intriguing mathematical structures to arise so far in this context are the inconsistent models of arithmetic.[3] These are interpretations of the language of arithmetic that model all the truths of the standard model of arithmetic, plus more (and so are inconsistent). The first part of this paper, [7], whose contents I will summarise later, provided a complete taxonomy of the finite inconsistent models.

In this second part of the paper I will discuss the general case. In the next section of the paper, I will summarise the relevant material from [7]. Following that, I will establish the general structure of all inconsistent models of arithmetic. Models can be chunked into blocks that I will call *nuclei*. Section 3 establishes the basic properties of nuclei. Section 4 establishes their internal structure. Proper nuclei contain successor sequences that I will call *chromosomes*. As we will see, there are two type of proper nuclei, those whose chromosomes are linear, and those whose chromosomes are cyclical. Section 5 establishes some further facts about

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[1] See, e.g., [5]

[2] For a survey of some of these, see [3].

[3] Some of the background to these is given in [7]. The models also have a (controversial) philosophical punch. See, e.g., [6], [4] and [8].

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the internal structure of nuclei. The subsequent sections of the paper discuss the order-type of nuclei. Section 6 demonstrates that the improper nuclei may have the order-type of any ordinal. Section 7 shows that they may have the order-type of the rationals, and Section 8 generalises this result to certain other order-types. In Section 9, I conclude with some comments and open questions.

§2. **Preliminaries.** The models that will concern us are interpretations of the paraconsistent logic LP. The language of the logic, L, is that of classical first order logic, including function symbols and identity. An LP interpretation is a pair $\langle D, I\rangle$, which is exactly the same as an interpretation of classical first-order logic, except that for every n-place predicate, P, $I(P)$ is a pair comprising the extension and anti-extension of P. I will write these as $I^+(P)$ and $I^-(P)$, respectively. $I^+(P)$ and $I^-(P)$ may overlap, but their union must be the set of all n-tuples of D. The extension of the identity predicate, '=', is always the set $\{\langle x, x\rangle ; x \in D\}$.

To give the truth and falsity conditions of the language I employ the standard dodge of supposing that it is augmented with a name for every member of D. Without loss of generality, we will take the names to be the members of D themselves, and adopt the convention that for every $d \in D$, $I(d)$ is just d itself. If the interpretation is A, I will call the augmented language L_A. I is extended to assign every term of L_A a denotation in the usual way. Every formula, α, of L_A is assigned a semantic value, $v_A(\alpha)$, in the set $\{\{1\}, \{1, 0\}, \{0\}\}$, by the following clauses. (Truth conditions are obtained by ignoring the material in square brackets; falsity conditions, by substituting it in the obvious way.) If α is atomic, $Pt_1 \dots t_n$:

$$1[0] \in v_A(\alpha) \Leftrightarrow \langle I(t_1)\dots I(t_n)\rangle \in I^{+[-]}(P)$$

The clauses for negation, conjunction and the universal quantifier are as follows:

$$1[0] \in v_A(\neg\alpha) \Leftrightarrow 0[1] \in v_A(\alpha)$$
$$1[0] \in v_A(\alpha \wedge \beta) \Leftrightarrow 1[0] \in v_A(\alpha) \text{ and [or] } 1[0] \in v_A(\beta)$$
$$1[0] \in v_A(\forall x\alpha) \Leftrightarrow 1[0] \in v_A(\alpha(x/d)) \text{ for all [some] } d \in D$$

Disjunction and existential quantification have the natural dual truth/falsity conditions. $\alpha \supset \beta$ is defined, in the usual way, as $\neg\alpha \vee \beta$. If A is an interpretation, α is *true* [*false*] in A iff $1[0] \in v_A(\alpha)$. If Σ is a set of sentences, A is a *model* for Σ, $A \models \alpha$, iff every member of Σ is true in A. Note that those interpretations where all predicates have disjoint extension and antiextension are isomorphic to standard interpretations of classical logic, and so may be identified with them.

Next, we have two lemmas about LP. (For their proofs, consult [7].) First, suppose that we have two interpretations. The first has interpretation function I_1; the second, I_2. The second is an *extension* of the first iff they are identical, except that for every predicate, P, $I_1^{+[-]}(P) \subseteq I_2^{+[-]}(P)$.

