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SIMPLIFIED SEMANTICS FOR BASIC RELEVANT LOGICS

INTRODUCTION

When C. I. Lewis pioneered modern modal logic, he proposed a number of systems (S4, S5, etc.) in proof-theoretic form. When suitable world-semantics for these systems were produced, it became clear that these systems were but the tip of an ice-berg. Moreover, in the light of the semantics, it became clear that the basic (normal) logic was none of those that Lewis had suggested, but the system now called *K*. This has the most general semantics, other (normal) systems being obtained by adding extra conditions on the binary relation, *R*.

To a certain extent, the history of relevant logics parallels this development. When Anderson and Belnap pioneered relevant logic, they proposed a number of systems (*E*, *R*, etc.) in proofs theoretic form. When suitable world-semantics for these systems were produced, it became clear that these systems were but the tip of an ice-berg. Moreover, in the light of the semantics, it became clear that the basic (affixing) logic was none of those that Anderson and Belnap had suggested, but the system now called *B* (or *BM* if we drop all constraints on $*$). This had the most general semantics: other (affixing) systems being obtained by adding extra conditions on the ternary relation, *R*.

The parallel diverges at this point, however; for whereas there are *no* conditions on the binary *R* for *K*, the ternary *R* for *B* is subject to several conditions – including a hereditariness condition. (See Routley *et al.*, 1982, ch. 4.) The point of this paper is essentially to show how these conditions can be removed, making the parallel exact again. In doing so we simplify the semantics of relevant logics substantially.

In the first part of the paper we will consider the basic positive logic, B^+. In the second half we will consider negation-extensions of B^+. There are two strategies for handling negation in relevant logic:

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one uses the Routley $*$-operation; the other uses four-valued semantics (Routley *et al.*, *loc, cit.*). We consider both approaches. The extant four-valued semantics for relevant logics contain a complication over and above constraints on R: they require *two* ternary relations (one to state truth conditions; the other to state falsity conditions). A feature of the present semantics is that only a single ternary relation is needed. Thus, the four-valued semantics are doubly simplified. Moreover, an interesting divergence emerges here. All negative systems add De Morgan laws to B^+. The basic negative system with the Routley $*$ adds, in addition, contraposition; that for the four-valued semantics adds, instead, double negation. (B itself, adds both.)

We concentrate in this paper on the semantics of the basic affixing relevant systems. It is clear that simplified semantics for all (affixing) relevant logics, along the lines given here, are to be expected. But since details are not as straightforward as might be expected, we leave this topic for another occasion.

1.1. AN AXIOM SYSTEM FOR B^+

First, let us start with an axiom system for B^+. The axioms are as follows (where standard scope conventions are in force):

A1 $\alpha \to \alpha$

A2 $\alpha \to \alpha \vee \beta \quad [\beta \to \alpha \vee \beta]$

A3 $\alpha \wedge \beta \to \alpha \quad [\alpha \wedge \beta \to \beta]$

A4 $\alpha \wedge (\beta \vee \gamma) \to (\alpha \wedge \beta) \vee \gamma$

A5 $(\alpha \to \beta) \wedge (\alpha \to \gamma) \to (\alpha \to \beta \wedge \gamma)$

A6 $(\alpha \to \gamma) \wedge (\beta \to \gamma) \to (\alpha \vee \beta \to \gamma)$

If $\alpha_1 \ldots \alpha_n/\beta$ is a rule scheme, we define its disjunctive form to be the scheme $\eta \vee \alpha_1 \ldots \eta \vee \alpha_n/\eta \vee \beta$. The rules for B^+ are the following plus their disjunctive forms:

R1 $\alpha, \alpha \to \beta/\beta$

R2 $\alpha, \beta/\alpha \wedge \beta$

R3 $\alpha \to \beta, \gamma \to \delta/(\beta \to \gamma) \to (\alpha \to \delta)$.

It should be noted that special cases of R3 (using A1 and R1) are:

Prefixing: $\gamma \to \delta/(\alpha \to \gamma) \to (\alpha \to \delta)$

Suffixing: $\alpha \to \beta/(\beta \to \gamma) \to (\alpha \to \gamma)$

Transitivity: $\alpha \to \beta, \beta \to \gamma/\alpha \to \gamma$

If Σ is a set of formulas and α is a formula then $\Sigma \vdash \alpha$ is defined in the standard (classical) fashion.

We note that the disjunctive forms of R1 – R3 are not normally included in axiomatisations of B^+. However, B and all the related systems we shall refer to in this paper are known to be prime, i.e., if $\vdash \alpha \vee \beta$ then $\vdash \alpha$ or $\vdash \beta$. (This is proved for B in Slaney, 1987. Simple variants of Slaney's metavaluation techniques establish the results for B^+, *BM* and *BD*.) The admissibility of the disjunctive rules in the standard axiomatisation follows straightaway. Before we turn to the semantics we pause to establish one useful bit of proof theory.

LEMMA 0. *If* $\alpha \vdash \beta$ *then* $\gamma \vee \alpha \vdash \gamma \vee \beta$.

Proof. The proof is by a quite orthodox induction over the length of proofs, and is omitted.

COROLLARY. *If* $\alpha \vdash \gamma$ *and* $\beta \vdash \gamma$ *then* $\alpha \vee \beta \vdash \gamma$.

