Notre Dame Journal of Formal Logic
Volume XVII, Number 3, July 1976
NDJFAM

MODALITY AS A META-CONCEPT

GRAHAM PRIEST

1 In this paper, we will construct and consider some basic properties of a certain modal logic, or, more correctly, family of modal logics, generated by a construction which is a standard linguistic tool, but has never been applied to modal logics before. All modal logics till now (with the exception of perhaps Lemmon's S0.5) have tried to serve two purposes at once:

i) To establish which truths are necessary, e.g., $\vdash_T p \vee \sim p$ and hence $\vdash_T L(p \vee \sim p)$ by the rule of necessitation.
ii) To establish the relationships between such necessary truths, e.g., $\vdash_T L \sim p \supset \sim Lp$.

In other words, the axioms of the logic have had to be sufficient to establish modal truths such as $\vdash_T L(p \supset q) .\supset. (Lp \supset Lq)$ and non-modal truths, such as the theses of the propositional calculus. We will show in this paper that these functions are better separated. Further, it is surprising that it has never been fully realized that modality is a meta-concept; for 'necessary' as applied to sentences/propositions is in fact a convenient corruption of 'necessarily true' (and similarly for 'contingent'). The sentence/proposition is not itself necessary: it could have been any other. The necessity for being what it is belongs to its truth-value. Now, we know very well that for an adequate formalization of the concept of truth, we have need of a meta-language, we need then, *a fortiori*, a meta-language for the formalization of the concept of necessary truth.

A glimpse of the danger of telescoping the object and meta-language is the following. Consider the sentence:

(A) 'This sentence is not necessarily true'. (Or its Quinean version 'gives a statement that is not necessarily true when appended to its own quotation'.)

Suppose this is false, then it is necessarily true, and hence true. Consequently, (A) cannot be false. Hence it must be true. Thus it is not necessarily true and must be contingently true. It must, therefore, be possible for it to be false, but as we have seen this is not the case.

Received January 12, 1971

What we will do in this paper bears a relationship to the above paradox similar to the relationship between what Tarski did in his famous paper on 'truth' [1] and the Liar paradox. We will show how, given any object language, we can construct a meta-language of sufficient power to talk about its (the object language's) modalities. Rather, however, than show the process in its full generality at first, which might be a little confusing, we will show how it is possible with an example. The generalization on this procedure will then become more obvious. One final point, however, before entering the body of the work; that is, at the time of the method's conception, we wished to exclude iterated modalities, because their exact nature was even less clear than that of other modal concepts. It became clear, though, when the language, $\mathcal{G}_1$, had been constructed, that this method, in fact, provided a sound conceptual basis for the iteration of modalities. But of this, more anon.

2 Let us now turn to our example. For an object language, we will take propositional calculus. This we do for two reasons: first the propositional calculus is itself a basis for many other logical languages, and is, in some sense, *the* basic logical language; secondly, the meta-language, $\mathcal{G}_1$, which it generates, is of some interest on its own account.

Object Language O We shall write '$\vdash_O$' for 'is a thesis of the object language, O'. Small letters, 'p', 'q', etc. will be used as syntactic variables for wffs of O. We chose the following axiom schemes for O (although any equivalent set would do):

01 $\vdash_O q \supset (p \supset q)$
02 $\vdash_O [p \supset (q \supset r)] \supset [(p \supset q) \supset (p \supset r)]$
03 $\vdash_O (\sim p \supset \sim q) \supset [(\sim p \supset q) \supset p]$

Our rule of deduction is *modus ponens*: from $\vdash_O p$ and $\vdash_O p \supset q$ infer $\vdash_O q$. The logical constants '$\supset$' and '$\sim$' are taken as primitive. These are supplemented by the usual definitions of '$\vee$', '$\equiv$', etc. O is the 2-valued propositional calculus.

Meta-language $\mathcal{G}_1$ The question now arises as to whether the meta-language, $\mathcal{G}_1$ should be translational or not, i.e., whether we should have in it wffs having the same meaning as wffs of the object language O, (as in [1]) either course is, in principle, possible, but we will choose the latter. This has the advantage of making what is going on much clearer: the dual linguistic structure is made plain, and the exact status of the rule of translation, T, is exhibited. We will write '$\vdash_{\mathcal{G}_1}$' for 'is a thesis of $\mathcal{G}_1$'. Capital letters, 'P', 'Q', etc. will be used as syntactic variables for wffs of $\mathcal{G}_1$.

Primitive symbols

$p, q, r,$	. . . wffs of O
$P, Q, R,$	. . . wffs of $\mathcal{G}_1$
$\sim, \vdash_O,$	. . . monadic operators
$\supset,$	. . . dyadic operator
$($, $)$,	. . . brackets

Formation rules:

1) If p is any wff of O, then $\vdash_O p$ is a wff of $\mathcal{G}_1$.
2) If P, Q are wffs of $\mathcal{G}_1$, so are $\sim P$ and $P \supset Q$.
3) Only formulas derivable from rule 1) and by successive applications of rule 2) are wffs of $\mathcal{G}_1$.