EXTENSION LEMMA. *If B is an extension of A then for any α of L_A, $v_A(\alpha) \subseteq v_B(\alpha)$.*

For the second lemma, let $A = \langle D, I\rangle$, be any interpretation, and let $\sim$ be any equivalence relation on D, which is also a congruence relation on the interpretations of the function symbols in the language. If $d \in D$, let $[d]$ be the equivalence class of d under $\sim$. The *collapsed interpretation*, $A^\sim = \langle D^\sim, I^\sim\rangle$, is defined as follows. $D^\sim = \{[d]; d \in D\}$. For every constant, c, $I^\sim(c) = [I(c)]$. For every n-place function symbol, f, $I^\sim(f)([d_1]\dots[d_n]) = [I(f)(d_1\dots d_n)]$. (This is well defined since $\sim$ is a congruence relation.) If P is an n-place predicate, $\langle[d_1]\dots[d_n]\rangle$ is in its extension in $A^{\sim}$ iff there are $e_1 \sim d_1, \dots, e_n \sim d_n$, such that $\langle e_1 \dots e_n \rangle \in I^{+}(P)$. The anti-extension of P is defined similarly.

COLLAPSING LEMMA. *For any formula, α, of L_A, $v_A(\alpha) \subseteq v_{A^{\sim}}(\alpha)$.*

The Collapsing Lemma tells us that in a process of collapse, truth values are never lost; anything true/false in the original interpretation is true/false in the collapsed interpretation. In particular, if $A \models \Sigma$ then $A^{\sim} \models \Sigma$.

Now and for the rest of this paper, fix L to be the language of arithmetic. There is one binary predicate (identity), one constant symbol, $\boldsymbol{0}$, and function symbols for successor, addition and multiplication, $'$, $+$ and $\times$, respectively. As usual, the numeral $\boldsymbol{n}$ is $\boldsymbol{0}$ followed by n primes. Let S be the standard (classical) interpretation of the language. A simple example of an inconsistent model of arithmetic is obtained by collapsing S under the congruence relation, $\sim$, defined as follows. Let $n \geq 0$ and $k \geq 1$. Then $x \sim y$ iff:

$$(x, y < n \text{ and } x = y) \text{ or } (x, y \geq n \text{ and } x = y \pmod{k})$$

The collapsed model has a tail of length n, and a cycle of period k. Its structure may be depicted as follows:

$$\begin{array}{ccccccccccc} 0 & \rightarrow & 1 & \rightarrow & \dots & \rightarrow & n & \rightarrow & .. & \rightarrow & i \\ & & & & & & \uparrow & & & & \downarrow \\ & & & & & & n+k-1 & \leftarrow & \dots & \leftarrow & i+1 \end{array}$$

For future reference, I will call this model $\widetilde{S}$.

§3. Nuclei and their periods.

In this section I will spell out the basic structure of any inconsistent model of arithmetic (which includes the consistent ones, since these area a special case).

Take any *LP* model of the set of sentences that hold in the standard model of arithmetic, $\mathscr{M} = \langle M, I \rangle$. I will call the denotations of the numerals *regular* numbers. Let $x \leq y$ be defined in the usual way, as $\exists z\; x + z = y$. It is easy to check that $\leq$ is transitive. For if $i \leq j \leq k$ then for some x, y, $i + x = j$ and $j + y = k$. Hence $(i + x) + y = k$. But $(i + x) + y = i + (x + y)$ (since $\mathscr{M}$ is a model of arithmetic). The result follows.

If $i \in M$, let $N(i)$ (the *nucleus* of i) be $\{x \in M; i \leq x \leq i\}$. In a classical model, $N(i) = \{i\}$, but this need not be the case in an inconsistent model. For example, in $\widetilde{S}$ the members of the cycle constitute a nucleus. If $j \in N(i)$ then $N(i) = N(j)$. For if $x \in N(j)$ then $i \leq j \leq x \leq j \leq i$, so $x \in N(i)$, and similarly in the other direction. Thus, every member of a nucleus defines the same nucleus.

Now, if N_1 and N_2 are nuclei, define $N_1 \preceq N_2$ to mean that for some (or all, it makes no difference) $i \in N_1$ and $j \in N_2$, $i \leq j$. It is not difficult to check that $\preceq$ is a partial ordering. Moreover, since for any i and j, $i \leq j$ or $j \leq i$, it is a linear ordering. The least member of the ordering is $N(0)$. If $N(1)$ is distinct from this, it is the next (since for any x, $x \leq 0 \vee x \geq 1$), and so on for all regular numbers.

Say that $i \in M$ *has period* $p \in M$ iff $i + p = i$. In a classical model every number has period 0 and only 0. But again, this need not be the case in an inconsistent model, as $\widetilde{S}$ demonstrate. If a nucleus has a period $p \geq 1$, I will call it *proper*.

