A Bedside Readers Guide to the Conventionalist Philosophy of Mathematics

by Graham Priest

0. The following paper falls into several loosely connected parts, which, however, represent a continuous train of thought. Part one is an analysis of the problems with which the philosophy of mathematic concerns itself; we consider it worthwhile to go back to first principles occasionally since in a subject as old and involved as this it is only too easy to lose the wood for the trees.

Parts two, three, four and five are a review of the most influential views held on the problems in the last century, in the light of our analysis. We do not follow arguments into gruesome detail, and these sections are intended to be a catalyst and touchstone for the views of the reader as much as an establishment of the final conclusions.

Section five, although part of the above review, lays the foundation for the most important section of the paper, section six, which is an analysis of the meaning for conventionalism of Gödel's incompleteness theorem.

1. All the problems concerning the philosophy of mathematics can be neatly summarised by the question:

Question 0   What is (pure) mathematics?

The question of course hides a multitude of sins and we must be much more precise.

Firstly, what is meant by 'mathematics'? The only answer we can give without begging the question is 'That which is done and has been done for the last four thousand years by mathematicians.' This answer will be more helpful than it looks. (Incidentally, if the above question came from a particular person, the best, and in a sense, only answer, would be to teach him or her mathematics. However, this would not sell many philosophy books.)

Knowledge of the nature of mathematics lies in an ability to do it. Which is why it has often been the case that some good has come out of mathematicians reflecting on the nature of mathematics, whilst this can rarely be said of any philosopher, who was not also a mathematician.

Now, what is it that mathematicians do? They are interested in establishing the truth or otherwise of certain statements. If we then had a complete answer to the following question, we would be in a position to form an answer to question zero.

Question 1

Why are the truths of mathematics true?

Any reasonable answer to the above question must also permit reasonable answers to the following questions.

Question 1 (a)

Why is it that such truths appear necessary and inviolable, and why are we unable to conceive them being false?

Question 1 (b)

How is it that we come to know such truths?

Question 1 (c)

Why is it that truths of mathematics can be applied in practical matters e.g. surveying, building bridges, sending rockets to the moon, etc. In short, why are they useful?

These are epistemological considerations which must clearly be answered. Yet it is surprising that most philosophies of mathematics fail to satisfy one or other of these. One pulls mathematics towards the transcendent, and the others to the immanent; almost, it seems, incompatibly so.

Now, how does a mathematician establish these truths? A small number of truths are basic. Their truth is axiomatic in both senses of the word. Any others have to be proved. That is, they have to follow logically from these basic truths (we use the word 'logically' in a loose, naive sense). That is, the sentence which states that the truth is implied by (a finite subset of) the axioms must be a logical truth.

Any adequate philosophy of mathematics must hence answer question 1, 1(a), 1(b), and 1(c) with respect to logical truths as well as purely mathematical truths.

This is a simple point, but again it is surprising that many of the philosophies of mathematics are found wanting here. From now on, when we speak of mathematical truths we include logical truths.

Now the naive answer to question one is that mathematical truths are so because they are true of certain objects such as numbers, functions; propositions, points, groups, models etc., i.e. these are what mathematics is about.

We must hence be able to answer:

Question 2

What exactly are the above objects, and in what sense do they exist?

Further, the impression we get that these objects do exist objectively and we do actually study them is so strong that question two must continue:

Question 2 cont'd.

And if they don't exist, why is it that we have such a strong impression that they do?

Question two, although subsidiary to question one, has been the main battle-ground for philosophers considering the subject.

We see then that questions one and two are the hard core of the problem. Now we have got this straight, let us look at some answers.

2. The first two philosophies we consider attack the problem via question two.

Platonism

We consider platonism first since it is prior to the others historically and also since it is closest to what a mathematician thinks he is doing when he is actually doing mathematics. It is also the most prevalent view amongst mathematicians past and present, and is usually accepted by them unquestioningly or uncritically. (In fact, there is a great tendency amongst present mathematicians to think of everything that is not mathematics as unworthy of serious attention by their superior minds and to become mathematical ostriches. But I digress.)

An outstanding exception to the above, however, is Gödel, and for a lucid and cogent account of platonism, see (1).

Now we have noted that when a mathematician is actually doing mathematics, he feels he is studying external, objective objects and discovering the relationships between them. This is precisely what platonism advocates, and this is its answer to question two.

Its answer to question one is that mathematical truths are true because they state correctly the relationships between these objects.