We note that

(i) The symbols '$\sim$', '$\supset$', etc. are used ambiguously, as logical constants of both O and $\mathcal{G}_1$. Strictly speaking, different symbols should have been used, but since no confusion can arise, it was thought simpler to have only the one set of logical constants.
(ii) The formation rules of $\mathcal{G}_1$ permit as wffs those, and only those, formulas that are pure, and of the first degree. That is, each O wff subformula, must be in the scope of one and only one modal operator. E.g., $\vdash_O p$, $\sim \vdash_O \sim q$ are wffs of $\mathcal{G}_1$, but $\vdash_O \sim \vdash_O p$, $p \supset q$, $p \supset \vdash_O p$ are not.

Definitions In $\mathcal{G}_1$, we have the usual definitions of '$\vee$', '$\equiv$', etc. plus the modal definitions:

D1 $\vdash_O p =_{df} Lp$
D2 $Mp =_{df} \sim L \sim p$

It will be seen that the symbol L, is not strictly needed. However, it makes things intuitively much clearer, to have both '$\vdash_O$' and 'L' as symbols of $\mathcal{G}_1$. (It might well be thought that *D1* is excessively strong; but remember, we are constructing a modal meta-language for a *particular* object language, and hence, this is tolerable. Further, it is very easy to relax *D1*, taking 'L' as primitive, and introducing the further axiom scheme: $\vdash_{\mathcal{G}_1} \vdash_O p \supset Lp$.)

Axiom schemes for $\mathcal{G}_1$ Our non-modal, or proper, axiom schemes for $\mathcal{G}_1$ are:

M1 $\vdash_{\mathcal{G}_1} P \supset (Q \supset P)$
M2 $\vdash_{\mathcal{G}_1} [P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]$
M3 $\vdash_{\mathcal{G}_1} (\sim P \supset \sim Q) \supset [(\sim P \supset Q) \supset P]$

Hence all tautological manipulations of wffs of $\mathcal{G}_1$ are permissible in $\mathcal{G}_1$. Further, we have the modal axiom schemes:

M4 $\vdash_{\mathcal{G}_1} \vdash_O (p \supset q) \supset (\vdash_O p \supset \vdash_O q)$
M5 $\vdash_{\mathcal{G}_1} \vdash_O p \supset \sim \vdash_O \sim p$

It is clear that *M4* and *M5* are equivalent to postulating *modus ponens* for O, and the consistency of O, respectively. When translated into more familiar language, with the help of *D1* and *D2*, they become:

M4^1 $\vdash_{\mathcal{G}_1} L(p \supset q) \supset (Lp \supset Lq)$
M5^1 $\vdash_{\mathcal{G}_1} Lp \supset \sim L \sim p$ or $\vdash_{\mathcal{G}_1} Lp \supset Mp$

Our rule of deduction for $\mathcal{G}_1$ is *modus ponens*: from $\vdash_{\mathcal{G}_1} P$ and $\vdash_{\mathcal{G}_1} P \supset Q$, infer $\vdash_{\mathcal{G}_1} Q$.

We now come to the very important:

Translation Rule This is the rule that correlates O and $\mathcal{G}_1$.

T: From $\vdash_{O} p$ (in O), deduce $\vdash_{\mathcal{G}_1} \vdash_{O} p$ (in $\mathcal{G}_1$), and vice versa: from $\vdash_{\mathcal{G}_1} \vdash_{O} p$ (in $\mathcal{G}_1$) deduce $\vdash_{O} p$ (in O).

This says that if p is a thesis of O, then $\vdash_{O} p$ is a thesis of $\mathcal{G}_1$ and vice versa. It is equivalent to saying that if p is provable, then 'p is provable' is provable. With the help of *D1*, **T** reads:

T[1]: From $\vdash_{O} p$ deduce $\vdash_{\mathcal{G}_1} Lp$, and from $\vdash_{\mathcal{G}_1} Lp$ deduce $\vdash_{O} p$.

We will now prove a few derived rules and theses of $\mathcal{G}_1$, to give some idea of the power, and the proof technique, of $\mathcal{G}_1$. We will take all tautologies for granted, writing 'Taut M' or 'Taut O' as they follow for *M1-3* or O 1-3.

DR1 *From* $\vdash_{O} p$ *deduce* $\vdash_{\mathcal{G}_1} Mp$.

(i) from $\vdash_{O} p$ infer $\vdash_{\mathcal{G}_1} \vdash_{O} p$ (**T**)
(ii) infer $\vdash_{\mathcal{G}_1} LP$ (*D1*)

But

(iii) $\vdash_{\mathcal{G}_1} Lp \supset \sim L \sim p$ (*M5*)

i.e.,

(iv) $\vdash_{\mathcal{G}_1} Lp \supset Mp$ (*D2*)

So from (ii), (iv) by *modus ponens*:

(v) from $\vdash_{O} p$ infer $\vdash_{\mathcal{G}_1} Mp$

G1 $\vdash_{\mathcal{G}_1} L(p \equiv q) \supset (Lp \equiv Lq)$

(i) $\vdash_{\mathcal{G}_1} L\,(p \supset q) \supset (Lp \supset Lq)$ (*M4*)
(ii) $\vdash_{\mathcal{G}_1} L(q \supset p) \supset (Lq \supset Lp)$ (*M4*)

but

(iii) $\vdash_{\mathcal{G}_1} [(P \supset Q).(R \supset S)] \supset (P.Q) \supset (R.S)]$ (Taut M)

hence

(iv) $\vdash_{\mathcal{G}_1} [L(p \supset q).L(q \supset p)] \supset [(Lp \supset Lq).\ (Lq \supset Lp)]$
(v) $\vdash_{\mathcal{G}_1} L(p \equiv q) \supset (Lp \equiv Lq)$ (*Def* '≡')

