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# AN ANTI-REALIST ACCOUNT OF MATHEMATICAL TRUTH

## INTRODUCTION

The aim of this paper is to give an account of the nature of the truth of mathematical statements that does not invoke a domain of abstract mathematical objects. The account is given in the first part of the paper. Some consequences and objections are considered in the second part.

## 1. REALISM IN MATHEMATICS

The central question in the philosophy of mathematics is 'What makes mathematical claims true?' or, stripping this question of its anthropomorphic overtones, 'What is it in virtue of which mathematical statements are true or false?' This paper attempts to answer the question.

Let us start by asking a similar question about empirical statements. What is it in virtue of which they are true or false? A standard (though by no means uncontroversial) answer is that they are so in virtue of real, actual objects, their properties and relations. Thus 'my pen is in my hand' is true because of a certain physical relationship which actually obtains between my hand and my pen, two actual objects. This sort of account is called 'realism'.

Now if we return to the analogous question for mathematical statements, we may be tempted to give the same sort of answer. Mathematical statements are true or false in virtue of the relationships that hold between, and the properties of, real objects. The objects in question may not be actual (i.e., they may not enter into causal relationships) but they are real nonetheless. Thus '$1+1=2$' is true in virtue of a relationship which obtains between the objects 1 and 2 (and perhaps + also). There is a realm of real though abstract objects, numbers, sets, points, groups, categories, etc., which statements of mathematics are about. Such an account is called 'mathematical realism' and often (badly) called 'platonism'.

Realism has always been a widely canvassed position in the philosophy of mathematics and it seems that there is a movement towards realism

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today, whether of the simple Gödelean kind (see Gödel 1947) or of the more sophisticated Quinean kind (see Smart 1969). Much of its popularity derives, I suspect, from the impact of model theory on logicians; for model theory is the realistic theory *par excellence*. To give an account of the truth and falsity of statements in the language of mathematics one needs to specify their truth conditions. The realist gives these conditions by reference to abstract mathematical structures. Model theory, at least in a simple way of looking at it, is precisely the study of the relationships between language and these structures or models. Thus the success of model theory has lent weight to the realist position.

Be that as it may, realism is not without its problems. I do not wish to give a critique of realism here. Its problems are well known. However, the sheer prima facie implausibility of these supra-sensible objects (to which it is easy to become blind) means that, other things being equal, an account of mathematical truth which avoids invoking real mathematical objects is preferable. It is my claim that such an account can be given. What is required is to give an account of the truth condition of mathematical assertions without the invocation of real mathematical entities. To this I now turn.

## 2. ARITHMETIC

First I will show how this can be done for the simple case of arithmetic. (Parsons 1971, p. 231 and Kripke 1976, p. 384 also observe that this method works.)

The language of arithmetic has one constant, 0, a one-place function $S$, the two two-place functions $+$ and $\times$. Terms of the language are formed, as usual, recursively from 0 using $S$, $\times$, and $+$. Any term with a string of $S$'s preceeding a 0 is called a *numeral*. Atomic formulas are of the form $\ulcorner t_1 = t_2 \urcorner$ where $t_1$ and $t_2$ are terms, or $f$, the absurdity symbol. The set of formulas $F$ is the closure of the set of atomic formulas under the three conditions.

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|---|---|---|
| (i) | If $\phi, \psi \in F$ | $\ulcorner \phi \supset \psi \urcorner \in F$ |
| (ii) | If $\phi, \psi \in F$ | $\ulcorner \phi \rightarrow \psi \urcorner \in F$ |
| (iii) | If $\phi \in F$ | $\ulcorner \exists x \phi(t/x) \urcorner \in F$ |

where $\phi(t/x)$ is $\phi$ with the variable $x$ replacing any occurrences of some term $t$ in $\phi$. (Note that all the formulas of $F$ are closed.) As usual we will identify $\ulcorner \neg \phi \urcorner$ with $\ulcorner \phi \supset f \urcorner$, $\ulcorner \phi \vee \psi \urcorner$ with $\ulcorner \neg \phi \supset \psi \urcorner$, $\ulcorner \phi \wedge \psi \urcorner$ with $\ulcorner \neg(\neg \phi \vee$