Now, apart from the fact that the exact nature of the existence of these metaphysical objects is very obscure, it is also the case that platonism is unsatisfactory with respect to question 1(a) and totally inadequate with respect to 1(b).

1(a) According to platonism "1 + 1 = 2" states a relationship between the objects one and two (and perhaps plus, also). It is, however, a de facto relationship in the external world. From whence therefore does it derive its inviolability?

1(b) Platonism advocates that we know of these truths by our 'mathematical intuition'. Apart from Plato (whose account it is now very difficult to take seriously) no one has attempted to explain exactly what this is; why it is wrong sometimes, right others; and how it works physiologically.

If we have got this sixth sense, then it should be possible to examine it scientifically. Yet this has never been suggested, let alone attempted, because it is absurd.

Further, if we accept 'mathematical intuition' we must also accept 'ethical intuition' which perceives justice, goodness etc., 'sociological intuition' which perceives societies, economies etc., and one for every other sort of universal we know.

Platonism must be rejected as an inadequate philosophy of mathematics.

Constructivism

We next consider intuitionism and other forms of constructivism between which, for our purposes, we need not distinguish. For an exposition of intuitionism see (2).

One of the basic problems of platonism was, we saw, how we come to know the truth about mathematical objects. For constructivism this is no problem, for mathematical objects are 'mental entities' which, in some sense, we construct ourselves. Constructivism then has no problem with question 1(b). The exact nature of these mental entities is a moot point and we will discuss question two no further.

The really awkward question for constructivism is, however, 1(a). If mathematical objects are subjective mental constructions, from whence derives their air of stability and objectivity? Why even do we all appear to effect the same construction? Very strange!

Further, if they really are my own constructions why can I not construct 1 + 1 to be equal to 3 if I want to; it would seem that I have less power over my conscious actions than I have over my reflexes.

But there is a much more serious objection yet; so far we have been playing the constructivist on his home ground.

It is well known that it is impossible to secure all of classical analysis, let alone mathematics, constructively, and so the constructivis fails to answer the question, since our definition of mathematics was 'that which is done by mathematicians'.

If a constructivist wants to call constructive mathematics 'true mathematics' then we won't quibble about words. But what then is it that a 'classical' mathematician is doing?

The situation is not symmetrical. When asked what it is that a constructivist is doing, a classical mathematician can reply that he is studying that part of mathematics that is obtainable constructively

Since mathematics goes beyond the bounds of constructive mathematics, however, a constructivist has no answer to a similar question about a classical mathematician.

We must dismiss constructivism as an inadequate philosophy of mathematics.

3. Logicism

It is clear to any working mathematician that he studies two sorts of systems (we leave the word deliberately vague and exclude logic for the moment).

Those such as number theory, set theory and Euclidean geometry were each originally assumed to possess a unique subject matter, viz: numbers, sets, lines and points. These systems were supposed to be categorical.

Others such as topology, group theory, functional analysis, Riemannian geometry, were never expected to be categorical. For example, group theory is the study of what all groups have in common. That is, what follows from the definition of what a group is.

In general, studies of the latter sort are studies of what follows from a basic set of conditions, which are usually generalisatic or slight modifications of conditions occurring naturally in analysis or geometry.

Hence in the former subjects, one need never know a list of axioms (e.g. number theory was done for many thousand years before Peano) which are essentially a post-hoc characterization or attempted characterization of what is already going on.

Whereas in the latter, one learns the axioms right at the start of the subject (e.g. the first chapter of any introduction to topology is a specification of the conditions that a structure must satisfy in order to be a topological space) and they are a pre-hoc characterization of what one is studying.

The latter sort of system is simpler philosophically. We saw that we could reduce the problem of mathematical truth to the pair of problems, (i) why are the basic truths true, and (ii) why is logic true? For the latter systems, the basic truths are true by our own fiat. The axioms of ring theory are true because that is what we define a 'ring' to be. Hence the problem reduces to the nature of logical truths.

This approach will not, however, work for systems of the former sort, where the axioms are not true-by-definition. Indeed, there is not even a complete axiom system for two of the former systems, nor a categorical system for any of them.

What has this to do with logicism?

Logicism is strictly out of place in this review since it is not a philosophy of mathematics, but a reduction program. For an exposition of logicism see e.g. (3).

Had logicism worked (and it was a very near miss), it would have proved that all mathematical truths are in fact truths of pure logic.

This is true, as we saw, for systems of the latter kind above. Logicism, if correct, would have shown this true for systems of the former sort also.