Proof. By the Lemma, $\alpha \vee \beta \vdash \gamma \vee \beta$ and $\gamma \vee \beta \vdash \gamma \vee \gamma$; but $\gamma \vee \gamma \vdash \gamma$ (A1, R2, A6, R1). The result follows by transitivity of deducibility.

1.2. SEMANTICS FOR B^+

An interpretation for the language is a 4-tuple $\langle g, W, R, I\rangle$, where W is a set (of worlds); $g \in W$ (the base world); R is a ternary relation on W; and I assigns to each pair of world, w, and propositional parameter, p, a truth value $I(w, p) \in \{1, 0\}$. Truth values at worlds are then assigned to all formulas by the following conditions:

$$1 = I(w, \alpha \vee \beta) \text{ iff } 1 = I(w, \alpha) \text{ or } 1 = I(w, \beta),$$

$$1 = I(w, \alpha \wedge \beta) \text{ iff } 1 = I(w, \alpha) \text{ and } 1 = I(w, \beta),$$

$$1 = I(g, \alpha \rightarrow \beta) \text{ iff for all } x \in W \text{ (if } 1 = I(x, \alpha) \text{ then } 1 = I(x, \beta)).$$

For $x \neq g$:

$$1 = I(x, \alpha \rightarrow \beta) \text{ iff for all } y, z \in W \text{ (if } Rxyz \text{ then (if } 1 = I(y, \alpha) \text{ then } 1 = I(z, \beta)).$$

Giving $\alpha \rightarrow \beta$ different truth conditions at g and elsewhere is the heart of the matter. At g $\alpha \rightarrow \beta$ receives S5 truth conditions. Elsewhere $\alpha \rightarrow \beta$ receives the standard ternary truth conditions. We note that we could give ternary truth conditions at all worlds, at the cost of introducing the modeling condition: $Rgxy$ iff $x = y$. This at once reduces the ternary relation to a binary one at g and also makes g access all worlds. (This was, in fact, how the simplified semantics were discovered.)

Semantic consequence is now defined in terms of truth preservation at g:

$$\Theta \models \alpha \text{ iff for all } \langle g, W, R, I \rangle$$

(if $1 = I(g, \beta)$ for all $\beta \in \Theta$ then $1 = I(g, \alpha)$).

1.3. SOUNDNESS

We can now demonstrate the soundness of the semantics.

THEOREM 1. *If* $\Sigma \vdash \alpha$ *then* $\Sigma \models \alpha$.

Proof. The proof is by a simple induction over the length of proofs. The details are straightforward. We do the induction case for the disjunctive form of R3 as an example. Suppose that $\eta \vee (\alpha \rightarrow \beta)$ and $\eta \vee (\gamma \rightarrow \delta)$ are true at g (i.e. $I(g, \eta \vee (\alpha \rightarrow \beta)) = 1$, etc). Then either η is true at g, and hence $\eta \vee ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \delta))$ is true at g, or else $\alpha \rightarrow \beta$ and $\gamma \rightarrow \delta$ are true at g. Let w be any world and let $(\beta \rightarrow \gamma)$ be true at w. We show that $\alpha \rightarrow \delta$ is true at w. Suppose that $w \neq g$, $Rwxy$ and α is true at x. Since $\alpha \rightarrow \beta$ is true at g, β is true at

x. Thus, since $\beta \to \gamma$ is true at w, γ is true at y; and since $\gamma \to \delta$ is true at g, δ is true at y. Thus, $\alpha \to \delta$ is true at w. The case for $w = g$ is similar. It follows that $(\beta \to \gamma) \to (\alpha \to \delta)$, and hence $\eta \vee ((\beta \to \gamma) \to (\alpha \to \delta))$, are true at g.

1.4. COMPLETENESS: KEY NOTIONS

We now establish completeness. The proof uses standard techniques (see, especially Routley *et al.*, *op cit.*, pp. 306–8, 336–9). We give it in full, partly because we are proving completeness of consequence rather than theoremhood, which is a bit stronger than usual; partly to unify some of the terminology used there; and partly to make the paper self contained. For starters, let us give the definitions of all key concepts.

(i) If Π is a set of sentences, let $\Pi_{\to}$ be the set of all members of Π of the form $\alpha \to \beta$.
(ii) $\Sigma \vdash_{\pi} \alpha$ iff $\Sigma \cup \Pi_{\to} \vdash \alpha$.
(iii) Σ is a Π-*theory* iff:
  (a) if $\alpha, \beta \in \Sigma$ then $\alpha \wedge \beta \in \Sigma$
  (b) if $\vdash_{\pi} \alpha \to \beta$ then (if $\alpha \in \Sigma$ then $\beta \in \Sigma$)
(iv) Σ is *prime* iff (if $\alpha \vee \beta \in \Sigma$ then $\alpha \in \Sigma$ then $\beta \in \Sigma$).
(v) If X is any set of sets of formulas the ternary relation R on X is defined thus:

$$R\Sigma\Gamma\Delta \text{ iff (if } \gamma \to \delta \in \Sigma \text{ then (if } \gamma \in \Gamma \text{ then } \delta \in \Delta))$$

