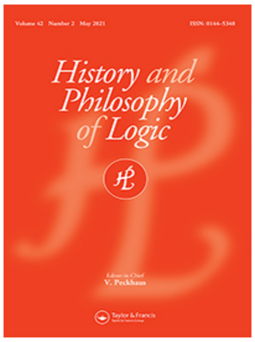

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Reflections on Orlov

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In 1928 Ivan Orlov published a remarkable paper which contains the first formulation of a relevant logic. The paper remained largely unknown to English-speakers until this discovery of relevant logic was shown by Kosta Došen in 1992. By that time the material on relevant logic was well known; however, there is more of interest in Orlov's paper than this: his remarks on intuitionism, the motivation for his construction, and its wider implications. This paper explains and reflects on these matters.

Keywords: Relevant logic; Ivan Orlov; Kosta Došen; intuitionism; paraconsistency; logic in Russia

1. Introduction

In *1928* Ivan Orlov produced a remarkable paper, 'The Calculus of Compatibility of Propositions'.[1] It included the first formulation of what was to become known as a relevant (or relevance) logic–indeed, of a modal relevant logic. The modal logic is (relevant) S4, though the honours for being first to produce *S*4 cannot go to Orlov, since this had already been done by C. I. Lewis in the United States,[2] though Orlov knew nothing of this.[3]

The ideas in the paper were not further developed by Orlov himself or taken up by others. The contents of the paper was presented to English-speaking philosophers by Kosta Došen only in *Došen 1992*.[4] But by this time there had been a great deal of work on relevant logic in the United States and Australia, and there was nothing new in Orlov's work.[5]

However, despite the fact that the material on relevant logic is now well known, the way that Orlov's discussion frames them is not. What follows are some reflections on this. We will look at Orlov's views on intuitionism and the way that they motivated the logic he

[1] As far as I know, none of Orlov's papers have yet appeared in English. I am indebted to Anya Yermakova for sharing her translation of Orlov's paper with me (*Yermakova 2019*). Quotations and page references to Orlov in what follows are taken from this.

[2] *Lewis 1914*, *1918*.

[3] For an account of Orlov, his life and his work, see *Bazhanov 2003*. A referee of an earlier draft of this paper–who I presume is Russian–commented: 'Orlov's 1928 article is the only purely mathematical work of the many works of the scientist. The rest of his works are devoted to the philosophy of mathematics, logic, physics, psychology, etc. In these works, Orlov was looking for a certain dialectics in nature, i.e. a "logic of natural sciences". All the works by Orlov were saturated with ideological content, which was completely absent in the 1928 article. Orlov did his best to propose a new logic suitable to introduce namely "logic of natural sciences". Orlov certainly seems to have had a change of heart concerning mathematical logic a few years before he published his paper. Some of that part of the story is told in § 1 of *Stelzner 2002*.

[4] The paper is also mentioned briefly in *Chagrow and Zakharyaschev 1992*.

[5] Dimitry Zaitsev tells me that Orlov's paper was reported by V. M. Popov in his doctoral dissertation (in Russian) in the early 1970s, and is described in *Popov 1986*. The referee mentioned in fn 3 drew my attention to the fact that *Bazhanov 2007*, pp. 246–247, reports that it was also noted by Zinoviev in 1962 and Routley in 1991. My inability to read Russian has prevented me from tracking down these references. Zaitsev also pointed out to me that Orlov's paper (with French abstract) appears as item 379 of the logic bibliography *Church 1936*. *Church 1951* is the first well known paper on relevant logic.

constructed. We will see that the motivation is incorrect, though not without its interest. Finally, we consider the natural extrapolation of Orlov's ideas into the area of which he could have had no real idea–paraconsistency.[6]

2. Background on relevant logic

But first some background.