$\ulcorner\neg\psi\urcorner$, and $\ulcorner\forall x\phi\urcorner$ with $\ulcorner\neg\exists x\neg\phi\urcorner$. Now for the truth conditions. First let us suppose that we have an account of the truth conditions of atomic sentences. We can give the truth conditions for other sentences recursively as follows:

$\ulcorner\phi\rightarrow\psi\urcorner$ is true iff if $\phi$ is true, then $\psi$ is true,
$\ulcorner\phi\supset\psi\urcorner$ is true iff $\psi$ is true or $\phi$ is not,
$\ulcorner\exists x\phi\urcorner$ is true iff $\phi(x/n)$ is true for some numeral $n$.

Of course these are just the standard truth conditions for first-order languages with an implication operator and substitutional quantifiers. We have now to give the truth conditions of atomic sentences. $f$ is easy enough.

$f$ is not true.

The truth conditions for other atomic sentences employ the notion of canonical form. The canonical form $t^*$ of a term $t$ is a numeral defined recursively as follows

$0^* = 0$,
$(St)^* = St^*$,
$(t_1 + t_2)^*$ = the term obtained by prefixing all the $S$'s at the beginning of $t_1^*$ to $t_2^*$,
$(t_1 \times t_2)^*$ = the term obtained by replacing every occurrence of $S$ at the beginning of $t_2^*$ by as many $S$'s as commence $t_1^*$.

The truth conditions of $\ulcorner t_1 = t_2\urcorner$ can now be given easily.

$\ulcorner t_1 = t_2\urcorner$ is true iff $t_1^*$ is the same as $t_2^*$.

A little thought is sufficient to show that $\ulcorner t_1 = t_2\urcorner$ is true iff $\ulcorner t_1 = t_2\urcorner$ is an arithmetically correct equation. For * just mimics standard computation procedures. This completes the specification of the truth conditions of the language of arithmetic. The obvious question now is 'Are they right?' We are faced here with an instance of a much more general problem. Given a part of a natural language, or a formal language that is meant to simulate it, and a proposed theory of its truth conditions, how can we tell whether the truth conditions proposed are correct? There is only one (fallible) answer to this question: the truth theory must fit the pretheoretical data. Specifically the theory must make true those things we have reason to believe are true and not make true those things we have good reason to believe are not. How are we to do this in the present case? We could proceed inductively by showing that '$1+1=2$' is true, '$1+2=3$' is true, '$1+1=3$' is false and so on. More generally we could take an axiom system which is generally agreed to capture a large part of arithmetic, such as Peano's, and prove that all its theorems are true under the proposed truth conditions. However, there is a more satisfactory way for us to argue here. The realist has a way of specifying precisely these sentences of arithmetic which are true, viz., those sentences which hold in (that mathematical fiction) the "standard model". Now it is easy to prove by a simple induction over formation that a sentence of arithmetic is true under the truth conditions I have given iff it holds in the "standard model". Thus we can argue *ad hominem* against the realist that the truth conditions are right. If his account fits the pretheoretic data, so does ours. Moreover, this fact also implies that Peano's axioms, etc., come out as true under the proposed truth conditions. Thus we have the other argument, not directed specifically at the realist, at our disposal too.

Hence it is reasonable to claim that we have achieved our goal. We have given the truth conditions of arithmetic in such a way that its sentences come out as intuitively right, but we have done this without invoking a realm of abstract mathematical objects; realism has been avoided.

## 3. SET THEORY

Arithmetical statements are a special case of mathematical ones. However, provided that the widely accepted principle that set theory is the universal theory of mathematics is correct, mathematical statements can be identified with set theoretic ones. I now wish to give the truth conditions for the language of set theory in a similar way.

The language of set theory contains two binary predicates $\in$ and $=$, and a term forming operator $\{\,|\,\}$. Terms and formulas are defined by a joint recursion.

If $\phi$ is any formula and $t$ any term, $\ulcorner\{z|\phi(t/z)\}\urcorner$ is a term.

