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ON AN ERROR IN GROVE'S PROOF

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Nearly a decade has past since Grove [1988] gave a semantics for the AGM postulates. The semantics, called sphere semantics, provided a new perspective of the area of study, and has been widely used in the context of theory or belief change. However, the soundness proof that Grove gives in his paper contains an error. In this note, we will point this out and give two ways of repairing it.

We follow Grove in matters of notation. To make this paper self-contained, we start by rehearsing some. Let L be a propositional language. Let M_L be the set of all maximally consistent sets of sentences of L.[1] If A is any sentence, $|A|$ is the set of all members of M_L containing A. If $X \subseteq M_L$, $t(X)$ is the theory $\cap X$. If Σ is a set of sentences, $\mathrm{Cn}(\Sigma)$ is the set of logical consequences of Σ. If T is a theory in L, $c(A)$ is a certain subset of M_L —intuitively, the smallest "sphere" containing all extensions of T, some of which contain A— and $T + A$ is defined as $t(|A| \cap c(A))$. For future reference, T / A, the *expansion* of T by A, is $\mathrm{Cn}(T \cup \{A\})$.

We can now state the problematic step in Grove's proof. On p. 161, in verifying the postulate +7, two successive lines of the proof are:

$$|A| \cap |B| \cap c(A) \subseteq |A| \cap |B| \cap c(A \wedge B)$$
$$|B| \cap |T+A| \subseteq |\mathrm{Cn}((A,B))| \cap c(A \wedge B)$$

The left-hand side of this step relies on the fact that:

$$|T+A| \subseteq |A| \cap c(A)$$

i.e.:

$$|t(|A| \cap c(A))| \subseteq |A| \cap c(A)$$

[1] Alternatively, M_L can be taken as set of all models of the language, but we follow Grove here.

But in general $|t(X)| \not\subseteq X$. Let X be the class of all maximal consistent sets minus any one of them. $t(X)$ is the set of all tautologies, since for any non-tautology there is more than one maximal consistent set which does not contain it. Hence $|t(X)| = M_L$. A similar problem besets the verification of +8.

The problem can be repaired as follows.[2] We do not replace $|A| \cap c(A)$ by $|T + A|$ at this stage of the proof, but simply carry it along. The proof then becomes:[3]

$$\begin{aligned} |A| \cap |B| \cap c(A) &\subseteq |A| \cap |B| \cap c(A \wedge B) \\ |A| \cap |B| \cap c(A) &\subseteq |A \wedge B| \cap c(A \wedge B) \\ t(|A \wedge B| \cap c(A \wedge B)) &\subseteq t(|B| \cap |A| \cap c(A)) \\ T + A \wedge B &\subseteq t(|A| \cap c(A)) / B \end{aligned}$$

By definition, $t(|A| \cap c(A)) = T + A$. Hence $t(|A| \cap c(A)) / B = T + A / B$, as required.

Now this new proof requires the function t to satisfy the condition that $t(S) / A = t(S \cap |A|)$ for any $S \subseteq M_L$. Grove states this as a property of t but does not prove it.[4] A proof is illuminating, and goes as follows.[5] Note that $A \rightarrow B$ is the material conditional $\sim A \vee B$.

Proof: Right-to-left: Suppose that for some sentence B, $B \in t(S \cap |A|)$. Then for all $x \in S \cap |A|$, $B \in x$. Hence, $A \rightarrow B \in x$. Moreover, since $\sim A$ entails $A \rightarrow B$, $A \rightarrow B \in x$ for all $x \in |\sim A|$. So $A \rightarrow B \in x'$ for all $x' \in S \cap (|A| \cup |\sim A|) = S$. Hence $A \rightarrow B \in t(S)$. Thus $B \in \mathrm{Cn}(S \cup \{A\}) = t(S) / A$ (by detachment for the material conditional). Since B is an arbitrary sentence, $t(S \cap |A|) \subseteq t(S) / A$.

Left-to-right: Suppose that for some sentence B, $B \in t(S) / A$. Then $B \in \mathrm{Cn}(t(S) \cup \{A\})$. So $A \rightarrow B \in t(S)$. Hence for all $x \in S$, $A \rightarrow B \in x$. Now for any $x' \in S \cap |A|$, $A \rightarrow B \in x'$. Also $A \in x'$. Hence $B \in x'$ by material detachment. Thus $B \in t(S \cap |A|)$. QED.

This proof is perfectly correct in classical logic, but it may fail in a non-classical logic, particularly one where material detachment, the disjunctive syllogism, fails, as it does in standard relevant and paraconsistent logics. In

[2]As was pointed out by Grove in correspondence.

[3]The proof appeals to the following property of t: For $S, S' \subseteq M_L$, if $S \subseteq S'$ then $t(S') \subseteq t(S)$. See Grove [1988], p. 158.

[4](3), p. 158.

[5]This is the heart of the proof of postulate +7 that is given in Priest, Surendonk and Tanaka [1996]. A similar proof was given by Grove in correspondence.

other words, the argument will fail if one generalises Grove's semantics to non-classical logics, as is entirely possible.[6]

Is there any way that the proof may be repaired more generally? The answer is yes. An elementary class is one of the form $|B|$ for some sentence B. It is well known that if X is an elementary class then $|t(X)| = X$. Moreover, the proof of this fact is a quite general, model-theoretic, one, and has nothing to do with classical logic.[7] Hence, if we require that every sphere in M_L be an elementary class, Grove's original argument goes through.[8]

This constraint means, in effect, that every sphere represents a theory that is finitely axiomatisable. This is not an implausible constraint if one takes it that spheres represent theories that are accessible to the belief-reviser in question, and that the reviser is a finite agent. In this way, Grove's construction may be liberated from the vicissitudes of classical logic.

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[4] K. Tanaka [1995], 'Paraconsistent Belief Revision', Honours Thesis, Department of Philosophy, University of Queensland.

[6] See Tanaka [1995].

[7] See, e.g., Bell and Slomson [1969], p. 141.

[8] This is the approach taken in Tanaka [1995].