DR2 *If* $\vdash_{O} (p \equiv q)$ *then* $\vdash_{\mathcal{G}_1} (Lp \equiv Lq)$

(i) from $\vdash_{O} (p \equiv q)$ infer $\vdash_{\mathcal{G}_1} L(p \equiv q)$ (**T**[1])
(ii) but $\vdash_{\mathcal{G}_1} L(p \equiv q) \supset (Lp \equiv q)$ (*G1*)

Hence by (i), (ii) and *modus ponens*

(iii) from $\vdash_{O} (p \equiv q)$ infer $\vdash_{\mathcal{G}_1} Lp \equiv Lq$

It is now very easy to establish by induction the rule of substitutivity of equivalents in $\mathcal{G}_1$ or more precisely:

S.E.1. *If* $\vdash_{O} (p \equiv q)$ *then* $\vdash_{\mathcal{G}_1} P \equiv Q$.

Where P and Q are wffs of $\mathcal{G}_1$ and are the same, except that P has p as an O wff subformula in one or more places, where Q has q.

S.E.2. *If* $\vdash_{\mathcal{G}_1} (R \equiv S)$ *then* $\vdash_{\mathcal{G}_1} (P \equiv Q)$.

Where P and Q are wffs of $\mathcal{G}_1$ and are the same, except that P has R as a $\mathcal{G}_1$ wff subformula in one or more places, where Q has S.

The above results are shown in the normal way, by induction on the degree of P and Q, using **DR2**, and the following tautological derived rules:

If $\vdash_{\mathcal{G}_1} (P \equiv Q)$ *then* $\vdash_{\mathcal{G}_1} (\sim P \equiv \sim Q)$.
If $\vdash_{\mathcal{G}_1} (P \supset Q)$ *then* $\vdash_{\mathcal{G}_1} (P \supset R) \supset (Q \supset R)$.
If $\vdash_{\mathcal{G}_1} (P \supset Q)$ *then* $\vdash_{\mathcal{G}_1} (R \supset P) \supset (R \supset Q)$.

G2 $\vdash_{\mathcal{G}_1} L(p.q) \equiv (Lp.Lq)$

(i) $\vdash_O p.q \supset p$ (Taut *O*)
(ii) $\vdash_{\mathcal{G}_1} L(p.q \supset p)$ ($\mathbf{T}^1$)
(iii) $\vdash_{\mathcal{G}_1} L(p.q) \supset Lp$ (by *M4* and *modus ponens*)

similarly

(iv) $\vdash_{\mathcal{G}_1} L(p.q) \supset Lq$
(v) $\vdash_{\mathcal{G}_1} [(P \supset Q).(P \supset R)] \supset [(P \supset (Q.R))]$ (Taut *M*)
(vi) $\vdash_{\mathcal{G}_1} L(p.q) \supset Lp.Lq$. (by (iii), (iv) and (v))
(vii) $\vdash_O p \supset (q \supset (p.q))$ (Taut *O*)
(viii) $\vdash_{\mathcal{G}_1} Lp \supset L(q \supset (p.q))$ (by $\mathbf{T}^1$, *M4* and *modus ponens*)
(ix) $\vdash_{\mathcal{G}_1} Lp \supset (Lq \supset L(p.q))$ (by *M4* and *modus ponens*)
(x) $\vdash_{\mathcal{G}_1} [P \supset (Q \supset R)] \supset [(P.Q) \supset R]$ (Taut *M*)
(xi) $\vdash_{\mathcal{G}_1} Lp.Lq \supset L(p.q)$ (by (ix) and (x))

Thus

(xii) $\vdash_{\mathcal{G}_1} L(p.q) \equiv (Lp.Lq)$ (for (vi), (xi) and the def. of '≡')

The above proofs should be sufficient to demonstrate the proof techniques at hand in $\mathcal{G}_1$. Since this is only a short paper, and the details of proof are somewhat tedious anyway, we will just list a selection of the most important theses of $\mathcal{G}_1$. In most cases, the actual proof is quite straightforward.

$G_1 3.a)$: $\vdash_{\mathcal{G}_1} L \sim p \equiv \sim Mp$
$G_1 3.b)$: $\vdash_{\mathcal{G}_1} \sim Lp \equiv M \sim p$
$G_1 3$: $\vdash_{\mathcal{G}_1} Lp \equiv \sim M \sim p$
$G_1 4$: $\vdash_{\mathcal{G}_1} M(p \vee q) \equiv (\sim Mp.\sim Mq)$
$G_1 5$: $\vdash_{\mathcal{G}_1} M(p \vee q) \equiv (Mp \vee Mq)$
$G_1 6$: $\vdash_{\mathcal{G}_1} L(p \supset q) \supset (Mp \supset Mq)$
$G_1 7$: $\vdash_{\mathcal{G}_1} (Lp \vee Lq) \supset L(p \vee q)$
$G_1 8$: $\vdash_{\mathcal{G}_1} M(p.q) \supset (Mp.Mq)$
$G_1 9$: $\vdash_{\mathcal{G}_1} L(\sim p \supset p) \supset Lp$
$G_1 10$: $\vdash_{\mathcal{G}_1} [L(q \supset p).\ L(\sim q \supset p)] \equiv Lp$
$G_1 11$: $\vdash_{\mathcal{G}_1} Lp \supset L(q \supset p)$
$G_1 12$: $\vdash_{\mathcal{G}_1} L \sim p \supset L(p \supset q)$

$G_1 13$: $\vdash_{\mathcal{G}_1} Lp \supset [Mq \supset M(p.q)]$
$G_1 14$: $\vdash_{\mathcal{G}_1} L(p \vee q) \supset (Lp \vee Mq)$

It is possible to define 'p strictly implies q' ($p \prec q$), in the usual way, as $L(p \supset q)$. This will do as an exposition of $\mathcal{G}_1$ and its object properties.