The atomic formulas are $f$, $\ulcorner t_1 = t_2\urcorner$ and $\ulcorner t_1 \in t_2\urcorner$ where $t_1$, $t_2$ are any terms. The set of all formulas $F$ is the closure of the set of atomic formulas under the conditions

(i) If $\phi, \psi \in F$ $\quad \ulcorner\phi \supset \psi\urcorner \in F$

(ii) If $\phi$, $\psi \in F$ $\ulcorner\phi \to \psi\urcorner \in F$
(iii) If $\phi \in F$ $\ulcorner\exists z\phi(t/z)\urcorner \in F$.

The definitions of $\wedge$, $\vee$, $\neg$ and $\forall$ are as usual. Assuming once again that we have an account of the truth conditions of atomic sentences, the truth conditions of the others is given as follows:

$\ulcorner\phi \to \psi\urcorner$ is true iff if $\phi$ is true, then $\psi$ is true,
$\ulcorner\phi \supset \psi\urcorner$ is true iff $\psi$ is true or $\phi$ is not,
$\ulcorner\exists x\phi\urcorner$ is true iff for some term $t$, $\phi(x/t)$ is true.

Again these are the standard truth conditions for a first-order language with an implication operator and substitutional quantification. The truth conditions for atomic sentences are as follows:

$f$ is not true,
$\ulcorner t_1 = t_2\urcorner$ is true iff (for every term $t_3$, $\ulcorner t_3 \in t_1\urcorner$ is true iff $\ulcorner t_3 \in t_2\urcorner$ is true),
$\ulcorner t_1 \in \{z \mid \phi(t_2/z)\}\urcorner$ is true iff for some term $t_3$, $\ulcorner t_1 = t_3\urcorner$ is true and $\phi(t_2/t_3)$ is true.

This completes the specification of the truth conditions, but again we must ask whether they are right. It would be nice to be able to argue *ad hominem*, as we did in the previous case, that the things made true by these truth conditions are just those things which hold in the "standard model" of set theory. However, I cannot do this. The standard model of set theory is much more problematical, even among mathematical realists, than that of arithmetic. Is the standard model to be identified with the cumulative hierarchy? And if so, how are we to specify in a satisfactory way the "width" and the "height" of the hierarchy? Moreover, even if there were agreement about this, it would still not help me, for the model would be but a model of *ZF* and, as will become clear, I regard *ZF* as just a (putatively consistent) fragment of set theory. Thus I must argue directly that my account of the truth conditions meshes with our pretheoretic intuitions. Now I take our pretheoretic intuitions about set theory to be encapsulated in something like the principles of naïve set theory. Hence if I can show that the proposed truth conditions determine the truth of these, I will have achieved my aim. In fact this is straightforward. It is easy enough to prove that all the theorems of naive set theory (formulated paraconsistently) are true under this interpretation. A proof of this can be found in the appendix.

Thus again we have laid down the truth conditions of sentences in such a way that things come out right, but without invoking a realm of abstract mathematical objects. This time, however, we have done it for set theory, and hence for the whole of pure mathematics.

## 4. NECESSITY AND CONVENTION

One of the things that this account of mathematical truth does is to explain, in a simple way, the necessity of mathematical statements. This is a particularly difficult point for realism to cope with. Empirical statements which state the relationships between physical objects are contingent. But if realism is correct, mathematical assertions also state *de facto* relationships between real objects. Whence then derives their necessity? Of course it could be claimed that necessity is *sui generis* to relations between abstract objects. However, this just labels the problem rather than explaining it. The difference in kind between empirical and mathematical statements falls naturally out of the account I have given. The truth conditions of empirical statements refer to physical objects. Thus changes in the relationships between these objects may alter the truth value of such statements. This is precisely in what their contingency consists. By contrast, the truth conditions of mathematical assertions refer to no real objects of any kind. Thus no change of the relationships between real things could change their truth value. This is precisely what constitutes their necessity.