A (propositional) logic is relevant if whenever $A \to B$ is a logical truth, A and B share a propositional parameter. 'Classical logic' (that is, the logic invented by Frege and Russell a few decades before Orlov wrote) deploys the notion of the material conditional. As hardly needs to be said, this delivers a spectacular violation of relevance. Orlov was dissatisfied with the material conditional, as was Lewis. In particular, for reasons that we will come to, he particularly balked at the principle of material equivalence:

$$MEq\text{: } A, B \models A \leftrightarrow B$$

As we now know, there are many relevant logics,[7] Orlov invented a very well known member of the family, R. Indeed, he invented the $S4$ish extension of this known as NR.[8]

Or to be precise, he invented the intensional fragments of these, where the only non-modal operators are $\to$ and $\neg$. The logic does not contain an extensional conjunction or disjunction, but it does contain an intensional conjunction and disjunction, $\circ$ and $+$. These can be defined, respectively as: $\neg(A \to \neg B)$ and $\neg A \to B$.[9] In modern terms, the first of these is sometimes called *fusion*. Orlov calls it *joint compatibility*.[10] And as the title of Orlov's paper indicates, it is his key notion. As he is aware, one can take it as primitive, and define $A \to B$ as $\neg(A \circ \neg B)$. For future reference, note that intensional excluded middle, $A + \neg A$. is valid in the logic.

There are many reasons to be dissatisfied with the irrelevance of a conditional as such.[11] But these were not Orlov's motivation. So let us now turn to what was.

3. Intuitionism

The 1920s cannot have been an easy time to be an intellectual in the Soviet Union; but those interested in logic certainly knew of a number of the things going on in the subject in other parts of Europe.

It is clear from the references in Orlov's paper that he knew about Russell and Whitehead's *Principia*, and some of the works of Hilbert and his school. He also knew of some of Brouwer's papers on intuitionism. Indeed, work on intuitionism seems to have been of particular interest to logicians in Moscow at this time. Thus, both Kolmogorov and Glivenko were engaging in intuitionist thought there.

[6] Although Orlov had an interest in a formal account of dialectical logic (see *Bazhanov* 2003). In some incontestable sense, dialectics involves contradictions, as does paraconsistency.

[7] For a brief account see, e.g. *Mares* 2012. Note that one can have a material conditional, $\neg A \vee B$, in a relevant logic, and it will satisfy *MEq*. But it does not detach. This is enough to finesse any problem that Orlov foresaw with it–as well as hamstringing it as a working conditional.

[8] On *NR*, see *Routley and Meyer* 1972.

[9] See *Došen* 1992.

[10] See *Došen* 1992. Došen notes (p. 341) that fusion is a connective that frequently appears in the formulation of substructural logics (including relevant logic). There is nothing in Orlov's paper to suggest that he foresaw such developments, however. Quite the contrary, as we shall see.

[11] Articulated, for example, in *Anderson and Belnap* 1975.

Intuitionistic ideas were certainly what was driving Orlov's construction. He says:[12]

> One can also assert that the development of the ... [GP: logical theory that follows] is a problem waiting to be addressed, of the necessity to reconcile symbolic logic with new techniques introduced by intuitionism.
> An important aspect of intuitionism lies in the fact that in works of intuitionists concepts used depend not directly on used propositions A, B, C, but on functions of the latter, of the following form: 'A is established', 'A is provable', 'A leads to absurdity', 'absurdity of A is absurd', etc.[13] That intuitionists employ similar functions is a fact, and this fact cannot be ignored in the development of mathematical logic. **However, the introduction of similar functions into classical mathematical logic leads to contradiction with the law 'tertium non datur': later it will be shown that it is not possible at all.** Meanwhile, the 'calculus of joint compatibility of propositions' allows for the introduction of said functions and symbolic operations on them. As a result, the position of intuitionism does not require the rejection of the law 'tertium non datur', that is, is in complete agreement with it.

In other words, Orlov says that he wants a way of expressing the ideas of intuitionism without giving up the Principle of Excluded Middle. How does he propose to do this?