3 We note that formulas such as $p \supset Mp$, $Lp \supset p$, and other such formulas that straddle $\mathcal{O}$ and $\mathcal{G}_1$ are not expressible in either $\mathcal{O}$ or $\mathcal{G}_1$. This is a direct result of our electing to construct a non-transitional meta-language. Clearly, had we chosen the other course we would have wished such things to be provable in the system, and it is quite straightforward to see how this could be done. What, in fact, do we lose by not having such straddled formulas as theses? The answer is 'surprisingly little' and the reason is this:

Whenever a truth of the philosophy of logical necessity is expressed by a straddled thesis of **T** or S5, etc., then that truth is expressed by the corresponding derived rule of inference in $\mathcal{O}/\mathcal{G}_1$.

We will not stop to prove this, but it is intuitively clear. E.g.,

(i) $Lp \supset p$; *if* $\vdash_{\mathcal{G}_1} Lp$ *then* $\vdash_{\mathcal{O}} p$ *is* $\mathbf{T}^1$.
(ii) $p \supset Mp$; *if* $\vdash_{\mathcal{O}} p$ *then* $\vdash_{\mathcal{G}_1} M$ *is* **DR1**.
(iii) $\sim p \supset \sim Lp$; *if* $\vdash_{\mathcal{O}} \sim p$ *then* $\vdash_{\mathcal{G}_1} \sim Lp$ *is a derived rule*:

Proof: $\vdash_{\mathcal{O}} \sim p$ (Hyp.)
so $\vdash_{\mathcal{G}_1} L \sim p$ ($\mathbf{T}^1$)
so $\vdash_{\mathcal{G}_1} \sim Lp$. (*M5*)

(iv) $[L(p \supset q).\, p] \supset .Mq$; *if* $\vdash_{\mathcal{G}_1} L(p \supset q)$ *and* $\vdash_{\mathcal{O}} p$ *then* $\vdash_{\mathcal{G}_1} Mq$ *is a derived rule:*

Proof: $\vdash_{\mathcal{G}_1} L(p \supset q)$ (Hyp.)
$\vdash_{\mathcal{O}} (p \supset q)$ ($\mathbf{T}^1$)
but $\vdash_{\mathcal{O}} p$ (Hyp.)
so $\vdash_{\mathcal{O}} q$ (*modus ponens*)
so $\vdash_{\mathcal{G}_1} Mq$, (**DR1**)

etc.

Further, it might be thought that we could not prove some wffs that were pure and of the first degree that ought to be theses, due to the non-appearance of straddled wffs. But this is not the case (see section **5** for the completeness of $\mathcal{G}_1$—in effect *M5* takes over this function of straddled wffs). Hence, we lose virtually nothing. Also, an interesting extension of $\mathcal{G}_1$ might be obtained by adding a new predicate 'T' to $\mathcal{G}_1$ and axiomatizing it, such that $\vdash_{\mathcal{G}_1} Tp$ has the force of 'p is true'. Then the equivalent of any straddled wff could be proved in this system, e.g., for $p \supset Mp$, we have $Tp \supset Mp$, etc. (Assuming that one wished to assume that p is true if $\vdash p$.) We will not pursue this line of thought further in the present paper, however.

4 A pertinent question now would be this. We know that it is possible to prove something of the form $\vdash_{\mathcal{O}} p$ by a proof that is partly in $\mathcal{G}_1$, i.e., the rule **T** is invoked to infer a wff of O from the statement of its necessity in $\mathcal{G}_1$. Is it possible to prove anything like this, that is not provable simply in O? An affirmative answer to this question would be somewhat disastrous, but fortunately would be false. We will postpone a proof of this result, however, until we have considered modal predicate logics, so that the proof may have a more general nature.

5 We now diverge slightly from the general train of the paper to consider some individual properties of the system $\mathcal{G}_1$.

Consistency That $\mathcal{G}_1$ is consistent, is easily provable, by the standard method of modal logic. That is, by mapping $\mathcal{G}_1$ onto the 2-valued propositional calculus by deletion of modalities. The proof is quite straightforward, and we refer readers to, for example, [2], pp. 41-42.