If $i \leq j$ and i has period p so does j. For $j = i + x$, so $p + j = p + i + x = i + x = j$. In particular, if p is a period of some member of a nucleus, it is a period of every member. We may thus say that p is a period of the nucleus itself. It also follows that if $N_1 \preceq N_2$ and p is a period of N_1 it is a period of N_2. In particular, then, if any nucleus is proper, all subsequent nuclei are proper.

Note, also, that the improper nuclei are simply singletons. For suppose that N is not a singleton. Let x, y be distinct members of N. Then $x \leq y$. Let $x + i = y$. i must be distinct from 0, so $1 \leq i$. Similarly, $y + j = x$, for some $1 \leq j$. Thus, $x + (i + j) = x$. But $1 \leq i + j$. Hence, the nucleus of x is proper.[4]

§4. **Chromosomes.** Every proper nucleus is closed under successors. For suppose that $j \in N$ with period $p \geq 1$. Then $j \leq j' \leq j + p = j$. Hence, $j' \in N$. In an inconsistent model, a number may have more than one predecessor, i.e., there may be more than one x such that $x' = j$, as $\widetilde{S}$ demonstrates.[5] But if j is in a proper nucleus, N, it has a unique predecessor in N. For let the period of N be q'. Then $(j + q)' = j + q' = j$. Hence, $j + q$ is a predecessor of j; and $j \leq j + q \leq j + q' = j$. Hence, $j + q \in N$. Next, suppose that x and y are in the nucleus, and that $x' = y' = j$. We have that $x \leq y \vee y \leq x$. Suppose, without loss of generality, the first disjunct. Then for some z, $x + z = y$; so $j + z = j$, and z is a period of the nucleus. But then $x = x + z = y$. I will write the unique predecessor of j in the nucleus as ${}'j$.

Now let N be any proper nucleus, and $i \in N$. Consider the sequence:

$$\ldots, {}''i, {}'i, i, i', i'' \ldots$$

(the members of the sequence may not all be distinct). Call this the *chromosome* of i. Note that if $i, j \in N$, the chromosomes of i and j are identical or disjoint. For if they have a common member, z, then all the finite successors of z are identical, as are all its finite predecessors (in N). Thus they are identical. Now consider the chromosome of i, and suppose that two members are identical. There must be members where the successor-distance between them is a minimum. Let these be j and $j'^{\cdots\prime}$ where there are n primes. Then $j = j + n$, and n is a period of the nucleus—in fact, its minimum non-zero period—and the chromosome of every member of the nucleus is a successor cycle of period n.

Hence, any proper nucleus is a collection of chromosomes, all of which are either successor cycles of the same finite (minimum) period, or are sequences isomorphic to the integers (positive and negative). Both sorts are possible in an inconsistent model. Just consider a collapse of a classical non-standard model by a congruence relation which leaves all the standard numbers alone and identifies all the others modulo p. If p is standard, the non-standard numbers collapse into a successor cycle; if it is non-standard, the nucleus generated has linear chromosomes.

Let me summarise the results of this section and the last as:

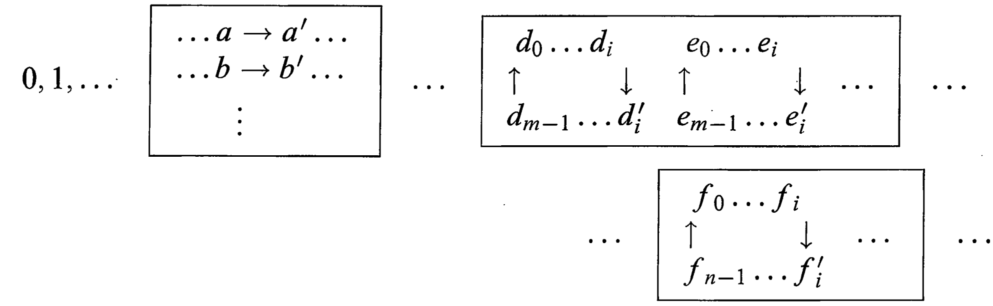
THEOREM 0. *The general structure of a model is a linear sequence of nuclei with periods inherited up the ordering. There are three segments (any of which may be empty). The first contains only improper nuclei. The second contains proper nuclei*

[4]If i is a member of a proper nucleus with period p, then $i \neq i$. For $i + p = i + j' = i$. But since we have a model of arithmetic $i + j' \neq i$. Hence $i \neq i$. In a collapsed model the members of the improper nuclei behave consistently. In an extension of a collapsed model, this need not be the case.