It is accepted now, however, that logicism failed: apart from any considerations of incompleteness, Russell had to import two extra-logical axioms, choice and infinity. Even if it had succeeded however, it would still have been left with the awkward problem of why logical truths are true.

Further, logicism is highly unsatisfactory with respect to question two. Frege assumed the existence of sets and Russell propositional functions, both of which are highly problematical abstrac objects.

Hence, logicism cannot provide a foundation for mathematics.

4. We next look at two philosophies that approach the problem via question one.

Empiricism

We give this name to the line of thought advocated by Mill and his followers, see e.g. (4). The starting point here is question 1(b) How do we know mathematical truths are true?

According to this school, all mathematical statements are empirical but confirmed to a very high degree. For example, when we see one object and one object put together we usually see two objects. We hence conclude that 1 + 1 = 2 in the way we conclude any other 'law of nature'.

Empiricism falls down very badly however on questions 1(a) and 2.

1(a) For if empiricism were correct, there ought to be some circumstances which would make us inclined to deny that 1 + 1 = 2. I can think of no such circumstances however. For example, if whenever we put one object next to another we always found that there was one object, we would not conclude that 1 + 1 = 1; we would reformulate our scientific laws to allow for some sort of fusion of objects.

So empiricism does not explain the apparent necessity of mathematical statements. This becomes very clear when we consider the truths of logic. It is not the case that '(p and q) implies p' is true, solely because whenever we can assert p and can assert q, we usually find that we can assert p.

2. Secondly, empiricism can only possibly explain why Δ_0 sentences are true (i.e. sentences with only predicates, constants, connectives and bounded quantifiers - in arithmetic, these are precisely the statements whose truth or falsity can be effectively established. If we have a true sentence of the form $\exists x\phi(x)$ and there is no effective way of finding such an x, then empiricism offers no explanation at all as to why this is true or even what it means.

Since the nature of the existence of mathematical objects is reflected in the meaning of the quantifiers, empiricism has no answer to question two.

Hence we must reject empiricism as an inadequate philosophy.

Formalism

We next consider formalism. This like constructivism covers a wide range of views, but here we will have to differentiate.

a) Hilbert's original formalism was not so much a philosophy of mathematics as a program for ensuring that mathematics never again received rude shocks such as Russell's paradox. See e.g. (12). However, there is the germ of a philosophical theory in the way he intended to carry this out.

He considered mathematics to be of two sorts, finitary and ideal. The distinction is well worn and needs no elaboration except to remark that forty years later we still do not know exactly where the boundary is, or indeed if there is one.

The rationale of finitary mathematics is simple. A mathematician can do what he likes provided it is consistent, the final arbiter being utility (in pure and applied mathematics) and intrinsic beauty.

Finitary reasoning on the other hand is very far from arbitrary, as can be seen from the fact that Hilbert insisted that all methods in metamathematics should be finitary, since these were the only ones that inspire absolute certainty.

But here Hilbert fails to go further. Why does finitary mathematics inspire such certainty? Why in short are the truths of finitary mathematics true and moreover, obviously true? Clearly this is just problem one.

So even if one accepts the distinction between finitary and ideal mathematics and accepts that a mathematician can do what he likes and invent whatever objects and realms of objects he likes subje only to consistency (and these are very big if's), one still has our original problems with a smaller more basic part of mathematics.

If one could guess what Hilbert thought about these problems (and indeed what has been advocated by others trying to justify finity reasoning) one would probably arrive at one of the following:

either i) All finitary statements are Δ_0 and hence we can look to empiricism for an answer.

So nine plus four is thirteen, because if I put down nine matches and next to them another four, I can count and see that I have thirteen. Now in view of the probable finiteness of the universe, even if the Albert Hall had been full of monkeys making matches since the beginning of time, there would still be but a finite number of matches in the universe. The identification of numbers with concrete objects will not, therefore, work.

Further we have already seen that empiricism can give no adequate answer to question 1(a), the necessity of mathematical truths.

or ii) One can take a more intuitionist line here since constructivist techniques generate all of finitary mathematics. We can say that numbers are mental constructions and that 1 + 1 = 2 is so because I have constructed it to be so.

Now we have already seen that this point of view does not explain the objectivity let alone the necessity of mathematical truths and so fails on question 1(a).

However, we will point out that this view fails on account of question 1(c) also. Why is it that the objects that I construct seem to tell me something about the external world?