(vi) $\Sigma \vdash_{\pi} \Delta$ iff for some $\delta_1 \ldots \delta_n \in \Delta$ $\Sigma \vdash_{\pi} \delta_1 \vee \ldots \vee \delta_n$
(vii) $\vdash_{\pi} \Sigma \to \Delta$ iff for some $\sigma_1 \ldots \sigma_n \in \Sigma$ and $\delta_1 \ldots \delta_m \in \Delta$
$$\vdash_{\pi} \sigma_1 \wedge \ldots \wedge \sigma_n \to \delta_1 \vee \ldots \vee \delta_m$$
(viii) Σ is Π-*deductively closed* iff (if $\Sigma \vdash_{\pi} \alpha$ then $\alpha \in \Sigma$).
(ix) If Φ is the set of formulas, $\langle \Sigma, \Delta \rangle$ is a Π-partition iff:
  (a) $\Sigma \cup \Delta = \Phi$
  (b) $\nvdash_{\pi} \Sigma \to \Delta$.

In all the above if Π is $\varnothing$, the prefix 'Π-' will simply be omitted. thus, a $\varnothing$-theory is simply a theory, etc.

1.5. EXTENSION LEMMAS

We now prove a number of lemmas. The first group concerns extensions of sets with various properties.

LEMMA 1. *If* $\langle \Sigma, \Delta \rangle$ *is a* Π*-partition then* Σ *is a prime* Π*-theory.*

Proof. Suppose $\alpha, \beta \in \Sigma$ but $\alpha \wedge \beta \notin \Sigma$. Then $\alpha \wedge \beta \in \Delta$. Hence $\vdash_\pi \Sigma \to \Delta$. Contradiction. Next, suppose that $\alpha \in \Sigma$ and $\vdash_\pi \alpha \to \beta$ but $\beta \notin \Sigma$. Then $\beta \in \Delta$, in which case $\vdash_\pi \Sigma \to \Delta$. Contradiction. Finally, suppose that $\alpha \vee \beta \in \Sigma$ but $\alpha \notin \Sigma$ and $\beta \notin \Sigma$ then $\alpha, \beta \in \Delta$. Hence, $\vdash_\pi \Sigma \to \Delta$. Contradiction.

LEMMA 2. *If* $\nvdash_\pi \Sigma \to \Delta$ *then there are* $\Sigma' \supseteq \Sigma$ *and* $\Delta' \supseteq \Delta$ *such that* $\langle \Sigma', \Delta' \rangle$ *is a* Π*-partition.*

Proof. Let $\alpha_0, \alpha_1, \ldots$ be an enumeration of the set of formulas, Φ. Define $\Sigma_i, \Delta_i, i \in \omega$, by induction. $\Sigma_0 = \Sigma$; $\Delta_0 = \Delta$.

If $\nvdash_\pi \Sigma_i \cup \{\alpha_i\} \to \Delta_i$ then $\Sigma_{i+1} = \Sigma_i \cup \{\alpha_i\}$ and $\Delta_{i+1} = \Delta_i$ otherwise $\Sigma_{i+1} = \Sigma_i$ and $\Delta_{i+1} = \Delta_i \cup \{\alpha_i\}$.

Then

$$\Sigma' = \bigcup_{i<\omega} \Sigma_i \text{ and } \Delta' = \bigcup_{i<\omega} \Delta_i.$$

Clearly $\Sigma' \cup \Delta' = \Phi$. It therefore remains to show that $\nvdash_\pi \Sigma' \to \Delta'$. By the compactness of $\vdash_\pi$ it suffices to show that for no i $\nvdash_\pi \Sigma_i \to \Delta_i$. This is proved by induction on i. It is true for $i = 0$ by definition. Suppose that it is true for $i = j$ but not $i = j + 1$. Then we must have both of:

$$\vdash_\pi \Sigma_j \cup \{\alpha_j\} \to \Delta_j$$

$$\vdash_\pi \Sigma_j \to \Delta_j \cup \{\alpha_j\}.$$

So for some conjunctions of members of Σ_j, σ_1, σ_2, and some disjunctions of members of Δ_j, δ_1, δ_2:

$$\vdash_\pi \sigma_1 \wedge \alpha_j \to \delta_1,$$

$$\vdash_\pi \sigma_2 \to \delta_2 \vee \alpha_j.$$

Now,	$\vdash_\pi \sigma_1 \wedge \sigma_2 \rightarrow \sigma_2$	(A3)
so	$\vdash_\pi \sigma_1 \wedge \sigma_2 \rightarrow \delta_2 \vee \alpha_j$	(R4)
but	$\vdash_\pi \sigma_1 \wedge \sigma_2 \rightarrow \sigma_1$	(A3)
so	$\vdash_\pi \sigma_1 \wedge \sigma_2 \rightarrow (\delta_2 \vee \alpha_j) \wedge \sigma_1$	(A5)
Thus,	$\vdash_\pi \sigma_1 \wedge \sigma_2 \rightarrow \delta_2 \vee (\alpha_j \wedge \sigma_1)$	(A4, R3)
Now since	$\vdash_\pi \sigma_1 \wedge \alpha_j \rightarrow \delta_1$	
	$\vdash_\pi \sigma_1 \wedge \alpha_j \rightarrow \delta_1 \vee \delta_2$	(A2, R3)
But	$\vdash_\pi \delta_2 \rightarrow \delta_1 \vee \delta_2$	(A2)
Hence	$\vdash_\pi \delta_2 \vee (\sigma_1 \wedge \alpha_j) \rightarrow \delta_1 \vee \delta_2$	(A6) (*)
So	$\vdash_\pi \sigma_1 \wedge \sigma_2 \rightarrow \delta_1 \vee \delta_2$	(R3)
i.e.	$\vdash_\pi \Sigma_j \rightarrow \Delta_j$	

Contradiction. We flagged one step (*) for future reference.