In some incontestable sense, truth in intuitionist mathematics is provability. Orlov proposes writing this as $\Box$, and says (*Orlov 1928*, p. 280):

> We claim that similar functions [GP: to $\Box$ and $\Box\neg$], even if without special symbolic denotation, in fact are introduced by intuitionists into mathematics. 'For any [given] system, every property [свойстсво] is either correct or impossible' is evaluated differently in purely formal mathematics and in works of intuitionists precisely because it is not understood in the same sense. If in the first case any property [of an object] is taken 'in itself', then in the second case the question of the provability of the condition is advanced. In the first case the question about the existence of a condition is treated in the sense of the expression 'A', while in the second, in the sense of '$\Box A$'.

In other words, in the mouth of an intuitionist, A really means $\Box A$. Similarly, he continues, $\neg A$ really means $\Box\neg A$. Thus:

> Brouwer examines the case where, from a property follows its negation, only as a particular case of 'absurdity', understanding this expression more broadly.[14] He uses the expression 'absurd' in those cases where it may be proven that A contradicts some axiom or some proven sentence. But our expression $\Box\neg A$ also has the same meaning. From here Brouwer's claim becomes clearer, that the correctness of a property and its absurdity form a complete disjunction only in a defined finite system. Even though the conditions of such a system can remain partially unknown, one needs only sufficient amount of time for their complete determination.[15]

In other words, Orlov takes classical mathematics to be unproblematic; but when intuitionists assert or deny a statement, they mean something different from classical mathematicians. What they mean can be captured by placing what they say in the scope of a $\Box$. In

[12] *Orlov 1928*, p. 263. I have updated Orlov's notation where appropriate. In particular, Orlov writes $\Box$ as Φ, and $\Box\neg$ as X. Yermakova's interpolations are simply in square brackets. My interpolations are marked thus [GP: ...]. The boldfacing is mine. Note this passage, we will come back to it.

[13] GP: A footnote refers to *Brouwer 1925a*.

[14] GP: A footnote refers to *Brouwer 1925b*.

[15] GP: A footnote refers to *Brower 1927*.

that way, their assertions can be embedded in classical discourse, and hence are compatible with Excluded Middle.

I note two things about Orlov's project *en passant*. First, he is in the process of giving an account of standard mathematical reasoning different from that of 'classical logic'; and in particular, he suggests replacing the material conditional with a relevant $\rightarrow$. For his project to be successful, it therefore needs to be shown that classical mathematics can be carried out by reasoning in a relevant logic. This is not at all easy. Work on relevant mathematics, aiming to show how one may interpret classical theories using a relevant logic ('classical recapture') has proved to be difficult.[16] Properties of the material conditional seem to be used essentially in reasoning in classical mathematics.

Secondly, Orlov's thesis is that intuitionist mathematics is classical mathematics 'under a box'. To a certain extent, this idea has been shown to be successful by the translations between intuitionist logic and classical $S4$ (not, NB, relevant $S4$), discovered by Gödel, and McKinsey and Tarski.[17] Such show that constructive reasoning can be understood classically. However, we now know that intuitionist mathematics is not just classical mathematics done using a constructive logic. It goes beyond this.[18] So there is still an issue of how this 'beyond' is to be understood classically.

Now, Orlov mentions neither of these matters. Here, I think he may be forgiven. Both of these things are clear with the wisdom of hindsight; but when he was writing, there was virtually no understanding of these matters.

4. The box

Back to Orlov's motivation. Orlov, then, needs a logic for 'it is provable that', and he takes this to work, plausibly enough, as does the $\Box$ of S4. In particular, it satisfies the familiar axioms and rules:[19]

- $\Box A \rightarrow A$
- $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
- $\Box A \rightarrow \Box\Box A$
- if $\vdash A$ then $\vdash \Box A$

Provability here, note, is not provability in some axiom system or other, but provability in the sense of 'establishing as true', naive provability.[20] Given this understanding, the first axiom is clearly true.[21] The second is a way of stating that *modus ponens* holds. The third says that if A is proved, this itself has been proved. The proof of A is, presumably, sufficient to establish this–or at the very least, running the proof through a proof-checker will provide the proof. The rule (Necessitation) says that if something has been established by the rules of proof in operation, it is proved; which is correct, assuming that the rules of proof are (as one would hope) sound.