Semantics for $\mathcal{G}_1$ A model for $\mathcal{G}_1$ is readily found with the help of the usual Kripke semantics for modal logics. In fact, it is quite easy to show that any S5 model is a model for $\mathcal{G}_1$. Since this is very similar to the standard cases we will not go into it in great detail, but we define a model as an ordered pair $\langle U, \vee \rangle$ where U is a set of worlds w_i and $\vee$ is an evaluation of wffs of O and $\mathcal{G}_1$, satisfying the normal S5 conditions. (See for example [2], pp. 71-75.) A wff of $\mathcal{G}_1$ is then true in the model iff it has the value 1 for all w_i under the evaluation $\vee$. It is now readily seen that the axiom schemes *M1-M5* are true in every model (they are all theses of S5) and that *modus ponens* preserves truth. The translational rule, **T**, cannot be said to preserve truth, as such, in a $\mathcal{G}_1$ model, since the wffs of O are not in $\mathcal{G}_1$, and hence, cannot be true in $\mathcal{G}_1$. What we can do, however, is to consider the S5 model as a dual model for both O and $\mathcal{G}_1$ taken together, i.e., as model for the language whose theses are the theses of O and the theses of $\mathcal{G}_1$. We then can say that the rule of translation preserves truth. Hence all the theses of $\mathcal{G}_1$ are true in any $\mathcal{G}_1$ model. Since also, in effect, a $\mathcal{G}_1$ model is a restriction of an S5 model to the wffs of S5 that are well-formed by the formation rules of $\mathcal{G}_1$, we have shown:

If P is a wff of S5, *that is well-formed according to the formation rules of* $\mathcal{G}_1$, *then P is true in an* S5 *model iff P is true in the corresponding* $\mathcal{G}_1$ *model.*

Completeness To prove the completeness of $\mathcal{G}_1$, what we will do is this, we will prove that

(A) *If P is a wff of* S5, *well-formed with respect to the formation rules of* $\mathcal{G}_1$, *then* $\vdash_{S5} P$ *iff* $\vdash_{\mathcal{G}_1} P$.

This shows that the theses of $\mathcal{G}_1$ are precisely the theses of S5, that are well-formed with respect to the formation rules of $\mathcal{G}_1$. We then invoke the completeness theorem for S5 ([2], p. 116), to show that $\mathcal{G}_1$ is complete with respect to its model, i.e., the S5 model restricted to $\mathcal{G}_1$ well-formed formulas. We prove (A) by furnishing a decision procedure for $\mathcal{G}_1$.

Decision Procedure We define Modal Conjunctive Normal Form (M.C.N.F.) as follows: P (a wff of $\mathcal{G}_1$) is in M.C.N.F. if P is a conjunction, possibly degenerate, of disjunctions, possibly degenerate, such that each component of the disjunction is a wff of O in the direct scope of one modal operator. E.g.:

$$(Mp \vee Lp).\ L(p \supset q)$$

(i) Every wff P of $\mathcal{G}_1$ can be reduced to M.C.N.F. P^1 such that $\vdash_{\mathcal{G}_1} P \equiv P^1$. This is done by replacing all occurrences of '$\sim L$' and '$\sim M$' by '$M\sim$' and '$L\sim$' respectively, ($G_1 3.a$ and b). We then treat all wffs of the form Lp and Mp, as atomic and proceed by propositional calculus methods. We know that all propositional calculus wffs can be reduced to conjunctive normal form.

(ii) Order each conjunct: $Lp_1 \vee Lp_2 \vee \ldots \vee Lp_n \vee Mq_1 \vee \ldots \vee Mq_m$ which is provably equivalent to

$$Lp_1 \vee \ldots \vee Lp_n \vee Mq, \text{ where } q = q_1 \vee q_2 \ldots \vee q_m. \qquad (G_1 5)$$

(iii) A particular conjunct is valid iff some $p_i \vee q$ is propositional calculus valid.
(iv) P^1 is valid if each conjunct is valid.
(v) P is valid if P^1 is valid.
(iv) and (v) are obvious. We will prove (iii).

Proof of (iii): a) Let $p_i \vee q$ be **PC** (propositional calculus) valid, i.e., $\vdash_{\mathcal{O}} p_i \vee q$. Hence $\vdash_{\mathcal{G}_1} L(p_i \vee q)$ by $\mathbf{T}^1$. Hence $\vdash_{\mathcal{G}_1} (Lp \vee Mq)$ by $G_1 14$ and *Modus ponens*. If q is null, then the last step is redundant. If p_i is null, then we have $\vdash_{\mathcal{G}_1} Mq$ by **DR1**. Finally, $\vdash_{\mathcal{G}_1} Lp_1 \vee Lp_2 \ldots \vee Lp_n \vee Mq$ ($\vdash_{\mathcal{G}_1} P \supset (P \vee Q)$), i.e., $\vdash_{\mathcal{G}_1} P^1$.
b) Let none of $p_i \vee q$ be **PC** valid. We then construct a $\mathcal{G}_1$ model in which P^1 ($= Lp_1 \vee \ldots \vee Lp_n \vee Mq$) is invalid and hence, unprovable. In our model $\langle U, \mathsf{V}\rangle$, let $U = \langle w_1 \ldots w_n\rangle$. Since all $p_i \vee q$ (and hence all p_i and q) are **PC** invalid, we can find an evaluation V, such that

$$\mathsf{V}(w_i q) = 0 \text{ for all } i.\ \ \mathsf{V}(w_i p_i) = 0 \text{ for all } i.$$

So we have that p_i is invalid in w_i, and hence Lp_i is invalid in $\langle U\mathsf{V}\rangle$. Further Mq is invalid in $\langle U\mathsf{V}\rangle$, since q is false in all w_i. Hence P^1 is invalid. Thus we see that either a wff in M.C.N.F. is invalid and hence, not provable, or it is valid and the above proof furnishes an effective proof in $\mathcal{G}_1$ starting from the **PC** valid $p_i \vee q$.