COROLLARY. *Let Σ be a Π-theory, Δ be closed under disjunction, and $\Sigma \cap \Delta = \varnothing$. Then there is a $\Sigma' \supset \Sigma$ such that $\Sigma' \cap \Delta = \varnothing$ and Σ' is a prime Π-theory.*

Proof. First, observe that $\nvdash_\pi \Sigma \rightarrow \Delta$, for otherwise there would be $\delta_1, \ldots, \delta_n \in \Delta$ such that $\delta_1 \vee \ldots \vee \delta_n \in \Sigma \cap \Delta$. The result follows by Lemmas 1 and 2.

LEMMA 3. *If $\Sigma \nvdash \Delta$ then there is $\Sigma' \supseteq \Sigma$, $\Delta' \supseteq \Delta$ such that $\langle \Sigma', \Delta' \rangle$ is a partition and Σ' is deductively closed.*

Proof. We repeat the construction of Σ', Δ' in Lemmas 2, but replacing occurrences of the form $\vdash_\pi X \rightarrow Y$ by ones of the form $X \vdash Y$. As before, it follows that $\Sigma' \nvdash \Delta'$. We leave the reader to work through the details. There is one subtlety, however, at the step we flagged (*). Given that:

$$\sigma_1 \wedge \alpha_j \vdash \delta_1 \vee \delta_2$$

$$\delta_2 \vdash \delta_1 \vee \delta_2$$

the Corollary of Lemma 0 is required to conclude that:

$$\delta_2 \vee (\sigma_1 \wedge \alpha_j) \vdash \delta_1 \vee \delta_2.$$

Now since $\Sigma' \nvdash \Delta'$ it follows that $\nvdash \Sigma' \rightarrow \Delta'$ by R1. Hence $\langle \Sigma', \Delta' \rangle$ is a partition. It remains to show that Σ' is deductively closed. Suppose that $\Sigma' \vdash \alpha$ but $\alpha \notin \Sigma'$. By construction, there is an i such that $\Sigma_i \cup \{\alpha\} \vdash \Delta_i$. But then $\Sigma' \vdash \Delta'$, contradiction.

COROLLARY. *If* $\Sigma \nvdash \alpha$ *then there is a* $\Pi \supseteq \Sigma$ *such that* $\alpha \notin \Pi$, Π *is a prime* Π*-theory and* Π *is* Π*-deductively closed.*

Proof. Let Δ in the Lemma be $\{\alpha\}$. Let Π be Σ'. Then by the Lemma $\Pi \supseteq \Sigma$ and $\alpha \notin \Pi$. By Lemma 1, Π is a prime theory. By the Lemma Π is deductively closed, and since $\Pi \vdash_\pi \alpha$ entails $\Pi \vdash \alpha$, it follows that Π is Π-deductively closed. It remains to show that Π is a Π-theory. Suppose that $\vdash_\pi \alpha \rightarrow \beta$ and $\alpha \in \Pi$; then clearly $\Pi \vdash \beta$. Hence $\beta \in \Pi$ by deductive closure.

1.6. COUNTER-EXAMPLE LEMMAS

The second group of lemmas establishes that there are certain theories with properties that are crucial in the recursion case for $\rightarrow$ in the proof of the main theorem.

LEMMA 4. *If* Π *is a prime* Π*-theory, is* Π*-deductively closed and* $\alpha \rightarrow \beta \notin \Pi$, *then there is a prime* Π*-theory,* Γ, *such that* $\alpha \in \Gamma$ *and* $\beta \notin \Gamma$.

Proof. Let $\Sigma = \{\gamma; \alpha \rightarrow \gamma \in \Pi\}$. Σ is a Π-theory. For suppose that $\gamma_1, \gamma_2 \in \Sigma$. Then $\alpha \rightarrow \gamma_1, \alpha \rightarrow \gamma_2 \in \Pi$. Thus $\vdash_\pi \alpha \rightarrow \gamma_1 \wedge \gamma_2$ (A5), so $\gamma_1 \wedge \gamma_2 \in \Sigma$ by Π-deductive closure. And suppose that $\vdash_\pi \gamma \rightarrow \delta$ and $\gamma \in \Sigma$. Then $\alpha \rightarrow \gamma \in \Pi$; hence $\alpha \rightarrow \delta \in \Pi$ by R3 and Π-deductive closure. Moreover, clearly $\alpha \in \Sigma$ and $\beta \vee \ldots \vee \beta \notin \Sigma$. Let Δ be the closure of $\{\beta\}$ under disjunction. Then $\Sigma \cap \Delta = \varnothing$. The result follows by the corollary of Lemma 2.