[16] To give just two examples: the problem of recapturing classical Peano Arithmetic in $R^{\#}$ (see *Friedman and Meyer 1992*); and the difficulty of recapturing standard results concerning cardinals and ordinals in relevant set theory, even with unrestricted set-theoretic comprehension (see *Weber 2010* and *2012*)–though here the problem is exacerbated by the fact that the set theoretic principles force the use of a relatively weak relevant logic.

[17] See *Priest 2008*, pp. 117, 119, 455.

[18] See, e.g. the discussion in *Bridges 2018*.

[19] Proposed in this form by Gödel only in 1933. See *Iemhoff 2019*, § 1.2.

[20] As *Priest 2006*, ch. 3, calls it.

[21] Commenting on this, Orlov says (p. 282, fn 24): 'In other words: "provability of A presumes truthhood of that sentence ". The reverse sentence $A \rightarrow \Box A$ has no place in our system. This way, we accept the existence of truths [that are] unprovable by the current state of science. This kind of admission, as is obvious, has nothing in common with "ignorabimus"'.

Perhaps one should demand other conditions on naive provability, but one would certainly not expect it to satisfy the distinctive S5 axiom:

- $\neg\Box A \rightarrow \Box\neg\Box A$

The mere fact that something is not provable does not deliver a proof of this fact.

We can now turn to Orlov's motivation for his invention of relevant logic. He explains this as follows (p. 286):

> The introduction of the said functions [GP: $\Box$ and $\Box\neg$] into classical mathematical logic is not possible, since the interpretation of the concept of 'following from', as material implication, erases meaning for all the expressions proven for the functions we introduced. Besides, in classical theory there are sentences assumed and inferred, that from our point of view cannot be evaluated in any way other than false. For example, from the sentence provable in classical theory 'all true sentences are equivalent', the following conclusion follows:
>
> $$A \leftrightarrow \Box A \leftrightarrow \Box\neg\Box A \leftrightarrow \Box\neg\Box\neg\Box A$$
>
> Such a conclusion makes the introduction of this type of functions devoid of any kind of meaning; in this case when constructing schemata of transfinite inferences, there would be no other way except the rejection of the law 'tertium non datur'.

What Orlov appears to be saying is that if one has *MEq* (and Excluded Middle), we have modal collapse: $A \leftrightarrow \Box A$. Došen puts the matter as follows:[22]

> In the last paragraph of his paper, Orlov comes to a rather odd conclusion. He thinks that the modal postulates of S4 can be added to OR [GP: the intensional fragment of *R*], but cannot be added to classical logic. Orlov's opinion is that his modal theorems become senseless if we interpret $\rightarrow$ as material implication, and he somehow infers from the classical principle 'all true propositions are equivalent' that we cannot add the modal postulates of S4 to classical logic without also adding $A \rightarrow \Box A$, which would, of course, make our modal system trivial and useless.

Došen calls this paragraph of Orlov the 'least fortunate point of a paper otherwise graced with acumen and foresight'.

Perhaps the judgment is harsh. Many things were clear by the time that Došen was writing that were not clear in the 1920s. However, Došen is right that Orlov is just wrong. As we now know, in S4 based on classical logic, one can have the material conditional–with its *MEq* – and Excluded Middle, without modal collapse. Indeed, once one has the $\Box$, one can define a notion of joint compatibility as $\Diamond(A \wedge B)$. Deploying this, one can define a non-material, non-relevant, strict conditional, $A \multimap B$, as $\neg\Diamond(A \wedge \neg B)$. This was the way that Lewis went.