[5]Although $(x' = y') \supset x = y$ holds in the model, we cannot necessarily detach to obtain $x = y$.

with linear chromosomes. The final segment contains proper nuclei with cyclical chromosomes. ⊣

We may depict the general structure of a model as follows:



In the diagram, $0, 1, \ldots$ (or, more precisely, their singletons) form the sequence of improper nuclei. The next box represents a nucleus with linear chromosomes; and the next two boxes represent nuclei with cyclical chromosomes.

Note that if N_1 is a cyclical nucleus with minimum non-zero period, p, and $N_1 \preceq N_2$, the minimum non-zero period of N_2, q, must be a divisor (in the usual sense) of p. For p is a period of N_2, so $q \leq p$. Suppose that q is not a divisor of p. For some $0 < k < q$, p is some finite multiple of q plus k. So if $x \in N_2$, $x = x+q = x+p+\ldots+p+k$. Hence $x = x+k$, i.e., k is a period of N_2, which is impossible. Hence, if, in the preceding diagram, m and n are the minimum periods of their nuclei, n is a divisor of m.

Most of the results of [7] follow simply from Theorem 0 by imposing the constraint of finitude on the model. In that case, there is a finite initial tail, and then a finite sequence of cyclic nuclei (which [7] calls 'cliques') of non-decreasing periods; each of these has a finite number of chromosomes. In [7], there is also a proof that a model of any such structure can be constructed.

§5. More on the internal structure of nuclei. In this section, I will make two sets of miscellaneous observations about the structure of nuclei.

The first concerns predecessors. Suppose that i and j are in nucleus N, that $x' = i$, $y' = j$, but that x, y are not in N. Now either $x = y \vee x \geq y' \vee y \geq x'$. In the last case, $i = x' \leq y \leq y' = j \leq i$. So y is in the nucleus, which is impossible. Similarly for the second case. Hence, $x = y$ and so $i = x' = y' = j$. Thus, each nucleus has at most one member with an external predecessor, and this predecessor is unique. Moreover, only the first proper nucleus, if there is one, can have a member with an external predecessor, since any external predecessor must be a member of another nucleus, and all proper nuclei are closed under successors. Hence, there is at most one number with multiple predecessors, and this has exactly two, one inside and one outside its nucleus.

Secondly, some comments about addition and multiplication. The behaviour of regular addends and multiplicands in models is completely determined by the recursive equations for addition and multiplication. The general behaviour is not so determined, and appears to be a complex issue. One thing that can be said, however, is that in a finite model, each proper nucleus is closed under addition and multiplication.

For addition, start by considering the first proper nucleus, F. This must contain some regular numbers. Let n be one such. Since the nucleus is closed under successors, $n+n \in F$, and so $n+n \leq n$. Now if $x \in F$, $x \leq n$, and so $x \leq x+x \leq n+n \leq n \leq x$. Next, we show by induction that this holds for all the proper nuclei. Suppose that $x \in N$. Then, by ordinary arithmetic, for some $y \leq x$, $y+y = x \vee y+y = x+1$. y cannot be in a prior nucleus since otherwise $y+y$ would not be in N, by induction hypothesis. Hence, $x \leq y$. So $x \leq x+x \leq y+y \leq x+1 \leq x$. So $x+x \in N$. Finally, suppose that for any nucleus, N, $x, y \in N$. Then $x \leq y$, so $y \leq x+y \leq y+y \leq y$, so $x+y \in N$.

The argument for multiplication is similar. In the first nucleus, F, $n.n \leq n$ for some regular n. Now if $x \in F$, $x \leq n$, and so $x \leq x.x \leq n.n \leq n \leq x$. By induction, this holds for all proper nuclei. Suppose that $x \in N$. For some y, $y \leq x \leq y.y \leq x+x$. (This is a fact of ordinary arithmetic.[6]) y cannot be in a prior nucleus since otherwise $y.y$ would not be in N, by induction hypothesis. Hence, $x \leq y$. So $x \leq x.x \leq y.y \leq x+x \leq x$. So $x.x \in N$. Finally, suppose that for any nucleus, N, $x, y \in N$. If x is 0 then $x.y = 0$ and so $x.y \in N$. Otherwise, let $x = i'$. Then $y \leq iy + y = (i+1)y = x.y$. Since $x \leq y$, $x.y \leq y.y \leq y$, so $x.y \in N$.