LEMMA 5. *If* Σ, Γ, Δ *are* Π*-theories,* $R\Sigma\Gamma\Delta$ *and* $\delta \notin \Delta$ *then there are prime* Π*-theories,* Γ', Δ', *such that* $\Gamma' \supseteq \Gamma$, $\delta \notin \Delta'$ *and* $R\Sigma\Gamma'\Delta'$.

Proof. Under the conditions of the Lemma, we first construct a Δ' such that $\delta \notin \Delta'$ and $R\Sigma\Gamma\Delta'$. Let Θ be the closure of $\{\delta\}$ under disjunction. $\Delta \cap \Theta = \emptyset$ since $\vdash \delta \vee \ldots \vee \delta \rightarrow \delta$ and Δ is a theory. By the corollary of Lemma 2, there is a $\Delta' \supseteq \Delta$ such that $\Delta' \cap \Theta = \emptyset$. Finally, since $R\Sigma\Gamma\Delta$ and $\Delta' \supseteq \Delta$, $R\Sigma\Gamma\Delta'$.

Next, we construct Γ' so that $\Gamma' \supseteq \Gamma$ and $R\Sigma\Gamma'\Delta'$. Let:

$$\Theta = \{\alpha; \exists\beta \notin \Delta' \; \alpha \rightarrow \beta \in \Sigma\}$$

Θ is closed under disjunction. For suppose that $\alpha_1, \alpha_2 \in \Theta$. Then there are $\beta_1, \beta_2 \notin \Delta'$, such that $\alpha_1 \rightarrow \beta_1, \alpha_2 \rightarrow \beta_2 \in \Sigma$. Since Δ' is prime $\beta_1 \vee \beta_2 \notin \Delta$; and $\alpha_1 \vee \alpha_2 \rightarrow \beta_1 \vee \beta_2 \in \Sigma$ since Σ is a theory – details are left as an exercise. So $\alpha_1 \vee \alpha_2 \in \Theta$. Moreover $\Gamma \cap \Theta = \emptyset$. For suppose $\alpha \in \Gamma \cap \Theta$. then $\exists\beta \notin \Delta'$ such that $\alpha \rightarrow \beta \in \Sigma$. But $R\Sigma\Gamma\Delta'$; contradiction.

Thus we can apply the corollary of Lemma 2 to obtain a prime Π-theory, Γ' such that $\Gamma' \supseteq \Gamma$ and $\Gamma' \cap \Theta = \emptyset$. It remains to show that $R\Sigma\Gamma'\Delta'$. Suppose that $\alpha \rightarrow \beta \in \Sigma$ and that $\alpha \in \Gamma'$. Then $\alpha \notin \Theta$. Hence $\beta \in \Delta'$.

LEMMA 6. *Let Σ be a prime Π-theory and $\gamma \rightarrow \delta \notin \Sigma$. Then there are prime Π-theories, Γ', Δ' such that $R\Sigma\Gamma'\Delta'$, $\gamma \in \Gamma'$, $\delta \notin \Delta'$.*

Proof. We show that there are Π-theories Γ, Δ such that $R\Sigma\Gamma\Delta$, $\gamma \in \Gamma$, $\delta \notin \Delta$. The result then follows by Lemma 5. Let:

$$\Gamma = \{\alpha; \vdash_\pi \gamma \rightarrow \alpha\},$$

$$\Delta = \{\eta; \exists\alpha \in \Gamma \; \alpha \rightarrow \eta \in \Sigma\}.$$

Clearly, $\gamma \in \Gamma$. Moreover, $\delta \notin \Delta$. For suppose $\delta \in \Delta$. Then there is an $\alpha \in \Gamma$ such that $\alpha \rightarrow \delta \in \Sigma$. Thus, $\vdash_\pi \gamma \rightarrow \alpha$, and so $\vdash_\pi (\alpha \rightarrow \delta) \rightarrow (\gamma \rightarrow \delta)$. Hence $\gamma \rightarrow \delta \in \Sigma$ since Σ is a Π-theory. It remains to show (i) that Γ is a Π-theory; (ii) that Δ is a Π-theory; (iii) that $R\Sigma\Gamma\Delta$.

(i) Suppose that $\alpha_1, \alpha_2 \in \Gamma$. Then $\vdash_\pi \gamma \rightarrow \alpha_1$ and $\vdash_\pi \gamma \rightarrow \alpha_2$. Thus, $\vdash_\pi \gamma \rightarrow \alpha_1 \wedge \alpha_2$ (A5), i.e. $\alpha_1 \wedge \alpha_2 \in \Gamma$. Suppose that $\alpha \in \Gamma$ and $\vdash_\pi \alpha \rightarrow \eta$. Then $\vdash_\pi \gamma \rightarrow \eta$ (R3), i.e. $\eta \in \Gamma$.

(ii) Suppose $\eta_1, \eta_2 \in \Delta$. Then there are $\alpha_1, \alpha_2 \in \Gamma$ such that $\alpha_1 \rightarrow \eta_1$, $\alpha_2 \rightarrow \eta_2 \in \Sigma$. Hence, by the application of various rules $\alpha_1 \wedge \alpha_2 \rightarrow \eta_1 \wedge \eta_2 \in \Sigma$. (Details are left as an exercise.) Thus, $\eta_1 \wedge \eta_2 \in \Sigma$. Now suppose that $\eta \in \Delta$ and $\vdash_\pi \eta \rightarrow \varphi$. Then there is an $\alpha \in \Gamma$ such that $\alpha \rightarrow \eta \in \Sigma$. Thus, $\alpha \rightarrow \varphi \in \Sigma$ (as above), i.e. $\varphi \in \Delta$.

