The Import of Inclosure: Some Comments on Grattan-Guinness

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1. Introduction

The standard paradoxes of self-reference, such as Burali-Forti's Paradox, the Liar Paradox, Berry's Paradox, have a common underlying structure. This is established in Priest (1994)—hereafter, *SPSR*. That paper also argues that, in virtue of this common structure, a common solution is appropriate, and advocates a dialetheic solution. In his paper (1998),[1] Grattan-Guinness makes a number of comments that relate to these claims. There is much to be learned from these comments, and much, also, that I would take issue with. But in this note I will simply comment on what I take to be the two most important issues that the comments raise: the scope of the Inclosure Schema and the Principle of Uniform Solution.

2. The scope of the Inclosure Schema

The structure underlying the standard paradoxes of self-reference is the *Inclosure Schema*.[2] Grattan-Guinness argues that satisfying the Schema is neither necessary nor sufficient for a paradox of self-reference. I will comment on both parts of this claim.

The Inclosure Schema concerns a diagonalising function, δ, and a totality of objects, Ω. The conditions that these satisfy are, loosely, as follows: applying δ to any subset of Ω, X, of a certain kind, produces an object that is not in X (Transcendence) but is in Ω (Closure). But Ω is itself of this kind; hence applying δ, to Ω, gives something that is both in and not in Ω.[3]

[1] Page references in what follows are to this, unless otherwise indicated.

[2] This is not the name it is given in *SPSR*. It is what it is called in Priest (1995), which provides a more extended, and more general, discussion of the Schema.

[3] According to Grattan-Guinness (pp. 823–4, and again, p. 824), the contradictions delivered by the Schema are "double contradictions", that is, of the form $\alpha \leftrightarrow \neg\alpha$. This is not so: they are simply of the form $\alpha \wedge \neg\alpha$.

Grattan-Guinness' example of a non-paradox of self-reference that fits the Schema (p. 828) is the well-known Barber Paradox.[4] His argument (fn. 5) that there is a version of this paradox that fits the form of the Schema is rather sketchy, and it is not clear to me what he has in mind exactly. But the details are not so important: the Barber certainly can be put into the form of the Schema, as is shown in Priest (1995, p. 173, fn.2). But there is more to the Schema than its form. Let me elucidate.

It is clear that the premises of an instance of the Inclosure Schema entail a contradiction ($\delta(\Omega)$ is both in and not in Ω). So an inclosure argument is valid. But one needs more than this for a paradoxical argument: the premises must also be true, at least prima facie—or none would suppose the situation paradoxical. It is this fact that rules out the Barber Paradox and its ilk as inclosure paradoxes. We have no good reason to suppose that there is a Barber of the required kind.

There is a possible reply here though. Could there not be good inductive evidence for the existence of such a barber?[5] (We interview the first man in the town, then the second ...) I do not think that such an argument would be cogent, but if one takes this possibility seriously, one can avoid it by putting a slightly stronger condition on the premises of an inclosure contradiction: not just that the premises are prima facie true, but that they are a priori so. For exactly this is true of the standard paradoxes of self-reference: Transcendence and Closure would appear to be a priori certified.The contradictions that these give rise to seem to be inherent in thought itself, intrinsic to our conceptual structures. This, indeed, is the *leitmotiv* of Priest (1995). The premises in the case of the Barber and its ilk, were they ever to be true, could only ever be so a posteriori.[6]

Grattan-Guinness' example of paradoxes of self-reference that do not fit the Schema (pp. 826–7) are paradoxes that he attributes to Löb (1955). In fact, paradoxes of this kind had already been noted by Curry in (1942), and paradoxes in this family now standardly bear his name. The existence

[4] On p. 830 he gestures towards others, but it is unclear to me what these are supposed to be. On p. 831 he also claims that "the seemingly consistent Gödel's incompletability theorem also fits the Schema". I do not know what, exactly, he has in mind here. There certainly is a paradox underlying the standard proof of Gödel's first incompleteness theorem. This is the "knower paradox", and it fits the Schema: see Priest (1995, p. 159).

[5] Nick Denyer has pressed this point on me on a couple of occasions.