For example, I know that if I take a matchstick out of my left hand pocket and another out of my right hand pocket and put them on the table in front of me, then failing some political nuclear insanity or other act of God, I shall see two matchsticks there; if "1 + 1 = 2" is true by my own fiat however, it is not clear why.

Hilbert's philosophy must rate as another near miss, but until a good justification for finitary reasoning can be found, it must remain inadequate.

b) There was a type of formalist philosophy in the early days of formalism which said that e.g. arithmetic is nothing but the study of some particular formal system of arithmetic and the truth is just provability in this system.

It is difficult to see how this could ever have been taken seriously since there are so many objections to it. We will list four:

1) Formalization is a post-hoc characterization of arithmetic.

2) This denies the existence of mathematical objects but does not explain why we have such a strong impression that they do exist. It does not answer therefore, question two.

3) This sort of theory cannot explain why logical truths are true since in any formal system of logic, logic is necessary to derive the theorems from the axioms.

4) This position is untenable in view of Gödel's proof that no formal system of arithmetic can capture all the true statements of arithmetic.

An extension of this philosophy to take account of 4) above has been proposed by Curry, see e.g. (6); he advocates that mathematics is the study of all formal systems.

Now we will not discuss whether this philosophy can give a satisfactory answer to question 1(c), i.e. why it is that some formal systems are so useful, or 1(b), why some of these systems force themselves on us so strongly; but we will point out that this philosophy falls foul of points 1, 2 and 3 above.

Condition 2 is fairly obvious. We will elaborate 1 and 3.

1) Axiomatization and formalization are things which happened to number theory and set theory a long time after they were first studied. They are not therefore basic to the subject. Even after a theory has been formalized, this makes no difference to the practising mathematician, e.g. the number theorist whose subject of study is numbers, not formal arithmetic.

As Curry himself says in (13): "Acceptability [of a formal system] is usually a matter of interpreting the theory in relation to some subject matter ... In an interpretation we associate them [predicates] with certain intuitive notions."

Arithmetic is precisely the study of the intuitive notions of number, addition etc., and not of formal systems of arithmetic.

Look at it another way. If mathematics is the study of formal systems, what sort of study is it? What sort of tools are we allowed to use? It is the mathematical study of formal systems. We come full circle.

3) The 'truths' of a formal system are those which are derivable in it, i.e. are logical consequences of the axioms. But what of logic?

According to a formalist, logical truths are those things provable in some formal system of logic, but this is impossible. For to follow Quine, if logic is to follow mediately from axioms, then logic is itself needed to infer logic from the axioms. In other words, a person cannot know what logic is by deriving it in some formal system, since this would require prior knowledge of logic.

For a fuller account see (5).

We must reject this view of the foundations of mathematics as not only inadequate but as putting the cart before the horse.

5. Finally in this review we consider conventionalism.

Conventionalism

Conventionalism is well in keeping with the general trend of 20th century philosophy.

It has at last been realised that the language we speak is not merely a passive vehicle in the communication of our thoughts, but is an active agent in determining how we can and do think. What cannot be said cannot be thought; there is no such thing as a concept without some description of it in the language. (See e.g. the appendix to (10).)

If it is false to say that rational thinking is precisely a subconscious manipulation of the thinker's native tongue, then the truth is certainly much nearer to this extreme than to the classical conception of the nature of thought. And this explains why logic, the theory of the structure of our language, was for so long conceived of as the laws of thought.

For an exposition of conventionalism we turn to the later Carnap, especially (9), Ayer (14) and an intelligent interpretation of the later Wittgenstein, e.g. some of the better parts of (8).

Now each of us has a native language which we use to talk and think, and the following is a summary of the aspects of natural language which are important for conventionalism.

Every language is governed by syntactic and semantic regularities or rules which are in the nature of conventions (see (10) for a good analysis of the nature of implicit convention) and which give the language both a syntactic and a semantic structure. With each word of the language we can associate a reasonably precise usage or meaning and words can be combined into sentences which, when uttered under certain circumstances may be correct (true) or not.

Conventionalism asserts that a truth of mathematics* or logic is true in virtue of the meanings of its constituent words. That is, the semantic rules which govern the use of its component words and the manner in which they are put together, ensure that it is always correct to assert the sentence. More picturesquely, the sentences of mathematics and logic are true because they picture the semantic structure of the language.