However, another question now suggests itself. We have seen what the truth conditions of statements of arithmetic and set theory are. But why are these the truth conditions? Or, less misleadingly, what grounds the truth of the statements stating the truth conditions themselves? The answer to this is simple but depends on arguments I have given elsewhere (Priest 1979). For the sake of the present paper I will merely summarize the relevant conclusions I came to there. A logical conditional (If $A$ then logically $B$) is true iff the corresponding rule of inference ($A/B$) is deductively valid. A sentence is analytic iff it validly follows from true logical conditionals. Which rules of inference are valid is, in a certain sense, a matter of convention. Specifically, validity depends (as Wittgenstein argued) on the concurrence of human actions and not on correspondence with some abstract logical object. Thus, in the same sense, analytic truths are conventionally true.

We can now answer the question of what it is in virtue of which statements expressing the truth conditions of arithmetic and set theoretic sentences are true: they are analytic. The statements of truth conditions are all (conjunctions of) logical conditionals and are all true because the corresponding rules of inference are valid. These rules are, in turn, valid since they underpin our practice of arithmetic and set theoretic inference. The inferences may be fairly immediate, as for example in the case of the truth conditions of quantified sentences, or they may be less immediate. Thus, for example, the rule of inference corresponding to the truth conditions for atomic arithmetic sentences is applicable only via the algorithmic computation of canonical forms – addition and multiplication. Be that as it may, the statements of truth conditions are analytic and, in the sense explained, conventionally true.

## 5. WELL-FOUNDEDNESS AND PARADOX

We have seen that as in arithmetic, truth conditions in set theory can be given without reference to a domain of real mathematical objects. However, there is an important difference between set theory and arithmetic that no doubt the reader will have been wanting to point out for some time. The truth conditions of arithmetic are well-founded whilst those of set theory are not: the truth conditions of $\in$ are given in terms of the truth of compound sentences, which is itself given ultimately in terms of, among other things, $\in$. Similar loops do not arise with the truth conditions of arithmetic. It might be thought that this vitiates my whole procedure. But it does not; for well-foundedness is not required for my enterprise. It is for some. For example, it was Tarski (1936) who first showed explicitly how to give truth conditions. For his purpose well-foundedness was necessary: he was giving a definition of 'is true in *L*', and an adequate recursive definition requires a base clause which ensures a ground. However, I am not trying to give a *definition* of 'is true'. I am stating the conditions under which certain sentences *are* true. (Compare defining 'win' and stating under what conditions certain games are won.) There is no reason why well-foundedness should be necessary for this.

It might be thought that if circles are allowed then the whole process of stating truth conditions is quite trivial. Indeed, why not just say that a sentence of set theory, $\psi$, is true if and only if $\psi$ is true. This obviously is trivial. However, there are important differences between my approach and this one. My purpose is to state the conditions under which set theoretic sentences are true and thereby explain their truth. Now given the trivial truth conditions, how is one to establish (in the meta-language) that the axioms of (naive) set theory are true? One can do this only by assuming those axioms themselves. In other words, the meta-language needs to be beefed up with set-theoretic principles – in fact the very set theoretic principles whose truth is at issue – for the process to work. This is why it is trivial. By contrast given my account of the truth conditions of set theoretic sentences, the truth of the axioms of set theory can be shown (in the meta-language) to follow *without the use of any additional set-theoretic principles*. (The proof of this is given in the appendix.) Thus it provides a genuine, noncircular explanation of the truth of these principles and here resides its nontriviality. Moreover, although the trivial truth conditions for $\psi$ are trivially correct, they are not informative enough, certainly for my purpose, and probably for any purpose. For to extract ontological commitment from truth conditions we need to look at how the truth conditions of formulas relate to those of their subformulas, especially for quantified sentences. This the trivial truth conditions fail to do. In other words the recursion itself carries important information.

However, the fact that the truth conditions of set theory are not well-founded is important, and sets set theory apart from arithmetic. For it brings forth the possibility that the truth and falsity of set theoretic statements are under- or overdetermined.

I have mentioned that a number of statements of set theory, the axioms of naive set theory, are determined by the truth conditions to be true. However consider the following sentence:

(1) $\{x \mid x \in x\} \in \{x \mid x \in x\}$.

This is true iff for some term $t$, $\ulcorner t = \{x \mid x \in x\} \urcorner$ is true and $\ulcorner t \in t \urcorner$ is true. But to determine whether for some term $t$, $\ulcorner t \in t \urcorner$ is true, we must first determine whether (1) is true. We have gone round in a loop. The conditions provide no determinate truth value for (1). (1), though having truth conditions, is neither true nor false, simply underdetermined.