So why does Orlov think that one cannot have this? He does not explain, so any answer has to be conjectural; but here is a suggestion. He reasoned as follows. Either A or $\neg A$. (As we noted, $A + \neg A$ holds in *R*.) Suppose that A. In an intuitionist context, this means that A is provable, that is, $\Box A$. Since both are true, $A \leftrightarrow \Box A$ by *MEq*. Suppose that A is false, $\neg A$. Again, in an intuitionist context, this means that A is refutable. That is, $\Box\neg A$. So $\neg\Box A$. Hence, $\Box A$ is false as well. Thus, again by *MEq*, $A \leftrightarrow \Box A$, in either case. All modal distinctions collapse.

[22] *Došen 1992*, p. 350.

If this *is* Orlov's reasoning, it betokens a misunderstanding of reasoning under assumption. Even in intuitionistic logic, to assume that A is not to assume that A is proved (so that one can apply Necessitation). In particular, the validity of a natural deduction proof of the form:

$$\frac{\begin{matrix}\mathcal{P}\\ A\end{matrix}}{\Box A}$$

depends crucially on whether the proof $\mathcal{P}$ has undischarged assumptions. But I think it fair to say that reasoning under assumption was not really well understood until Gentzen's work, a decade or so after Orlov.[23]

I note that, like many logicians at this time, Orlov does not appear to be very clear about the distinction between the conditional and deducibility. He simply reads $\rightarrow$ as 'follows from'. For the same reason, he is unclear about the distinction between an axiom and a rule of inference. Thus, his statement of *modus ponens* is as follows (p. 266):

> Axioms, as well as propositions inferred from axioms, can be omitted from the makeup of symbolic formulae, if they serve as premises for some kind of inferences.

His statement of the rule of Necessitation is equally unclear (p. 282):

> Given that we regard all axioms and all statements deduced from them as provable and therefore true, then every expression assumed in the form of an axiom, or deduced in preceding paragraphs, can be written in the form of the function $\Box A$.

Perhaps this confusion feeds in to his error.

5. Modal collapse

So as we now know, one has to give up neither Excluded Middle nor *MEq* to be able to accommodate a provability $\Box$. However, Orlov's mistake–if that was his mistake–is no simple mistake. It is tracking something which really does threaten modal collapse in an intuitionist context. This may be seen in two ways.

For the first, the semantics of the intuitionist connectives are generally agreed to be captured by the BHK (Brouwer–Heyting–Kolmogorov) semantics,[24] according to which:

- a proof of $A \rightarrow B$ is a construction which converts a proof of A into a proof of B

With this understanding, $A \rightarrow \Box A$ would appear to be valid. For, as noted, given a proof of A, that very proof (or at least its verification) shows that A is provable, and so is a proof of $\Box A$. Of course, we have the converse direction trivially. So we have modal collapse.

The second way to see this is to consider the Kripke semantics for propositional intuitionist logic (which are perfectly acceptable to an intuitionist if these are formulated with underlying intuitionist logic). In these semantics there are worlds which represent stages of investigation; and, intuitively, the things true at a world are exactly those things that are provable at that stage.[25] Given this, if we introduce a provability $\Box$ into the language, the

23 See *von Plato* 2014.
24 See *Iemhoff* 2019, § 3.1.
25 See, e.g. *Priest* 2008, ch. 6, esp. 6.3.6.

natural truth conditions are:[26]

- $w \Vdash \Box A$ iff $w \Vdash A$

The validity if $A \leftrightarrow \Box A$ is immediate.[27]

The collapse that Orlov wished to avoid, then, seems to be inherent in intuitionism itself. Whether that collapse is something that an intuitionist should simply embrace, or whether it is something they should wish to avoid, and if so, how, I leave for intuitionists to worry about.[28]

6. Paraconsistent logic

Let us turn instead to a topic that was not on Orlov's agenda: paraconsistency. Intuitionist mathematics is driven by the thought that truth is to be identified with provability, and that there may be things that are neither provable nor refutable: $\neg\Box A$ and $\neg\Box\neg A$. Paraconsistent mathematics is not driven by the thought of identifying truth with provability, though it is certainly open to a paraconsistent mathematician to adopt this position. But it is very much driven by an interest in theories where something may be both provable and refutable: $\Box A$ and $\Box\neg A$–in a sense, the dual of the situation we have been dealing with so far.