We note, further, that this decision procedure works for all $\mathcal{G}_1$—well-formed (by the formation rules of $\mathcal{G}_1$) formulas of T, S4 and S5. We simply replace the appeal to $\mathbf{T}^1$ by an appeal to the rule of necessitation. Further, the model in part b) of the proof is also a valid T, S4, or S5 model. We have hence shown:

If P is $\mathcal{G}_1$—well-formed, then $\vdash_{\mathcal{G}_1} P$ iff $\vdash_{\mathrm{T}} P$ iff $\vdash_{\mathrm{S4}} P$ iff $\vdash_{\mathrm{S5}} P$.

The completeness of $\mathcal{G}_1$ to its model follows trivially from the completeness of S5.

6 We now return to the main theme of the paper, which is the general method of constructing $\mathcal{G}$-meta-languages. The method appears to be this:

Take an object language. Furnish a set of proper (non-modal) axioms for $\mathcal{G}$. Add the two modal axioms *M4* and *M5*, and the translational rule **T**. This generates a $\mathcal{G}$-meta-language.

It will be noticed that the proper axioms of $\mathcal{G}$ are left open to choice. Often, they will be the axioms of $\mathcal{O}$, as in our example, but not necessarily so. For example, suppose we wish to construct a $\mathcal{G}$-meta-language for the intuitionist propositional calculus, or for a particular many-valued calculus. Then for the proper axioms of $\mathcal{G}$ we might wish to take the axioms of the object language or we might wish to take, say, the axioms of the ordinary two-valued, propositional calculus. Either choice would produce a $\mathcal{G}$-meta-language, and which one one chose would depend purely on philosophical considerations.

Another interesting way in which our original $\mathcal{G}_1$ system could be extended, is to a modal predicate logic. Here, different systems arise, again, depending on what sets of axioms we choose for $\mathcal{O}$ and $\mathcal{G}$. If the object language is the predicate calculus, and the proper axioms of $\mathcal{G}$ are those of the propositional calculus, then we see that only modalities *de dicto* can arise. If, however, we do it the other way round, only *de re* modalities arise.

Probably, the most interesting system is generated by taking the axioms of the predicate calculus (without identity) for both $\mathcal{O}$, and the proper axioms of $\mathcal{G}$. This way we get both sorts of modalities. Without making this paper too lengthy it is impossible to go much further into any of these systems, but we will give one proof, in the last cited system, just to show what sort of structures are available to us. We add the following axioms to 01-03:

04 $\vdash_{\mathcal{O}} (x)\alpha \supset \alpha'$ *where* α' *is* α *with all occurrences of* x *replaced by some individual constant or variable, (possibly* x *itself).*

05 $\vdash_{\mathcal{O}} (\alpha \supset \beta) \supset (\alpha \supset (x)\beta)$ *provided* x *is not free in* α.

This makes $\mathcal{O}$, the predicate calculus. We add two similar axioms *M6* and *M7* to *M1-3.*

$G_2 1$: $\vdash_{\mathcal{G}} L(x)\alpha \supset (x)L\alpha$

Proof:

(i)	$\vdash_{\mathcal{O}} (x)\alpha \supset \alpha$	(04)
(ii)	$\vdash_{\mathcal{G}} L((x)\alpha \supset \alpha)$	($\mathbf{T}^1$)
(iii)	$\vdash_{\mathcal{G}} L(x)\alpha \supset L\alpha$	(*M4*)
(iv)	$\vdash_{\mathcal{G}} L(x)\alpha \supset (x)L\alpha$	(*M7* and *modus ponens*)

It is interesting to note, however, that even in this system, the Barcan formula $(x)L\alpha \supset L(x)\alpha$ is not provable, but may be taken as an extra axiom. A little thought will show that this is equivalent to postulating the ω-completeness of O. The rule of inference corresponding to the Barcan formula, however, is provable. We prove:

If $\vdash_{\mathcal{G}} (x)L\alpha$, *then* $\vdash_{\mathcal{G}} L(x)\alpha$

(i)	$\vdash_{\mathcal{G}} (x)L\alpha$	(Hyp.)
(ii)	$\vdash_{\mathcal{G}} (x)L\alpha \supset L\alpha$	(*M6*)
(iii)	$\vdash_{\mathcal{G}} L\alpha$	(by *modus ponens*)
(iv)	$\vdash_{O} \alpha$	($\mathbf{T}^1$)
(v)	$\vdash_{O} (x)\alpha$	(by universal generalization)
(vi)	$\vdash_{\mathcal{G}} L(x)\alpha$	(by $\mathbf{T}^1$)

A similar result is provable in the usual predicate extensions of T and S4.

Another interesting extension is gained by introducing identity into the object language. In normal modal predicate logics, this leads to paradoxical results, such as

$$a = b \supset L(a = b)$$

i.e., all identities are necessary! This is a direct result of the fact that in the usual axiom scheme for identity, i.e., $a = b \supset (A \equiv B)$ where A and B are the same, except that A has a in some or all the places where B has b, 'A' and 'B' are allowed to range over all wffs of the language, which, of course, include modal wffs. In our two-tier system, however, this does not occur, since A and B range over wffs of O, only. In normal, so-called 'contingent identity systems' this restriction appears totally *ad hoc*, but in our two-tier system, it is a natural consequence of the linguistic structure. We see clearly now how the $\mathcal{G}$-construction throws light on several awkward problems of modal predicate logic, such as the nature of *de re* and *de dicto* modalities, and the 'paradoxes' of identity.