(iii) Suppose that $\varphi \rightarrow \psi \in \Sigma$ and $\varphi \in \Gamma$, then $\psi \in \Delta$, by definition of Δ.

1.7. COMPLETENESS

We are now in a position to prove the completeness of the axiom system.

THEOREM 2. *If* $\Theta \models \alpha$ *then* $\Theta \vdash \alpha$.

Proof. We prove the contrapositive. Suppose that $\Theta \nvdash \alpha$. By the corollary of Lemma 3 there is a $\Pi \supseteq \Theta$ such that $\alpha \notin \Pi$, Π is a prime Π-theory and Π is Π-deductively closed. Define the interpretation $\mathfrak{A} = \langle \Pi, X, R, I \rangle$, where X is the set of all prime Π-theories (R being restricted to X^3) and I is defined thus. For every world, Σ and propositional parameter, p:

$$I(\Sigma, p) \;=\; 1 \text{ iff } p \in \Sigma.$$

We show that this condition holds for an arbitrary formula, β:

$$I(\Sigma, \beta) \;=\; 1 \text{ iff } \beta \in \Sigma. \qquad (**)$$

It follows that $\mathfrak{A}$ is a counter-model for the inference, and hence that $\Sigma \nvDash \alpha$.

The proof of (**) is by recursion over the formation of β. The recursion cases for $\wedge$ and $\vee$ are as follows:

$$\begin{aligned} 1 = I(\Sigma, \gamma \wedge \delta) \quad & \text{iff } 1 = I(\Sigma, \gamma) \text{ and } 1 = I(\Sigma, \delta) \\ & \text{iff } \gamma \in \Sigma \text{ and } \delta \in \Sigma \quad \text{(Rec. Hypothesis)} \\ & \text{iff } \gamma \wedge \delta \in \Sigma \quad (\Sigma \text{ is a theory}) \end{aligned}$$

$$\begin{aligned} 1 = I(\Sigma, \gamma \vee \delta) \quad & \text{iff } 1 = I(\Sigma, \gamma) \text{ or } 1 = I(\Sigma, \delta) \\ & \text{iff } \gamma \in \Sigma \text{ or } \delta \in \Sigma \quad \text{(Rec. Hypothesis)} \\ & \text{iff } \gamma \vee \delta \in \Sigma \quad (\Sigma \text{ is a prime theory}) \end{aligned}$$

The case for $\rightarrow$ splits into two, depending on whether or not $\Sigma = \Pi$:

$$\begin{aligned} 1 = I(\Pi, \gamma \rightarrow \delta) \quad & \text{iff } \forall \Gamma \in X \text{ (if } 1 = I(\Gamma, \gamma) \text{ then } 1 = I(\Gamma, \delta)) \\ & \text{iff } \forall \Gamma \in X \text{ (if } \gamma \in \Gamma \text{ then } \delta \in \Gamma) \quad \text{(Rec. Hypothesis)} \\ & \text{iff } \gamma \rightarrow \delta \in \Pi \quad (\Gamma \text{ is a } \Pi\text{-theory, and Lemma 4}) \end{aligned}$$

If $\Sigma \neq \Pi$:

$1 = I(\Sigma, \gamma \rightarrow \delta)$ iff $\forall \Gamma, \Delta \in X$ (if $R\Sigma\Gamma\Delta$ then (if $1 = I(\Gamma, \gamma)$ then $1 = I(\Delta, \delta)$))

iff $\forall \Gamma, \Delta \in X$ (if $R\Sigma\Gamma\Delta$ then (if $\gamma \in \Gamma$ then $\delta \in \Delta$)) (Rec. Hypothesis)

iff $\gamma \rightarrow \delta \in \Sigma$ (Definition of R and Lemma 6)

Hence, the result is proved.

2.1. NEGATION

We now turn to negation-extensions of B^+. Negation extensions of B^+ all add De Morgan Laws:

A7 $\neg(\alpha \wedge \beta) \leftrightarrow \neg\alpha \vee \neg\beta$,

A8 $\neg\alpha \wedge \neg\beta \leftrightarrow \neg(\alpha \vee \beta)$.

The basic negative system when negation is handled with the Routley $*$-operator, adds, in addition, contraposition.

R4 $\alpha \rightarrow \beta / \neg\beta \rightarrow \neg\alpha$.

We call this system *BM*. (We note that this axiomatisation contains slight redundancies. Contraposition, together with positive axioms suffices to prove each of A7 and A8 in one direction. Details are left as an exercise.) To model *BM*, an interpretation is extended to a 5-tuple $\langle g, W, R, *, I \rangle$ where g, W, R, and I are as before, and $*$ is a one place function from W to W. The truth conditions for negation are:

$$1 = I(w, \neg\alpha) \text{ iff } 1 \neq I(w, \alpha^*).$$

It is a simple matter to check that *BM* is sound with respect to these truth conditions. Completeness is scarcely more complicated. If Σ is a set of sentences we define:

$$\Sigma^* = \{\alpha; \neg\alpha \notin \Sigma\}.$$

The canonical model, $\mathfrak{A}$ of Theorem 2 is now just extended to $\langle \Pi, X, W, *, I \rangle$. All that needs to be checked is that this is well-defined, that is, that if Σ is a prime Π-theory, so is Σ^*.