Or, if one doubts the meaningfulness of the above explanation (say if one shares the doubts of Quine in (11)), we can put it this way. The statements of logic and the basic statements of mathematics are true because we are determined that they are to be true (i.e. it

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* By mathematics we here mean arithmetic, set theory and Euclidean geometry. We saw in section 3 that all other branches of mathematics could be reduced to logic.

is always correct to assert them) and to remain so, come what may; there is no evidence that we would allow to count against them.

This is the answer of conventionalism to question one. The answers to questions 1(a), (b) and (c), follow.

1(a) Mathematical truths are necessarily true because the semantic rules of the language alone are sufficient to ensure their truth. Under no possible physical circumstances could they therefore be false.

To put it another way, they are necessarily true (i.e. could not possibly be false) because we will never allow them to be false. We cannot conceive of them being false since, as there are no circumstances we could count as falsifying them, there are none we can conceive of.

It is often said that such truths are true by convention, i.e. the conventions we have governing the use of words. This is a very misleading way to put it since this could in fact be said about any true sentence.

Further, it has led to the objection that conventionalism cannot show mathematical truths to be necessary since we could well have had other conventions. This objection misses the point however. Mathematical truths are fixed in truth value for exactly the same reason that a frame of reference in mechanics or co-ordinate geometry is fixed in position.

1(b) We come to know the truths of logic and basic mathematics because they are part of our native tongue: we learn which sentences of logic, arithmetic, set theory and geometry are true when we learn to speak and to use the words 'and', 'not', 'two', 'collection', 'point', etc. correctly, i.e. as everybody else uses them.

Now at this point it is worth noting that with a few moot cases, whatever natural language one learns first, one learns the same mathematics and logic. That is, all natural languages seem to have the 'same' (i.e. isomorphic) logico-mathematical parts. (But this is not all they have in common of course.) This may point to the fact as Chomsky asserts that we all have an innate tendency (possibly physiological) to speak certain sorts of language. Or it might indicate that all the world languages share a common root. However, this is not relevant to the present discussion.

1(c) Why is it that mathematics has practical application?

Natural language has developed by linguistic evolution† where the criterion for the retention of something new, as in animal evolution, is its usefulness. It is not therefore surprising that

† And is of course still evolving. Language is not static and fixed but is in a constant state of flux. The same can be said for that part of language which is mathematics; it also is constantly developing.

mathematics being part contained in natural language, part an extension of it, partakes of this usefulness.

But more precisely, we have seen that logico-mathematic truths are true independent of any non-linguistic considerations. In a sense, therefore, they have no factual content. How can they therefor be useful?

The answer is this. Given any correct factual description of a situation, by using the logico-mathematical truths (which embody the semantic rules of the language) we can establish other correct factual descriptions. These may not be logically new but will probably be psychologically new and therefore give us new information about the situation.

However, there is more to it than this. Everything we have so far said about conventionalism is true of the whole analytic part of language, not just the mathematical part. So why mathematics?

The answer to this lies in the origins of the subject. Mathemati as practised by the Babylonians and Egyptians was a practical subject, concerned with measuring fields, building pyramids, counting herds of animals, etc. So mathematics was that part of language whose function was to describe practical matters. Mathematics therefore had practical applications for the same reason that the theory of music has musical applications.

Some of modern mathematics has inherited this function. One only has to look to see that the most useful parts of mathematics are those which have developed from the oldest parts of mathematics viz. arithmetic and geometry. The newer bits like the theory of transfinite numbers are relatively useless.

2) We must now consider how conventionalism deals with question two: the existence of mathematical objects.

Questions of the existence of mathematical objects must be divided into two sorts. Those which are answerable within the framework of our language and those which are not.

For example, we can ask whether there are prime numbers greater than 100. We have well defined procedures for determining whether there are or not. Or we can ask whether there are any numbers (as we can ask whether there are any unicorns) and the answer is yes since e.g. seventeen is a number.

We cannot transcend the language however, and try to talk about 'reality'. To ask whether numbers really do exist, in the metaphysical sense is a meaningless question to which no clear sense can be attributed.

We must then ask why it is that mathematical objects 'seem to exist in reality', i.e. why platonism is the naive philosophy of mathematics.

This is so for the same reason that we naively believe that physical objects 'really exist'. Namely, because they are part of the structure that is imposed on the world by the form of language we use, which governs the way we think.

Perhaps a well worn analogy will help to clarify matters. In some respects, the language 'game' is rather like a game of chess.