§6. Nuclear ordering of ordinal order-type. Let us now turn to the question of the order-type of the nuclei. In a collapsed model the sequence of improper nuclei is identical (up to isomorphism) to the initial section of the model from which it is collapsed. Thus, the improper nuclei in an inconsistent model can have any structure which is an initial section of a classical model of arithmetic. Whether they can have any other structure is as yet unknown.

More can be said about the structure of the proper nuclei. In this section I will show that the proper nuclei can have the order-type of any ordinal. In subsequent sections, we will see that they can also have non-well-ordered order-types. The hard work of this section is packed into the following lemma.

LEMMA 1. *For every ordinal, α, there is a classical model of arithmetic, $\mathcal{M}_\alpha$, with domain D_α, such that for all $\beta < \alpha$, $\mathcal{M}_\alpha$ is a (classical) elementary end-extension of $\mathcal{M}_\beta$ (i.e., $\mathcal{M}_\alpha$ is an elementary extension of $\mathcal{M}_\beta$, and all the members of D_α not in D_β are greater than all members of D_β). Further, if $\alpha < \beta$ then $D_\alpha - D_\beta$ is non-empty, and for limit λ, $D_\lambda = \bigcup_{\alpha<\lambda} D_\alpha$.*

PROOF. The proof is by transfinite induction. $\mathcal{M}_0$ is the standard model of arithmetic. Given $\mathcal{M}_\alpha$, by a result of McDowell and Specker, it has a proper end-extension. (See [1], p. 244, or [2], p. 96.)[7] Let this be $\mathcal{M}_{\alpha+1}$. It is clear that this satisfies the conditions. Now suppose that the result holds for all $\beta < \lambda$. Then $\{\mathcal{M}_\beta; \beta < \lambda\}$ is a chain of elementary extensions. Let $\mathcal{M}_\beta$ be its union. By a standard result ([1], p. 79) this is an elementary extension of each $\mathcal{M}_\beta$ satisfying the appropriate condition. ⊣

[6]It can be established by induction. It is clear for $x = 0$. So let $x > 0$. Suppose that $y \leq x \leq y^2 \leq 2x$. If $x < y^2$ then $y \leq x+1 \leq y^2 \leq 2x \leq 2(x+1)$. Hence, y is the number. If $x = y^2$ then $y+1 \leq x+1 \leq y^2+1 \leq (y+1)^2 = y^2+2y+1 \leq 2(y^2+1) = 2(x+1)$. Hence, $y+1$ is the number.

[7]Theorem 1 can be proved without this result, by constructing a non-end extension with the compactness theorem. But the rest of the proof has then to be made more complex.

THEOREM 1. *For every ordinal, α, there is an inconsistent model of arithmetic in which the proper nuclei (which are, in fact, simple cycles) have order-type α.*

PROOF. Consider the model $\mathscr{M}_\alpha$ of the Lemma. For $\beta < \alpha$, let $C_\beta = D_{\beta+1} - D_\beta$. It is easy to check that the C_βs partition the non-standard numbers of D_α into α disjoint non-empty segments, each of which is closed under $\times$ (and *a fortiori*, $+$ and $'$). Let $n \in D_0$ and define a relation, $\sim$, on D_α as follows. $x \sim y$ iff:

$$(x, y \in D_0 \text{ and } x = y) \text{ or } (\exists \beta < \alpha,\ x, y \in C_\beta \text{ and } x = y \ (\text{mod}\ n))$$

It is easy to see that $\sim$ is an equivalence relation, and also to check that it is a congruence on the arithmetic operations. Hence we can construct the collapsed interpretation, $\mathscr{M}_\alpha^\sim$. In this, D_0 collapses into a segment comprising ω improper nuclei, and for $\beta < \alpha$, each C_β collapses into a proper nucleus (with one cyclical chromosome, of period n). Hence, the proper nuclei of $\mathscr{M}_\alpha^\sim$ have order-type α, as required. ⊣

§7. **Nuclear ordering of rational order-type.** In an inconsistent model, the proper nuclei do not have to have a discrete order-type. Call a linear ordering *rational-like* if it is dense, with no first or last member. In this section I will show that there are inconsistent (in fact, collapsed) models where the proper nuclei have rational-like order-type.

A natural thought as to how to construct an inconsistent model with proper nuclei of such order-type is to take a non-standard classical model of arithmetic, and collapse it using an equivalence relation that turns every block of non-standard numbers of type $\omega^* + \omega$ into a nucleus. This will not work, however, since the blocks are not closed under arithmetic operations, and so the equivalence relation involved is not a congruence relation. We can, however, show that any non-standard classical model of arithmetic can be partitioned into segments closed under arithmetic operations ordered in a rational-like way, and then collapse.