However, worse is to come. For it sometimes happens that the loops, like a möbius strip, go via an inversion. For by the truth conditions of $\in$

(2) $\ulcorner \{x \mid x \notin x\} \in \{x \mid x \notin x\} \urcorner$ is true only if $\ulcorner \exists y(y = \{x \mid x \notin x\} \wedge y \notin y) \urcorner$ is true.

But this implies that $\ulcorner \{x \mid x \notin x\} \notin \{x \mid x \notin x\} \urcorner$ is true (by the properties of identity) and since ⌜$\{x|x \notin x\} = \{x|x \notin x\}$⌝ is true, it follows that ⌜$\exists y(y = \{x|x \notin x\} \wedge y \notin y)$⌝ is true. Whence again by the truth conditions of $\in$,

⌜$\{x|x \notin x\} \in \{x|x \notin x\}$⌝ is true.

Hence (2) is true if and only if it is not true. It is therefore both true and not true. Just as in the previous case the truth conditions underdetermine the truth value of a sentence, so in this case, the truth conditions overdetermine truth value. For (2) is determined to be both true and false.

We have seen that the existence of paradoxical and truthvalueless sentences in set theory is a natural consequence of the non-well-foundedness of the truth conditions of set theoretic statements. No similar possibility can arise in the case of arithmetic. However, that the truth conditions of set theory should over- or underdetermine the truth value of certain statements is not surprising. After all, there was no Hilbert around to kibitz the human behaviour on which mathematical consistency depends. Perhaps it is more surprising that arithmetic *is* consistent.

At this point the reader may be worried that the truth conditions of set theory do too much determining – that they really determine everything to be true. This worry can be set aside. For dialectical set theory – naive set theory with an underlying paraconsistent logic – has been proved to be nontrivial by Brady (198+). Thus the contradictions of naive set theory do not spread through the whole theory but are limited. However, since even limited inconsistency may worry people, it is probably worth spending a little time on it.

Let us start with incompleteness. Consider for example the priority-to-the-right rule of French driving. This convention determines that at a junction one must give way to any car immediately to one's right. This convention is perfectly workable and determines what should happen with 2 or 3 cars arriving simultaneously at a crossroad. But what happens when four cars arrive simultaneously is not determined. Just as in the case of the set theoretic truth conditions, the priority conditions get into a loop from which there is no exit. *A* must give way to *B* who must give way to *C* who must give way to *D* who must give way to *A* who must . . . . This may show that there is need for an additional convention to govern the four-car case. However, it also shows that incomplete conditions arise quite naturally and that this does not deprive them of all application.

So much for the underdetermined case. Returning to the overdetermined case, let us suppose that we have the following neo-French priority rule:

> At a junction: if one person is a woman, she has priority. The oldest person has priority.

The absent-minded administrator who made the law, we may suppose, added the second clause to take care of the case when both drivers are women or both are men, without realising that in certain circumstances it is inconsistent with the first. If the male driver of one car is older than the female driver of the other it gives them both priority. But notice that this inconsistency does not ruin all applications of the priority rule. In three cases out of four it determines a unique priority. That the rule is inconsistent may be a good reason for amending it, but the important point is that limited inconsistency does not lead to total inconsistency and total uselessness.