Leaving aside extensions of our original system now, we will finally consider an extension of the technique itself, to generate iterated modalities. Suppose we construct a meta-meta-language $\mathcal{G}\mathcal{G}$, axiomize it with the same axioms as $\mathcal{G}$, and add a rule of translation between $\mathcal{G}$ and $\mathcal{G}\mathcal{G}$, of the form:

$\mathbf{T}_1$ $$\vdash_{\mathcal{G}} P \textit{ iff } \vdash_{\mathcal{G}\mathcal{G}} \vdash_{\mathcal{G}} P.$$

Then, writing L_1 for $\vdash_{\mathcal{G}}$ just as we wrote L for $\vdash_{O}$, we have wffs in $\mathcal{G}\mathcal{G}$ such as

$$L_1 Lp \supset L_1 Mp.$$

In this manner, we can construct an infinite hierarchy of metalanguages such that every pure wff (i.e., wff in which every O sub-wff is in the scope of the same number of modal operators) is a wff of some member of the hierarchy. The subscripts of the 'L' symbols then become optional and can be dropped, giving the 'L' typical ambiguity. This technique has one very interesting consequence, which is as follows: The construction of modalities in our $\mathcal{G}$ systems is compatible with a conventionalist view of logical necessity. That is, that a necessary truth is necessary because of the way we use words, i.e., it is true either by convention or definition (i.e., the axioms of O) or by logical consequence of conventions (logical consequence being truth-preserving by definition).

However, returning to $\mathbf{T}_1$, we see that we have:

$$\vdash_{\mathcal{G}} P \textit{ iff } \vdash_{\mathcal{G}\mathcal{G}} LP.$$

Now 'Lp' is a wff of $\mathcal{G}_1$ and so substituting for 'P' in the above, we have:

$$\vdash_{\mathcal{G}} Lp \textit{ iff } \vdash_{\mathcal{G}\mathcal{G}} LLp.$$

This is the equivalent, in our infinite hierarchy of languages, of the S4 principle, which is usually reckoned to be inconsistent with a conventionalist point of view. We see, however, that this is not so.

7 We now return to the question posed in section four: can we prove any wff of O that was not already provable, when we build a $\mathcal{G}$-meta-language onto O? The answer to this question for the system $\mathcal{G}_1$ is clearly 'no', since we know that O (the 2-valued PC) is maximally consistent. Suppose, however, we chose for O a weaker system, say an n-valued logic, but retain a 2-valued logic for the proper axioms of $\mathcal{G}$? The answer is no longer obvious. We will prove the following:

Theorem 1 *Let O be any consistent system governed by modus ponens, and let the proper axioms of $\mathcal{G}$ be any subsystem (possibly not proper) of the 2-valued predicate calculus. Then if there is a proof of any wff p of O, there is a proof of p wholly in O.*

We note first that we cannot let the proper axioms of $\mathcal{G}$ be an arbitrary consistent system, since this would let in such things as the consistent system whose only axiom is $\vdash (p \supset p) \supset \sim q$. (There is no wff p, such that both $\vdash p$ and $\vdash \sim p$ in this system.) Hence we restrict our attention to subsystems of the 2-valued predicate calculus. This is no drastic loss of freedom, however, since all the systems we are normally interested in e.g., the intuitionist or many-valued propositional and predicate logics, are subsystems of the 2-valued predicate logic. Further, it would be very difficult to take seriously any system that had theses that were not theses of the 2-valued predicate logic, as an adequate formalization of the terms 'not', 'if . . . then', 'for all', etc. Also, it is fairly clear that the proof could be extended to any consistent first-order logic. Secondly, we note that to prove Theorem 1, it suffices to prove the result when $\mathcal{G}$ is the full 2-valued predicate logic. Theorem 1 then follows trivially, since if $\vdash_{O} p$ were available in some subsystem, it would certainly be available in the full system. We prove then, the result, when the proper axioms of $\mathcal{G}$ are those of the full 2-valued predicate calculus. The first step in the proof is to observe that what we are in fact trying to show is that 'Lp' can never be proved in $\mathcal{G}$ when 'p' is not provable in O alone. Now, it is clear that no wff of the form 'Lp' is provable in $\mathcal{G}$ without the rule **T** appearing somewhere in the proof. If this were not so, we could merely put '$\sim p$' for 'p' everywhere in the proof and obtain a proof of '$L \sim p$' and hence, by *M5*, and *modus ponens*, we have a proof of '$\sim Lp$', making $\mathcal{G}$ inconsistent.

This suggests treating the axioms of $\mathcal{G}$ as transformation rules on a certain input. Namely, those wffs of the form Lp which are obtained directly from O, by one application of the rule **T**. This in turn suggests a Gentzen-type system. What we do then is to construct a system of natural deduction at least as strong as the $\mathcal{G}$ meta-logic whose proper axioms are those of the 2-valued predicate logic.

Let γ be any set of rules adequate for a system of natural deduction equivalent to the 2-valued predicate calculus containing at least the cut rule,

$$\frac{A \vee B \qquad \sim B \vee C}{A \vee C},$$

the exchange rule,

$$\frac{A \vee B \vee C \vee D}{A \vee C \vee B \vee D},$$

and the dilution rule,

$$\frac{A}{A \vee D}$$

(see, e.g., [3], pp. 81-103, where a different, though essential equivalent formalization is used). We consider the system Γ of natural deduction, whose rules are those of γ, and the initial wffs (axioms) of whose proof trees are (i) wffs of the form Lp where $\vdash_O p$, or (ii) wffs of the form $\sim Lp$ where not $\vdash_O p$ and no others.