In what follows, A will be any non-standard classical model, and I will use the letters n, m, as variables for the natural numbers (in A). Consider the relation defined on the non-standard numbers: $a \simeq b$ iff $(a \leq b$ and $\exists n\ b < a^n)$ or vice versa.

LEMMA 2. *$\simeq$ is an equivalence relation, and the equivalence classes are sections (i.e., if a and c are in a class and $a < b < c$ then b is in the class) and are closed under arithmetic operations.*

PROOF. $\simeq$ is obviously reflexive and symmetric. For transitivity, suppose that $a \simeq b$ and $b \simeq c$. Without loss of generality, suppose that $a \leq b$, in which case, $b < a^n$. Now either $b \leq c$ or $c < b$. In the first case, $a \leq c$, and $c < b^m < a^{nm}$, so $a \simeq c$. In the second case $b < c^m$. Now, either $a \leq c$ or $c < a$. In the first case since $c < b < a^n$ we have $a \simeq c$. In the second case, $a \leq b < c^m$ and hence we have $a \simeq c$ again.

Next, we show that the classes are sections. Suppose that a and c are in a class and that $a < b < c$. Then $c < a^n < b^n$, as required.

Since each equivalence class is a section, to demonstrate arithmetic closure, it suffices to show that the class is closed under multiplication, which is done as follows. Suppose that a, b are in the same equivalence class. Then either $a \leq b$ and

$b < a^n$, or vice versa. Without loss of generality, suppose the former. Then $a \leq ab$ and $ab < aa^n = a^{n+1}$. Hence, a and ab are in the same class. ⊣

Next, let the equivalence class of a under $\simeq$ be $\|a\|$; define $\|a\| < \|b\|$ iff a and b come from different equivalence classes and $a < b$. This is well defined. For suppose that a and b come from different equivalence classes, that $a < b$ and that $c \simeq a$. (The argument for $c \simeq b$ is the same.) Suppose, for *reductio*, $b \leq c$. Then $a < c$, and $b \leq c < a^n$, which is impossible since a and b are from different classes.

LEMMA 3. *The relation* $<$ *on equivalence classes is a strong linear rational-like ordering.*

PROOF. $<$ is clearly anti-symmetric and connected. For transitivity, suppose that $\|a\| < \|b\| < \|c\|$. Then $a < b < c$. But a and c must come from different classes or $b < c < a^n$, so a and b would come from the same class. Hence, $\|a\| < \|c\|$.

The argument for denseness goes as follows. Suppose that $\|a\| < \|b\|$. Consider the formula, $\varphi(y)$, defined as follows: $\exists x(\boldsymbol{a}^y < x < x^y < \boldsymbol{b})$ (where exponentiation is defined in a standard fashion, and boldface is used for naming). This is satisfied by every finite n. For take x to be a^{n+1}: $a^n < a^{n+1} < (a^{n+1})^n = a^{n(n+1)} < b$. (The last part it true since a and b are in different equivalence classes.) Robinson's Overspill Lemma ([2], 6.2) applied to $\varphi(y)$ entails that for some non-standard c and d, $a^c < d < d^c < b$. It follows that d is in a different equivalence class from a and b, and that $\|a\| < \|d\| < \|b\|$.

That $<$ has no greatest or least member follows in exactly the same way. ⊣

One final lemma completes all the hard work. Let n be a fixed natural number. Define the relation $\sim$ on the domain of A as follows. $a \sim b$ iff:

$$(a \text{ and } b \text{ are standard and } a = b) \text{ or } (a \simeq b \text{ and } a = b \ (\text{mod}\ n))$$

LEMMA 4. $\sim$ *is an equivalence relation on the numbers in* A*, and also a congruence relation for successor, addition and multiplication.*

PROOF. Given Lemma 2, $\sim$ is clearly an equivalence relation. For congruence:

(Successor) Suppose that $x \sim y$. If x and y are standard, the result is immediate. If they are non-standard, the result follows, since equivalence classes under $\simeq$ are closed under successor (Lemma 2).