In fact inconsistent rules or conditions are something we live with. A country with a consistent legal code is unusual. And there is no reason to give up the whole ball game when we find an isolated inconsistency. We can of course always decide to change the rules. However, this is not obligatory and in the case of set theory, where inconsistent conditions will not lead to death on the roads, there is no particular reason why they should be changed. There may even be good reasons why they should not be. Inconsistent theories may well have more mathematical interest than consistent ones. Indeed, just such a situation seems now to have arisen in the foundations of category theory. After the discovery of inconsistency in set theory, set theorists cast around for a theory that was strong enough to handle all reasonable set theoretic constructions but which was (apparently) consistent. By the middle years of this century it appeared that Zermelo Fraenkel set theory, or some variant of it, was such a theory. Of course there are set theoretic constructions (such as forming the universal set and operating on it) which are intuitively perfectly acceptable but which cannot be handled in *ZF*. However, it appeared that such constructions were not necessary for working mathematics. With the development of category theory this has turned out not to be the case. The inability of *ZF* to represent the categories of all sets, all groups, etc., and do with them the things that category theorists want to do has been a decided embarassment. (For a further discussion see Fraenkel, Bar-Hillel, and Levy 1973). There are some decidedly *ad hoc* solutions to the problem, such as the Gröthendieck hierarchy, but these are in the last instance only temporizing measures which push the problem further back (up?) but do not solve it. The category of *all* groups remains as embarrassing as ever. (For a further discussion see Bell 1981.) However, naïve, inconsistent, set theory is exactly the set theory that is required as a foundation for category theory. The universal set can be formed and operated on in the normal way. Category theory can be only harmfully restricted by the insistence upon consistency. Thus there are good reasons for staying inconsistent. This is but a small part of the case for paraconsistency. For more of it see Priest (1979a) and Routley (1977).

Let us now return to and reiterate the main point of this section. When giving the semantics of a language, the well-foundedness of the truth conditions is sufficient (other normal conditions being met) for the well-definedness of the truth predicate, in the sense that it guarantees the existence of a unique (consistent) set which is its extension. In the absence of well-foundedness we have no guarantee that this is the case. Indeed, in set theory it is *not* the case. The extension of the truth predicate is both under- and overdetermined. However, this is just a fact about the truth conditions of ordinary set theoretic discourse and the one, moreover, which explains the paradoxes of set theory.

## 6. OBJECTIONS

The contents of the previous two sections show that the account I have offered has an explanatory power which speaks strongly in its favour. I will end by defending it against two objections. These both concern substitutional quantification. For a long time there were many general worries about substitutional quantification. However, most of these have been defused by Kripke (1976). Hence I will concentrate on those problems which arise specifically as a result of my use of it.

(i) This objection can be found in Quine (1973), pp. 118–20 and Kripke (1976), p. 385. I have argued that by using substitutional quantification we can give the truth conditions of mathematical sentences without having to invoke, or be committed to, a domain of abstract mathematical objects. But now consider the truth conditions themselves. These are given in a language. Moreover this language refers to linguistic objects. This is particularly clear in the case of the truth conditions for quantifiers, which are of the form 'there is a numeral . . .', 'there is an abstract . . .'. Furthermore, the linguistic objects involved must be types, not tokens. For otherwise there would not be enough for the truth conditions to come out right. But linguistic types are just as abstract as real mathematical objects. Indeed, if we code syntax in the usual way we can take syntax to be just a branch of number theory itself. Hence the anti-realist victory is an empty one. My anti-realism is committed to the existence of objects which are, in principle, no different from those I wish to avoid.

Let us grant for a second the claim that I am committed to the existence of linguistic types. Does it follow that the use of substitutional quantification has no point? That answer is 'No'. For it remains true that mathematics *per se* has no commitment to abstract objects. It is the second-order discourse about the language of mathematics which is so committed. This in itself is significant for it produces an important relocation of an old problem. Moreover, the abstract objects to which it is committed are of a simple and perspicuous kind. Even if we identify syntax with arithmetic, it still remains the case that we have "reduced" the ontology of mathematics (now including the discourse about mathematics) to the natural numbers – a significant reduction.

But now let us examine the claim that the language in which I give the truth conditions of arithmetic and set theory is committed to the existence of abstract objects. The part of the language which is causing problems is that part which appears to refer to linguistic objects. I have not given a formal specification of this but we can take it to be the language for the first-order theory of syntax. Is this committed to the existence of abstract objects? This depends on how *its* truth conditions are to be given. Can we give its semantics without refering to a domain of abstract objects in which terms find their denotation and over which quantifiers range? At this point the observation that we can view syntax as a subtheory of arithmetic becomes double edged. For we know that we can give an anti-realist account of arithmetical truth conditions. We can, *a fortiori*, give one of elementary syntax therefore.