Orlov's logic is, in fact, well placed to handle such a situation, since R is, in fact, a paraconsistent logic. Neither of the following is a logical truth:

- $A \to (\neg A \to B)$
- $(A \circ \neg A) \to B$

and when the intensional fragment is extended to contain extensional conjunction and disjunction ($\wedge$) and $\vee$), the following is not a logical truth either:

- $(A \wedge \neg A) \to B$

Hence, R can be the underlying logic of inconsistent but non-trivial theories.[29]

In particular, we may add axioms to those of NR, its intensional fragment – or the first-order extensions of these–which entail contradictions, but not everything. Those axioms have to be in the language of the relevant logic, of course, and this is not the place to go into the topic of inconsistent but non-trivial mathematical systems based on relevant logics. (In fact, the properties of the conditional in R put constraints on what axioms may deliver

[26] Note that these conditions preserve the heredity constraint required by the semantics.

[27] In standard intuitionist modal logics, there are two accessibility relations: one for the intuitionist machinery and one for the modal machinery (see *Božić and Došen 1984*). There is no modal collapse in these. The point here is that modal collapse seems to hold in virtue of understanding $\Box$ as provability, and understanding what holds at a world as what is provable there.

[28] I note that *Dummett 2002*–who has as good a claim to be an intuitionist as anybody–appears to endorse the identity of truth and assertibility (that is, in the case of mathematics, provability). *Rumfitt 2002* objects, precisely on the ground of modal collapse. I note also that *Artemov and Protopopescu 2016* argue that if $\Box$ is interpreted as 'it is known that', then for an intuitionist, one has $A \to \Box A$, but not the converse, since knowledge may hold in virtue of something other than the possession of a proof.

[29] Paraconsistency is usually defined as the invalidity of the inference $A \wedge \neg A \vdash B$. Orlov's paper does not, however, define a consequence relation, $\vdash$. There is no sense in which an inference is valid other than that the corresponding conditional is a logical truth. Early Anglo work in relevant logic also concentrated on logical truth, not valid inference, so much the same could be said about this. Of course, it is well aware of the distinction between axioms and rules of inference; but the rules of inference are conceptualised in terms of logical-truth-preservation. The standard rules:

$$\frac{A \quad A \to B}{B} \qquad \frac{A \quad B}{A \wedge B}$$

do, however, preserve truth at the normal words of a Routley-Meyer semantics.

this. We will see a case in a moment.) But to illustrate the idea in principle, let p and q be different propositional parameters, and let us add p and $\neg p$ to the axioms of *NR*. q does not follow. The proof is simple. Take a Routley-Meyer model for the relevant logic *NR*.[30] The model has a base (normal) world, 0; and any world, x, has a mate, x^*, used in the truth conditions for negation. Let $I(A, x)$ be the value of A at world x (T or F). Consider any model where:

- $I(p, 0) = T$
- $I(p, 0^*) = F$
- $I(q, 0) = F$

Then 0 a model of the logic at which p and $\neg p$ hold, but q does not.

As is clear, what *does* follow from our axioms (by Necessitation) are $\Box p$ and $\Box\neg p$. So we have the situation dual to the one Orlov considers. Moreover, since we have $\Box p \to p$, and the conditional contraposes, we also have $\neg\Box p$. So we have a contradiction of the form $\Box p$ and $\neg\Box p$. This may be thought odd; what can it be for something to be provable and not provable? Certainly if one thinks of something's being provable simply as its occurring at the end of a certain sequence of formulas, this does seem odd.[31] And it might motivate moving to a relevant logic where contraposition for $\to$ fails. Such cannot happen in Routley-Meyer semantics. Rule contraposition ($A \to B \vdash \neg B \to \neg A$) is guaranteed by the $*$ semantics for negation. However, there are relevant logics of a different kind where it does fail.[32]

But on the understanding of provability Orlov has in mind, it is not so implausible. Recall that the sense of provability at issue here is *establish as true*, and so it is implicitly semantic. In this sense, one may reasonably understand *it is provable that* A as: A and $\langle A\rangle$ is warranted. Hence, if $\neg A$, it follows that it is not provable that A. So 'it is provable that A' can be false, even if is true as well.