(Addition) Suppose that $x_1 \sim x_2$ and $y_1 \sim y_2$. Suppose that one of the xs or ys is standard, say the xs. Then the result follows since all the equivalence classes under $\simeq$ are closed under addition (Lemma 2). So suppose that both are non-standard. Clearly $x_1 + y_1 = x_2 + y_2$ (mod n). It remains to show that the sums come from the same blocks. If the xs and the ys themselves come from the same block, the result follows since the blocks are closed under addition (Lemma 2). Suppose, then, that they come from different blocks. Without loss of generality, suppose that $x_1 < y_1$. Now $y_1 < x_1 + y_1 < y_1 + y_1$. Since y_1's block is a section, and closed under addition (Lemma 2), it follows that $x_1 + y_1$ is in the same block as y_1. Similarly, since $x_2 < y_2$, $x_2 + y_2$ is in the same block as y_2. Thus $x_1 + y_1$ and $x_2 + y_2$ are in the same block, as required.

(Multiplication). The argument for this is essentially the same. ⊣

We can now prove the main result.

THEOREM 2. *Any non-standard classical model,* A*, has a collapse under which the nuclei* (*which are, in fact, cycles*) *have a rational-like ordering.*

PROOF. Since $\sim$ is a congruence relation, we can collapse under it using the Collapsing Lemma. The natural numbers collapse into a tail, and each equivalence class under $\simeq$ collapses into a nucleus (in fact, a cycle) of period n. If it can be shown that the relation $\leq$ on the equivalence classes under $\simeq$ coincides with the relation $\preceq$ on nuclei, the result follows, by Lemma 3.

So let a and b be numbers; let $\|a\|$ and $\|b\|$ be their equivalence classes under $\simeq$, and let C_a and C_b be the cycles into which the blocks collapse. If $\|a\| = \|b\|$, then $C_a = C_b$, and so $C_a \preceq C_b$. If $\|a\| < \|b\|$, then $a < b$, and $[a] \leq [b]$ in the collapsed model. Hence, $C_a \preceq C_b$. Conversely, suppose that $C_a \preceq C_b$. Then in the collapsed model $[a] \leq [b]$. So for some i, $a + i \sim b$. Hence $a + i \simeq b$, and $a + i$ and b are in the same block. Hence, $\|a\| \leq \|b\|$. ⊣

§8. Continuous embeddings. In this section, I will generalise Theorem 2, to show that there are inconsistent models where these nuclei have any order-type that can be embedded in the rationals in a certain way. Let I be any linearly ordered set and f an order-embedding of I into T. We will say that f is *continuous* iff whenever J is an initial segment of I and $f[J]$ $(= \{f(i); i \in J\})$ is bounded above in T, there is a $j \in I$ such that $f(j)$ is the least upper bound of $f[J]$. Let $\mathscr{Q}$ be the order-type of the rationals, then it is an easy exercise to see that ω^* and $\omega^* + \omega$ (but not $\omega + \omega^*$) have a continuous embedding in $\mathscr{Q}$.

THEOREM 3. *Let I be any linearly ordered set which has a continuous embedding in $\mathscr{Q}$. For each countable classical non-standard model of arithmetic, A, there is a collapsed model where the nuclei (in fact, cycles) have the same order-type as I.*

PROOF. Consider the equivalence classes of A under the equivalence relation $\simeq$ of the previous section. By Lemma 3, this is a dense linear order with no first or last member. It is therefore isomorphic to $\mathscr{Q}$ (as is well known[8]). Let f be a continuous embedding from I into this. Define the function g, with domain I as follows: if $i \in I$, $g(i) = \bigcup\{x; x \geq f(i) \text{ and for all } j > i, f(j) > x\}$. In other words, $g(i)$ collects up all the members of $f(i)$ together with all members of later equivalence classes that are not in the image of some later j. Let $G = \{g(i); i \in I\}$.

The members of G are clearly disjoint, and $\bigcup G$ is closed upwards under $<$. For suppose that $a \in \bigcup G$ and $a < b$. For some i, $f(i) \leq \|a\| \leq \|b\|$. Let $x = \|b\|$. Consider $J = \{i \in I; f(i) \leq x\}$. $f[J]$ is obviously bounded above. Hence, there is a $j \in I$ such that $f(j)$ is the least upper bound of $f[J]$. Clearly, $f(j)$ must be $\leq x$; moreover, there can be no $i \in I$ such that $f(j) < f(i) \leq x$. Hence, $x \subseteq g(j)$ and $b \in \bigcup G$.

Each $g(i)$ is a section. For suppose that $a \leq b \leq c$, and $a, c \in g(i)$. Then $f(i) \leq \|a\| \leq \|b\| \leq \|c\|$. Hence $b \in g(i)$. Finally, each $g(i)$ is closed under arithmetic operations. Since it is a section, it is sufficient to check that it is closed under multiplication. So suppose that $x, y \in g(i)$, and $x \leq y$. Then $y < xy \leq y^2$, and since $\|y\|$ is closed under multiplication, $xy \in \|y\| \subseteq g(i)$.