[6] There are, of course, versions of the standard paradoxes where some of the premises required to establish Transcendence and/or Closure, and so these claims themselves, are a posteriori; for example, where a premise refers to a sentence by a description such as "the only sentence on such and such a page", and it is an empirical fact that it, itself, is that sentence. Such a posteriori elements, though, are always accidental in a certain sense. They are ways of securing an effect that could be secured by a priori means.

of Curry paradoxes is, in fact, pointed out and discussed *SPSR*.[7] As is noted there, whether or not Curry paradoxes fit the Schema depends on how the conditional involved is interpreted. If it is interpreted as a material conditional, Curry Paradoxes fit the Schema. If not, they do not. Such paradoxes are, therefore, of a different kind.

It is agreed, then, that there are paradoxes, puzzling arguments, that involve self-reference and that do not fit the Schema. But this is hardly news. Many paradoxes involve self-reference in one way or another, but no one would suppose that they belong with the Liar and its ilk. Grattan-Guinness himself gives a list of examples of this kind on pp. 829–30. As an illustration, just consider the person—a fallibilist—who claims that one can never be certain of anything. This, of course, is self-reflexive. It follows that the fallibilist who claims this, if right, cannot be certain of it. What one makes of the matter is another issue. Those who think that to claim anything is to claim certitude can infer that the fallibilist cannot consistently maintain their own views. Fallibilists will prefer to maintain that one can assert things without being certain. But whatever one makes of the matter, the utterance is not an inclosure contradiction. Or consider another example: there are versions of the surprise exam paradox that are self-referential.[8] (The information given to students is that there will be an exam next week, but that they will not be able to infer when from this information.) Again, such paradoxes are different from inclosure contradictions: they turn on the use of backwards induction, and the assumptions implicit in this.

3. The Principle of Uniform Solution

I turn now to the other main issue raised by Grattan-Guinness' comments. This concerns the Principle of Uniform Solution (PUS): same kind of paradox, same kind of solution.[9] Grattan-Guinness regards this as contentious (p. 828). For my part, I regard it as little more than a truism. If two paradoxes have different solutions, this itself would seem to show that

[7] Section 6. See also Priest (1995, Sc.11.8).

[8] See, for example, Halpern and Moses (1986).

[9] Conversely, if two paradoxes are of different kinds, there is no reason one should expect them to have similar solutions. This is why it is no surprise if the Curry paradoxes which do not fit the Inclosure Schema have a different kind of solution from Inclosure Paradoxes. As Curry himself puts it: "the root of the difficulty lies in … [the axioms for] implication" (1945, p. 117). It should also be noted that solving the Curry paradoxes by denying the principle of Contraction ($\alpha \rightarrow (\alpha \rightarrow \beta) \vdash \alpha \rightarrow \beta$) will not also solve all inclosure paradoxes. For some of them, such as Berry's, do not use the principle. See Priest (1987, Sc.1.8).

they are of different kinds. What is not at all truistic is what constitutes a kind. So let me say more about this.[10]

Let us start with an analogous principle: same kind of illness, same kind of cure—if two people have the same illness, they are to be cured in the same way. In a superficial sense, this is obviously false. For example, the same illness can be treated with two different drugs. But in a more profound, and more important, sense, the principle is clearly correct: if we have one illness in the two people, this must be due to the same cause. So the two people must be cured in the same of way, namely, by attacking that cause.[11]

In the preceding example, the notion of a kind of illness is relatively unproblematic. The kind in question is a natural kind, to be individuated by its causal structure. When we are talking of kinds of paradox, rather than illness, we are no longer talking about kinds that can be individuated causally; but, it seems to me, we still have an operative notion of kind. The cause explains its illness. In mathematics, there are no causes, but there certainly are explanations. For example, we explain why numbers have a certain property by remarking that they form a group, and that all groups have this property. What constitutes an explanation in mathematics, and how this notion relates to that of proof, for example, is a hard question, and I do not know the answer. Yet mathematicians recognise an explanation when they see one; and I see one in the shape of the Inclosure Schema: an inclosure is what explains why the paradoxes arise.[12] This is why such paradoxes are of a kind.