7. Paradox

Whilst we are on the topic of provability, let us consider, further, the fact that this is a notion that is itself notoriously beset with paradox. Thus, there is 'Gödel's Paradox', concerning a sentence which says of itself that it is not provable.[33] Let G be the sentence 'it is not provable that G'. If this is provable, it is true, and so not provable. Hence it is not provable. But we have just proved this; so it is.

NR is of course negation-consistent. (It has a model in which the base world, 0, is consistent.) The major reason why it cannot represent Gödel's Paradox is that the language has no way of constructing such a self-referential sentence. However, if it were augmented by the apparatus of arithmetic and a truth predicate satisfying the T-schema ($T\langle A\rangle \leftrightarrow A$, for all A), there would be. For given standard techniques of diagonalisation, we could construct a sentence, H, which is $\neg\Box T\langle H\rangle$ (where $\langle H\rangle$ is a name for H).[34] Hence, $T\langle H\rangle \leftrightarrow H \leftrightarrow \neg\Box T\langle H\rangle$, and we can take G to be $T\langle H\rangle$.[35]

30 An appropriate version of these can be found in *Routley and Meyer* 1972. I follow the notation there.

31 Though as a matter of fact, matters are not so straightforward. See *Priest* 2006, ch. 17, esp. 17.8.

32 Notably, the 4-valued semantics. See, e.g. the logic N_4 of *Priest* 2008, ch. 9. See also *Priest and Sylvan* 1992 and *Restall* 1993.

33 See *Priest* 2006, ch. 3.

34 If this is done using Gödel numbering, and arithmetic is formalised with the usual vocabulary one can construct only a sentence, G, that is materially equivalent to $\neg\Box T\langle G\rangle$. However, if the language contains function symbols for further primitive recursive functions, and especially the diagonal function, one can construct a sentence, G, that is literally identical with $\neg\Box T\langle G\rangle$. See *Priest* 2006, pp. 48f.

35 Note that $\Box$ is a provability operator, not a provability predicate. The paradoxical nature of a provability predicate was pointed out by *Montague* 1963. I note that there is a modal 'logic of provability' in which the $\Box$ works as does the proof predicate

Rather than go into all the details here, let us proceed as follows. Fix a propositional parameter. Call this G. We now add the axioms:[36]

- $\vdash G \to \neg\Box G$
- $\vdash \neg\Box G \to G$

For future reference, call these the G-Axioms. Since $\vdash \Box G \to G$, $\vdash \Box G \to \neg\Box G$. Hence, $\vdash \neg\Box G$–by Excluded Middle (in the extensional vocabulary) or *Consequentia Mirabilis* $((A \to \neg A) \to \neg A)$, in the purely intensional vocabulary. Hence $\vdash G$, and so $\vdash \Box G$.

Although the theory, thus augmented, is inconsistent, it is non-trivial. A simple proof of this proceeds in two sages. Take any propositional parameter, p, distinct from G, and suppose that there is proof of it. Consider the map which turns a formula of the language of NR into the language of R by deleting boxes. All the axioms and rules of NR map into axioms and rules of R. The G-Axioms transform into $G \to \neg G$ and $\neg G \to G$. Hence, if there is a proof of p in NR, there is a proof of p from these axioms in R.

We can now show that this is not the case. The easiest way to do this is to use the simplified semantics of R.[37] In this, if n is a normal world then: if $Rnxy$ then $x = y$. Now take an interpretation of R and a normal world, n, in it, where:

- p is false at n
- G is true at one and only one of every pair of worlds, w and w^*.