(i) Suppose that $\alpha, \beta \in \Sigma^*$, i.e. $\neg\alpha, \neg\beta \notin \Sigma$. Then $\neg\alpha \vee \neg\beta \notin \Sigma$ (since Σ is prime). Hence $\neg(\alpha \wedge \beta) \notin \Sigma$ (De Morgan). Thus, $\alpha \wedge \beta \in \Sigma^*$.

(ii) Suppose that $\alpha \in \Sigma^*$ and $\vdash_\pi \alpha \rightarrow \beta$. Then $\neg\alpha \notin \Sigma$ and $\vdash_\pi \neg\beta \rightarrow \neg\alpha$ (R4). Hence $\neg\beta \notin \Sigma$, i.e. $\beta \in \Sigma^*$.

(iii) Suppose that $\alpha \vee \beta \in \Sigma^*$. Then $\neg(\alpha \vee \beta) \notin \Sigma$. Hence $\neg\alpha \wedge \neg\beta \notin \Sigma$ (De Morgan), i.e. $\neg\alpha$ or $\neg\beta \notin \Sigma$. So $\alpha \notin \Sigma$, or $\beta \notin \Sigma$.

2.2. DOUBLE NEGATION

The system *B*, itself, is obtained from *BM* by the addition of the axiom:

A9 $\alpha \leftrightarrow \neg\neg\alpha$

(making A7 and A8 completely redundant). To obtain semantics for *B* we simply require that an interpretation $*$ satisfy the condition $w = w^{**}$. It is an easy matter to check that this verifies A9. In the completeness proof we need only check that $*$, as defined in the previous section, satisfies this condition. This is done thus:

$$\begin{aligned} \alpha \in \Sigma^{**} & \text{ iff } \neg\alpha \notin \Sigma^* \\ & \text{ iff } \neg\neg\alpha \in \Sigma \\ & \text{ iff } \alpha \in \Sigma \qquad \text{(A9)} \end{aligned}$$

2.3. FOUR-VALUED SEMANTICS

The second way of handling negation that we mentioned in the introduction is through a four-valued semantics. In this case, the basic logic is obtained by adding to the positive system both De Morgan laws and Double Negation. Let us call this system *BD*. (So *BD* is B^+ plus A7, A8 and A9.)

A semantic interpretation for *BD* is the same as that for B^+ (with, *nb*, a single ternary relation and no modeling conditions), except that I assigns to each world and propositional parameter a truth value in the set $\{\{1\}, \{0\}, \{1, 0\}, \varnothing\}$. The truth conditions are as in 1.2, except that '$1 = I$' is replaced by '$1 \in I$'. The truth conditions for negation are:

$$1 \in I(w, \neg\alpha) \text{ iff } 0 \in I(w, \alpha).$$

We also need to give falsity conditions now as well. These are as follows:

$$0 \in I(w, \neg\alpha) \quad \text{iff } 1 \in I(w, \alpha),$$

$$0 \in I(w, \alpha \vee \beta) \quad \text{iff } 0 \in I(w, \alpha) \text{ and } 0 \in I(w, \beta),$$

$$0 \in I(w, \alpha \wedge \beta) \quad \text{iff } 0 \in I(w, \alpha) \text{ or } 0 \in I(w, \beta).$$

Whether or not a conditional is false is to be arbitrary. Hence we take I to assign O to the value of conditionals *ad lib.*

Again, it is an easy matter to check that *BD* is sound with respect to these truth and falsity conditions. The details are left as an exercise.

2.4. COMPLETENESS

Completeness is hardly more complicated. The only difference is that in the proof of the main theorem (1.7) we define the I of the canonical model as follows:

$$1 \in I(\Sigma, p) \quad \text{iff } p \in \Sigma$$

$$0 \in I(\Sigma, p) \quad \text{iff } \neg p \in \Sigma$$

$$0 \in I(\Sigma, \alpha \to \beta) \quad \text{iff } \neg(\alpha \to \beta) \in \Sigma.$$

It then needs to be shown that the following condition is true of all formulas:

$$1 \in I(\Sigma, \beta) \quad \text{iff } \beta \in \Sigma$$

$$0 \in I(\Sigma, \beta) \quad \text{iff } \neg\beta \in \Sigma$$

and then everything works as before. The new recursion steps to check are those for the truth condition for negation and all the falsity conditions, except those for the conditional, since this is given by definition. The new material is as follows.

$$\begin{aligned} 1 \in I(\Sigma, \neg\gamma) \quad & \text{iff } 0 \in I(\Sigma, \gamma) & \\ & \text{iff } \neg\gamma \in \Sigma & \text{(Rec. Hypothesis)} \end{aligned}$$

$$\begin{aligned} 0 \in I(\Sigma, \neg\gamma) \quad & \text{iff } 1 \in I(\Sigma, \gamma) & \\ & \text{iff } \gamma \in \Sigma & \text{(Rec. Hypothesis)} \\ & \text{iff } \neg\neg\gamma \in \Sigma & \text{(A9)} \end{aligned}$$