An obvious question is whether this move is the first one in a regress which is vicious. The answer is that although the regress may be infinite there is no reason to suppose that it is vicious. Actually I would assert that the regress is only apparent anyway. Although I have given the truth conditions of a formal language in English, the exercise is to be understood as giving the truth conditions for a fragment of English, viz., that in which pure mathematical statements are expressed. This is to be seen as part of the more general enterprise of giving the truth conditions of English in English itself. (Of course I expect this to lead to inconsistency, but not to triviality.) Thus the regress never gets going, since each move takes us back to our starting point – English.

(ii) The second objection concerns substitutional quantification and cardinality. A number of people have thought that the fact that there are uncountably many sets but only countably many abstracts shows that substitutional quantification in set theory is incorrect. For example, Quine (1962), p. 181, suggests that given a formula $\phi$ of one free variable $x$ it could be that there are objects satisfying $\phi$ even though there is no term $t$ such that $\phi(x/t)$ is true. Hence using substitutional quantification the truth value of $\ulcorner \exists x \phi \urcorner$ would come out wrong. Obviously this criticism will not cut much ice against someone who questions, as I am doing, the existence of real mathematical objects.

However, some may still feel uncomfortable with the situation. In part this may rest on a confusion. After all, someone who advocates using substitutional quantification to give the truth conditions of set theory is not saying that sets *are* abstracts – whatever that might mean. (Quine himself makes this point and comes close to making the next one (1973), pp. 113–14.) However, there is a deeper point of interest. Assuming that we can code linguistic entities in set theory in the usual way and that the theory can express its own denotation relation (of course this second assumption makes no sense classically but it does paraconsistently) we can prove, in the theory itself,

$$\ulcorner \exists x \neg \exists y \text{ ($y$ is an abstract and $y$ denotes $x$)} \urcorner$$

by the usual cardinality argument. Now by the truth conditions of existentially quantified sentences, it follows that for some term $t$

(1) $\ulcorner \neg \exists y$ ($y$ is an abstract and $y$ denotes $t$)$\urcorner$ is true.

But if $\# t$ is the code of $t$, we can also prove

$$\ulcorner \# t \text{ is an abstract and } \# t \text{ denotes } t \urcorner.$$

Thus the theory is inconsistent (which we knew anyway). However, this just provides another example of the overdetermination of truth by truth conditions (in this case the truth conditions of set theory and of the denotation relation). We met this before in §5. Hence this situation shows nothing about the illegitimacy (or otherwise) of substitutional quantification. We can even prove (1) for some term $t$ without appealing to the truth conditions for substitutional quantification. Let $t$ be $\ulcorner \{\alpha | \alpha$ is a von Neumann ordinal and $(\forall\beta \leq \alpha)$ ($\beta$ is denoted)}[1]. $t$ denotes the least undenoted ordinal. This is, of course, just König's paradox. A direct appeal to substitutional truth conditions therefore gives us a way of proving something we knew to be true anyway – the existence of something both denoted and undenoted!

## 7. CONCLUSION

I have given an anti-realist account of the truth conditions of the sentences of mathematics. Strictly speaking I should say 'pure mathematics'. For while the language of set theory considered is sufficient for the expression of pure mathematics, it is not at all obvious that it is sufficient for that of applied mathematics. For example, it is a legitimate question whether in doing physics we need to consider not just "pure sets" but also "mixed sets", such as functions from physical objects to numbers. To settle this issue we would need to specify a language for applied mathematics (or perhaps one for each of its applications) and establish whether its truth conditions can be given in a way which does not refer to abstract mathematical objects. This is no light undertaking, and certainly not one I will take on now, though I believe it can be done. However, it is interesting to note that Quine, a philosopher very sensitive to this kind of issue, has recently come to the conclusion that a language of pure sets *is* sufficient for all of science (1976, pp. 501ff.). If this is right then applied mathematics poses no extra problems. For we have already seen that pure set theory can be understood anti-realistically.