Thus, G is a partition of some terminal section of the numbers into a disjoint, arithmetically closed sections. Define an order on G in the natural way: $g(i) < g(j)$ iff $i < j$. Clearly, this is an order isomorphism. We now repeat the proof of Theorem 2, except that the members of G play the role of the equivalence classes under $\simeq$.

[8]See, e.g., [1], p. 176.

Let n be any finite number, and define a relation $\sim$ on numbers as follows. $x \sim y$ iff:

$$(x, y \notin \bigcup G \text{ and } x = y) \text{ or (for some } i \in I, x, y \in g(i), \text{ and } x = y (\text{mod } n))$$

It is easy to see that $\sim$ is an equivalence relation. And as in the proof of Lemma 4, it is a congruence relation.

Collapse A under this relationship by the Collapsing Lemma. The result is a model with a tail comprising (the equivalence classes of members of) the complement of $\bigcup G$. For each $i \in I$, $g(i)$ collapses into a cycle. If the relation $\leq$ on G coincides with $\preceq$ on the cycles into which G collapses, we have the required result. But this follows as in the proof of Theorem 2. ⊣

§9. **Conclusion.** This paper has established many important aspects of the structure of inconsistent models of arithmetic. In particular, it has shown that such models fall into three segments: the first contains improper nuclei; the second contains proper nuclei with linear chromosomes; the third contains proper nuclei with cyclical chromosomes. The nuclei have periods which are inherited up the ordering. We have also seen that the improper nuclei can have the order-type of any ordinal, of the rationals, or of any other order-type that can be embedded in the rationals in a certain way.

I will finish with some observations and open questions. First, the observations. I have followed the standard treatment of the language of first order arithmetic in taking successor, addition and multiplication (and only those) to be expressed by function symbols. This, however, is arbitrary to a certain extent. First-order arithmetic could be formulated just as well with no function symbols, but with a binary predicate to express successor, and ternary predicates to express addition and multiplication. If arithmetic were formulated in this way, then collapse under any equivalence relation would give an inconsistent model, and the inconsistent models would have no interesting structure, as far as I can see.

At the other extreme, we could formulate arithmetic with many more function symbols, say one for each primitive recursive function. This would make collapse much more difficult. For example, in any model of arithmetic it would be impossible to collapse the natural numbers in any but a trivial way. Just consider the predecessor function, p (where $p(0) = 0$ and $p(n+1) = n$). The collapse of the natural numbers cannot now have a tail, since predecessors are unique, but it cannot be a cycle either (other than the trivial one), since the predecessor of 0 must be 0. (Question: could there be a non-trivial collapse of non-standard models under these conditions?)

Assuming that successor, addition and multiplication are represented by function symbols, as is done in this paper, is an intermediate course of action. In fact, only the representability of the successor and addition functions are essential to the arguments of this paper, as can easily be checked. So we may jettison the representability of multiplication without loss. Jettisoning the representability of the addition function would not seem to leave enough machinery to do anything very interesting. (This is because virtually all arguments involve the ordering $\leq$. This is defined in terms of addition, and the arguments employ its functionality essentially.)

Next, the open (and interrelated) questions:

1. What order-types can be the order-type of the proper nuclei in a collapsed model, other than those established in Theorems 1, 2 and 3? Can it be, e.g., any linear order?

2. Can a nucleus have an infinitely descending sequence of periods? Must nuclei always be closed under addition and multiplication?

3. All the inconsistent models that we have seen are constructed by collapsing classical models—or at least, by collapsing them and then extending the collapse. Are all the inconsistent models to be obtained in this way?[9] I conjecture that they are.

4. The only collapsed models that we have seen are produced by a certain kind of equivalence relation. A classical model is partitioned into a number of disjoint sections closed under arithmetic operations; except for the first block, each block is collapsed with identity modulo some number (possibly identifying some of the blocks in the process). Are there any other kinds of collapsed models?

This paper establishes, I hope, that the theory of the structure of inconsistent models of arithmetic is just as rich and interesting as that of the structure of the consistent models (indeed, more so, since the consistent models are a special case). As is clear, there is still more to be learned about these models; in particular, a complete taxonomy is still to be found.[10]

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[9]Note that the collapse a classical model can itself be collapsed, but the result is the same as collapsing the original model under the composed equivalence relation. Hence double collapse produces no new models.

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