At any world, w, G is true at w iff G is false at w^* iff $\neg G$ is true at w. Hence $G \to \neg G$ and $\neg G \to G$ are true at n. But p is not true at n. So it cannot follow from these statements.

Finally, observe the following. If we had a truth predicate satisfying the T-Schema and the self-reference provided by arithmetic, then, by the same sort of construction as before, for any sentence, A, we could find a sentence C_A such that:

- $\vdash C_A \to (\Box C_A \to A)$
- $\vdash (\Box C_A \to A) \to C_A$

By a familiar Curry Paradox argument, we now have the following. By the first of these:

- $\vdash \Box C_A \to (\Box C_A \to A)$
- $\vdash \Box C_A \to A$

(This step holds because Contraction is valid in R: $(D \to (D \to E)) \to (D \to E)$.) So by the second conditional:

- $\vdash C_A$
- $\vdash \Box C_A$
- $\vdash A$

The theory is trivial.

Thus–as is well known–there are certain inconsistent theories of which NR cannot be the underlying logic without triviality. There are, however, weaker relevant logics–crucially,

in Peano Arithmetic. In this, the condition $\Box A \to A$ is weakened to $\Box(\Box A \to A) \to \Box A$ (in the light of Löb's Theorem), precisely in such a way as to avoid the contradictory conclusion. On Provability Logic, see *Verbrugge 2017*.

36 It would be more natural to take this to be a simple biconditional. However, in R one may define $A \leftrightarrow B$ as either $(A \to B) \wedge (B \to A)$ (as is usual) or $(A \to B) \circ (B \to A)$ (as does Orlov). These do not behave in exactly the same way. Using two conditionals avoids such complications.

37 See *Priest 2008*, 10.5.

ones that do not contain Contraction – which can be the underlying logic of theories of this kind.[38] However, to discuss these matters here would take us well away from Orlov.

8. Conclusion

As we have now seen, the interest of Orlov's paper goes well beyond the invention of relevant logic. It remains the case, however, that his invention of relevant logic was an accident, based as it was on a mistake. However, it is hardly the first occasion in the history of ideas when a wrong idea has given rise to something important. If one were a Hegelean, one might call this an example of the cunning of reason.

There remains the question of why Orlov's work was not taken up. Perhaps it is understandable that it was not taken up by non-(Russian speakers); for the paper was never translated into English, and few non-Russian logicians could speak Russian. Why it was not taken up by Russians is more puzzling. Perhaps they noticed the mistake? Or perhaps, as I think more likely, it was just an idea before its time.[39] In any case, had the paper received the attention it deserved, the history of both relevant logic and modal logic–and maybe even paraconsistent logic–would have been notably different. The history of Russian logic would certainly have been different.[40]

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[38] See *Priest 2002*, § 8.2.

[39] The referee mentioned in fn 3 expressed the following view, which I am happy to pass on. 'At the end of the article, the author expresses surprise at the fact that Orlov's ideas did not develop in the USSR. If the author had been more thoroughly familiar with the literature on the history of logic in the USSR, this surprise would not have arisen: mathematical logic in the USSR as a whole (including, of course, its non-classical subtrees) was considered as alien to dialectical thinking, as an example of a bourgeois metaphysics. Therefore, the development of logic was to a large extent mainly inhibited until mid and late 1950's'.

[40] A talk based on the paper was given at the 19th Trends in Logic Conference (dedicated to Orlov), Higher School of Economics, Moscow, October 2019, and the the Department of Philosophy I, Ruhr University of Bochum, January 2021. I am grateful to a number of the people in the audiences for helpful comments and suggestions. Thanks also go to Hartry Field for helpful comments on an earlier draft of this essay, and to two anonymous referees for this journal. My main indebtedness is, of course, to Anya Yermakova, for sharing her translation with me.

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