It is interesting to note that in the same paper Quine has his final word (that I know of) on substitutional quantification. His conclusion is that the "only remaining cause for hesitation over the substitutional version [of quantification in set theory] is impredicativity" (1976, p. 504, fn. 3). The question of impredicativity is just the question of the non-well-foundedness of the truth conditions. I discussed this in §5 and argued that it was no problem, at least for a paraconsistentist. Anyway the truth conditions I have proposed for pure mathematics refer to no realm of abstract mathematical objects. The problems of realism are therefore avoided. Not only this, but also the account I have given explains both the nature of mathematical necessity and the location of the paradoxes in set theory. Hence I claim it to be a better account than realism.*

## APPENDIX

The appendix shows that the rules and axioms of naive set theory are true under the truth conditions given in §3. It is assumed that the logic of the meta-language is a paraconsistent one such as that given in Priest (1980) or the *DL* of Routley (1977). (It is perhaps worth noting that although Routley uses *DL* including applications of *modus ponens* to develop naive set theory, *modus ponens* is only validity preserving, not truth preserving in the semantics of *DL*. No similar problem besets the logic of my [1980]). The axioms and rules are as follows:

(A) $t_1 = t_1$

(B) $\forall x(x \in t_1 \leftrightarrow x \in t_2) \rightarrow t_1 = t_2$

(C) $t_1 = t_2 \twoheadrightarrow \phi(t_3/t_1) \leftrightarrow \phi(t_3/t_2)$

(D) $t_1 \in \{z | \phi(t_2/z)\} \twoheadleftarrow\!\!\twoheadrightarrow \phi(t_2/t_1)$

where $\twoheadrightarrow$ is the meta-linguistic sign for a rule of inference.

The proofs are as follows.

(A) $\ulcorner t_3 \in t_1 \urcorner$ is true iff $\ulcorner t_3 \in t_1 \urcorner$ is true. Hence (A) is true.

(B) Suppose $\ulcorner \forall x(x \in t_1 \leftrightarrow x \in t_2)\urcorner$ is true.
Then for all $t_3$, $\ulcorner t_3 \in t_1 \urcorner$ is true iff $\ulcorner t_3 \in t_2 \urcorner$ is true. Hence $\ulcorner t_1 = t_2 \urcorner$ is true. This verifies (B).

(C) This is proved by induction over the formation of $\phi$. The basis cases are as follows: Suppose $t_1 = t_2$,

(i) then $t_3 \in t_1$ iff $t_3 \in t_2$. Consequently ($t_3 \in t_4$ iff $t_3 \in t_1$) iff ($t_3 \in t_4$ iff $t_3 \in t_2$).
Hence $t_1 = t_2 \twoheadrightarrow (t_1 = t_4 \leftrightarrow t_2 = t_4)$.

(ii) Let $t_3$ be $\{z | \phi(t_4/z)\}$, then $t_1 \in t_3$ iff for some $t$, $t = t_1$ and $\phi(t_4/t)$ iff ($t' \in t$ iff $t' \in t_1$) and $\phi(t_4/t)$ iff ($t' \in t$ iff $t' \in t_2$) and $\phi(t_4/t)$ iff $t_2 \in t_3$.
Hence $t_1 = t_2 \twoheadrightarrow (t_1 \in t_3 \leftrightarrow t_2 \in t_3)$.

(iii) Since $t_1 = t_2 \rightarrow (\forall x)(x \in t_1 \leftrightarrow x \in t_2)$, then $t_1 = t_2 \twoheadrightarrow (t_3 \in t_1 \leftrightarrow t_3 \in t_2)$ follows.

The basis case for $f$ is trivial and the induction cases for $\supset, \rightarrow, \exists$ are straightforward. This verifies (C).

(D) If $t_1 \in \{z \mid \phi(t_2/z)\}$, then for some $t$, $t_1 = t$ and $\phi(t_2/t)$.
By (C) $\phi(t_2/t_1)$.
Hence $t_1 \in \{z \mid \phi(t_2/z)\} \rightarrow \phi(t_2/t_1)$.
If $\phi(t_2/t_1)$, then since $t_1 = t_1(A)$, $t_1 \in \{z \mid \phi(t_2/z)\}$.
Hence $\phi(t_2/t_1) \rightarrow t_1 \in \{z \mid \phi(t_2/z)\}$.

This verifies (D).

## NOTE

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