$0 \in I(\Sigma, \gamma \wedge \delta)$	iff $0 \in I(\Sigma, \gamma)$ or $0 \in I(\Sigma, \delta)$	
	iff $\neg\gamma \in \Sigma$ or $\neg\delta \in \Sigma$	(Rec. Hypothesis)
	iff $\neg\gamma \vee \neg\delta \in \Sigma$	(Σ is a prime theory)
	iff $\neg(\gamma \wedge \delta) \in \Sigma$	(A7)
$0 \in I(\Sigma, \gamma \vee \delta)$	iff $0 \in I(\Sigma, \gamma)$ and $0 \in I(\Sigma, \delta)$	
	iff $\neg\gamma \in \Sigma$ and $\neg\delta \in \Sigma$	(Rec. Hypothesis)
	iff $\neg\gamma \wedge \neg\delta \in \Sigma$	(Σ is a theory)
	iff $\neg(\gamma \vee \delta) \in \Sigma$	(A8)

2.5. CONTRAPOSITION

How to obtain a semantics for the logic B along these lines (with a single ternary relation) is still an open problem. It might be thought that we could obtain a suitable contrapossible implication by simply defining it as follows:

$$\alpha \Rightarrow \beta \;=\; (\alpha \rightarrow \beta) \wedge (\neg\beta \rightarrow \neg\alpha).$$

The truth conditions for this conditional are clearly:

$1 \in I(g, \alpha \Rightarrow \beta)$ iff for all $x \in W$, if $1 \in I(x, \alpha)$

then $1 \in I(x, \beta)$ and if $0 \in I(x, \beta)$ then $0 \in I(x, \alpha)$.

For $x \neq g$:

$1 \in I(x, \alpha \Rightarrow \beta)$ iff for all $y, z \in W$, if $Rxyz$

then (if $1 \in I(y, \alpha)$ then $1 \in I(z, \beta)$)

and (if $0 \in I(y, \beta)$ then $0 \in I(z, \alpha)$).

And, indeed, this definition comes close. As may be checked, the conditional with these truth conditions satisfies A1–A4, A7–A9, R1, R2, and R4; but not A5, A6 or R3. (Though it does satisfy them if the main conditional in A5, A6 and the conclusion of R3 – but not the others – is taken to be $\rightarrow$ instead of $\Rightarrow$. What logic the $\wedge$, $\vee$, $\neg$, $\Rightarrow$ fragment is, is an interesting open question. It is a logic without affixing but with substitutivity of equivalents, such as the logic of Priest, 1980.)

Since the failure to verify A5, A6 and R3 comes entirely from the failure of falsity to be preserved backwards for certain conditionals, it is natural to suppose that some modification of the falsity conditions for conditionals will rectify the matter. And in any case, falsity conditions of the kind used for *BD* are philosophically unsatisfactory since they fall foul of the principle of compositionality. For the semantic value of a sentence (that does not contain context-dependent or indexical phrases) must be a function of the semantic values of its components and the way they are put together, or we would not be able to understand the meanings of totally new sentences. And clearly, falsity conditions of the kind in question violate this principle.

Two ways of giving compositional falsity conditions for $\rightarrow$ come to mind immediately. The first is that used in the double-ternary four-valued semantics, and is as follows:

$0 \in I(g, \alpha \rightarrow \beta)$ iff $\exists x \in W\,(0 \in I(x, \beta)$ but not $0 \in I(x, \alpha))$.

For $x \neq g$:

$0 \in I(x, \alpha \rightarrow \beta)$ iff $\exists y, z \in W\,(Rxyz, 0 \in I(y, \beta)$ but not $0 \in I(z, \alpha))$.

The second is the following conditions, whose naturalness arguably makes the logic they generate the basic affixing relevant logic on the four-valued approach:

$0 \in I(g, \alpha \rightarrow \beta)$ iff $\exists x \in W\,(1 \in I(x, \alpha)$ and $0 \in I(x, \beta))$.

For $x \neq g$:

$0 \in I(x, \alpha \rightarrow \beta)$ iff $\exists y, z \in W(Rxyz, 1 \in I(y, \alpha)$ and $0 \in I(z, \beta)))$

(see Priest, 1987, ch 6).

With either of these falsity conditions $\Rightarrow$ satisfies all the axioms and rules for *B*, as may easily be checked. However, both conditions verify invalid inferences in *B*. For example, the first verifies the formula:

$$(\alpha \Rightarrow \beta) \vee \neg(\alpha \Rightarrow \beta)$$

and the second verifies the rule:

$$\alpha, \neg\beta / \neg(\alpha \Rightarrow \beta).$$

Therefore B is not complete with respect to these semantics. The variety of falsity conditions illustrates an important fact about B and similar depth-relevant logics. This is that they are very uninformative concerning the properties of *negated* entailments, and as such may be augmented by a number of plausible principles. (To a lesser extent, the same is true of most standard stronger relevant logics.) At any rate, determining a complete proof-theory for semantics with the various falsity conditions is an interesting open problem.

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