To facilitate the proof, we prove two lemmas:

Lemma 1 *If* $\vdash_{\mathcal{G}} p$ *then* $\vdash_\Gamma P$, *and hence, if not* $\vdash_\Gamma p$, *then not* $\vdash_{\mathcal{G}} P$.

Proof: If P is a thesis of $\mathcal{G}$, obtained by a direct application of the rule **T**, from O, then P is an axiom (i.e., possible initial formula) of Γ.

The axiom schemes *M1-3, 6, 7*, of $\mathcal{G}$, are theses of Γ by hypothesis on γ.

The schemes *M4* and *M5* are theses of Γ.

The proof is as follows: Either Lp or $\sim Lp$ is an axiom. If $\sim Lp$ is, then $\vdash_\Gamma \sim Lp \vee \sim L \sim p$ by dilution. If Lp is, then $\vdash_O p$, and since O is consistent, $\sim p$ is not provable, and hence $\sim L \sim p$ is an axiom. Thus again we have $\vdash_\Gamma \sim Lp \vee \sim L \sim p$ by dilution and exchange. Hence, in either case $\sim Lp \vee \sim L \sim p$ is provable, and this is clearly the equivalent in Γ of *M5* (in disjunctive form) in $\mathcal{G}$. Similarly either $L(p \supset q)$ or $\sim L(p \supset q)$ is an axiom of Γ. If $\sim L(p \supset q)$ is, then $\vdash_{\Gamma} \sim L(p \supset q) \vee \sim Lp \vee Lq$ by dilution. If $L(p \supset q)$ is, then either Lp or $\sim Lp$ is an axiom. Suppose $\sim Lp$ is, then $\vdash_{\Gamma} \sim L(p \supset q) \vee \sim Lp \vee Lq$ by dilution and exchange. Otherwise Lp is an axiom. Hence by construction, we have that $\vdash_{\mathcal{O}} p \supset q$ and $\vdash_{\mathcal{O}} p$, and thus, by *modus ponens*, $\vdash_{\mathcal{O}} q$. Therefore, Lq is an axiom of Γ, and $\vdash_{\Gamma} \sim L(p \supset q) \vee \sim Lp \vee Lq$, by dilution and exchange. Hence, whatever the case, $\sim L(p \supset q) \vee \sim Lp \vee Lq$ is provable, and this is the equivalent in Γ of *M4* (in disjunctive form) in $\mathcal{G}$.

Finally, an application of *modus ponens* in $\mathcal{G}$ is truth-preserving between the equivalent formulas in Γ because of the cut rule. Hence we have established that: $\vdash_{\mathcal{G}} P \Longrightarrow \vdash_{\Gamma} P$.

We now prove:

Lemma 2 *If* $\vdash_{\Gamma} Lp$ *then* Lp *is an axiom,* i.e., $\vdash_{\mathcal{O}} p$.

Proof: We know that we can choose γ, such that all its rules have conclusions of degree higher than (i.e., more connectives and quantifiers than) their premises, except for the cut, dilution, and exchange rules. Hence, if Lp is a thesis of Γ, it is either an axiom, or it is the conclusion of a cut. We know, however, that if there is a proof of P in Γ, then there is a cut free proof. (This is Gentzen's Haupstatz.) It follows that if $\vdash_{\Gamma} Lp$ then Lp is an axiom, i.e., $\vdash_{\mathcal{O}} p$.

Theorem 1 is a trivial consequence of Lemma 1 and Lemma 2.

A Note on Constructivity The above proof is, of course, not constructive, but a constructive proof is available. We construct the system of natural deduction Γ^0 whose rules are those of γ, plus:

M4^0
$$\frac{A \vee L(p \supset q) \qquad Lp \vee B}{A \vee Lq \vee B}$$

and

M5^0
$$\frac{A \vee L \sim p}{A \vee \sim Lp}.$$

We allow as axioms of Γ^0

(i) any wff of the form $A \vee \sim A$,
(ii) any wff Lp where $\vdash_{\mathcal{O}} p$,

and no others.

It is quite straightforward to prove that:

a) $\vdash_{\Gamma^0} P \Longleftrightarrow \vdash_{\mathcal{G}} P$.
b) *If* $\vdash_{\Gamma^0} Lp$ *then* Lp *is an axiom.*

The proof of (b), however, requires that the cut theorem for Γ^0 be established. This, though not a complicated matter, is relatively lengthy, and hence we chose to give the nonconstructive proof.

8 With this result, which is a guarantee that we have not been doing anything illegitimate syntactically in the previous six sections, we conclude this paper. I am indebted to Sue Haack for a number of helpful comments on earlier draughts of this paper.

REFERENCES

[1] Tarski, A., "The concept of truth in formalized languages," in *Logic, Semantics, Metamathematics*, The Clarendon Press, Oxford (1956), pp. 152-278.

[2] Hughes, G. E., and M. J. Cresswell, *An Introduction to Modal Logic*, Methuen and Co., London (1968).

[3] Gentzen, G., *The Collected Papers of Gerhard Gentzen*, ed. by M. E. Szabo, North-Holland, Amsterdam-London (1969).

Bedford College
London, England