Let me elucidate these remarks by talking, first, of another family of paradoxes, certain paradoxes of infinity. A number of these were known to the medievals. Here are three.[13]

1. If the world is infinite in time past then the number of days before today is equal to the number of days before yesterday. But there are obviously fewer of these.

[10] The following takes further the discussion of the issue in Priest (1995, Scs. 9.5, 11.5).

[11] As medical research develops, it is not uncommon to find that a syndrome that had been given a single name is, in fact, a multiplicity of separate illnesses. They are different illnesses because they have different causes—though ones with similar effects. This appears to be the case with schizophrenia and maybe cancer, for example.

[12] As *SPSR* puts it (p. 32), it is the structure of an inclosure that *produces* the contradictions.

[13] Some of the history of these paradoxes, with further references, is given in Priest (1995, Sc. 2.6).

2. If the world is infinite in time past then the number of months before now must be 12 times the number of years before now, but this is already infinite, and so is as large as can be.
3. There are more natural numbers than even natural numbers; yet there must be the same number of each, since they can be put into one-to-one correspondence.

It was not clear in the middle ages that these paradoxes were members of a kind. Certainly, solutions offered for some of them, did not apply to others. For example, John Philoponus suggested that the solution to the first two was that time past was not infinite. (Indeed, he used these as arguments for that claim.) This clearly has no relevance to paradox number 3.

With the wisdom of hindsight, however, we can now see that they are the same kind of paradox. This is because they are all examples of a single phenomenon, namely, that an infinite set can have a proper subset that is the same cardinal size as itself. This fact also provides the solution to all the paradoxes. And for just this reason, Philoponus' solution, even if he is right about the length of time past, is beside the point. (Just as those are beside the point who maintain that in some inclosure paradoxes, the totality, Ω, does not exist.) How do we know that this phenomenon is behind each paradox? Simply because the fact explains each of its manifestations. Once one understands this fact about cardinality, one sees why each of the examples arises.

Now, in exactly the same way, all inclosure contradictions are generated by the same underlying mechanism: an operation that diagonalises out of totalities of certain kinds—subsets of Ω—whilst giving an object that is still in Ω, the contradiction arising when we diagonalise out of Ω itself. This is the mechanism that underlies each of the contradictions, explaining why, essentially, it arises. As Priest (1995, p. 149), puts it: once one understands how it is that a diagonaliser manges to "lever itself out" of a totality and produce a novel object of the same kind, it becomes clear *why* a contradiction must arise at the limit.

That an inclosure is the underlying mechanism of each paradox may not be obvious to the casual observer: one has to examine the details. But the underlying mechanism of the medieval paradoxes of infinity cited was not obvious at the time either. And once one does grasp the details, it is hard, I think, to resist the conclusion that the existence of an inclosure provides the explanation of the paradoxes. The inclosure paradoxes, then, are of a kind, just as much as illnesses with the same cause, and the paradoxes of infinity that I discussed.

4. Conclusion

If this is right, then all such paradoxes require the same solution. As *SPSR* notes, and Priest (1995) shows in details, standard solutions to the paradoxes of self-reference do not provide what is required. A dialetheic solution does. Grattan-Guinness requests more details of this solution (p. 831). I have given these in Priest (1987) and elsewhere. There is therefore no need to say more on the matter here.

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REFERENCES

Curry, H. B. 1942: "The Inconsistency of Certain Formal Logics". *Journal of Symbolic Logic*, 7, pp. 115–7.

Grattan-Guinness, I. 1998: "Structural Similarity or Structuralism? Comments on Priest's Analysis of the Paradoxes of Self-Reference". *Mind*, 107, pp. 823–34.

Halpern, J. Y. and Moses, Y. 1986: "Taken by Surprise: the Paradox of the Surprise Test Revisited". *Journal of Philosophical Logic*, 15, pp. 281–304.

Löb, M. H. 1955: "Solution of a Problem of Leon Henkin". *Journal of Symbolic Logic*, 20, pp. 115–8.

Priest, G. 1987: *In Contradiction*. Dordrecht: Martinus Nijhoff.

——1994: "The Structure of Paradoxes of Self-Reference". *Mind*, 103, pp. 25–41.

——1995: *Beyond the Limits of Thought*. Cambridge: Cambridge University Press.