Now when we play chess, all we do is move wooden pieces on a wooden board. As any chess player knows, however, this is not what he 'sees' when he looks at a board during a game. The symbols of the game become more than just pieces of wood. Pieces take on their moves, so that e.g. a bishop becomes a pair of diagonals across the board. The whole thing takes on a separate existence as a complex three dimensional lattice (the third dimension being the order of moves) of which certain paths are highlighted as possible strategies.

So it is with language. Language is, in fact, only a series of physical occurrences, made and observed. But to the user these signs are more than this. The words and sentences take on their meanings, and this is why to a mathematician, what he is doing is somewhat more than merely manipulating words.

The difference is that between a person who speaks Spanish and a person who merely reads sentences out of a phrase book. They may both utter exactly the same sounds but the mental processes involved are somewhat different.

We see then that conventionalism is the only philosophy we have considered which addresses all the problems of the philosophy of mathematics squarely. It is therefore the only adequate philosophy.

6. In the last part of the paper I will consider the consequences for conventionalism of Gödel's incompleteness theorem.

Now platonism as a philosophy of mathematics has been much weakened recently by the discovery of various incompleteness results for axiomatic arithmetic and set theory.

From a conventionalist point of view however, there is no reason why mathematics should be decidable, categorical, syntactically complete or even that it should be able to prove its own consistency: but it is often said in an attempt to refute conventionalism that in any language sufficient for a large part of arithmetic, there are true statements which are not provable.

Now, if as conventionalism asserts, a mathematical truth is one that is assertable (provable) in our ordinary language, this could not be the case.

As it stands, this objection is fallacious, but we will consider it to see if it can be made to stick.

A) Firstly, Gödel's first incompleteness theorem holds only for a consistent, axiomatic, formal theory, and any natural language is very far from being this.

It seems reasonable, however, to suppose that a natural language (for the sake of definiteness let us take English as it is spoken in London in 1972) could at least in principle be formalized. That is, its grammar regularized, ambiguities eliminated, etc. But it is not at all clear that it, or even the logico-mathematical part we are interested in, could be axiomatized.

And, if conventionalism is correct and if all else in Gödel's theorem holds, we must indeed be in the situation where it cannot possibly be axiomatic.

This would mean that somehow in learning our native tongue we learn to assert a non recursively enumerable set of sentences. This does not seem very likely, but if it were true it would be a very significant result, since assuming Church's thesis, it would show that language learning is non-effective.

B) We can go another way however. Let us assume that it is possible to make English into a consistent axiomatic formal system. The conditions for Gödel's theorem are then satisfied. It follows that there is a sentence ϕ such that neither it nor its negation is provable. Futhermore it is also claimed that ϕ is true.

There are many different proofs of this theorem and many different ϕ's. Some proofs start by assuming that the true and false sentences form an exclusive and exhaustive covering of the set of sentences of arithmetic. In other words, it starts by assuming that either ϕ or $\neg \phi$ is true. This is not a very convincing argument for the truth of ϕ.

Let us, however, consider Gödel's original proof. He constructs an unprovable sentence ϕ and the argument that it is true usually goes something like this:

"Look at ϕ. ϕ asserts in effect (wave of hands) its own unprovability. Since it is not provable, then what it says is true."

The above argument is usually carried out in as matter-of-fact a way as possible to try to cover up the non-rigorousness of the argument.

The weak point of the argument is of course the "in effect". How exactly does a purely arithmetic statement manage to make a non-arithmetical statement? If this were the only hope of proving the truth of ϕ then we would be on very shaky grounds. Fortunately, however, there is a much better way.

ϕ is of the form $\forall x\ \psi(x)$ and we can prove that for every numeral n, $\psi(n)$ is provable. Hence for every member of the standard model of arithmetic ψ is true. By the definition of truth, $\forall x\ \psi(x)$ is therefore true in the standard model, i.e. ϕ is true.

This is, I think the best that can be done towards proving the truth of ϕ.

What are conventionalists to make, however, of the appeal to the concept of 'the standard model'?

We can make sense of talk of numbers inside our language where it is an internal question, but as we saw, all talk of 'actual numbers', the 'standard model' as an external question is meaningless. Hence we may reject this proof too as fallacious. (We note in passing that exactly the same argument is sometimes offered to prove that if Fermat's last theorem is independent of Peano Arithmetic then it is true.)

C) The truth is, I think, less straight forward than either of these possibilities.

Let us suppose that we do accept the proof of the truth of ϕ given in B). How exactly did we manage to prove it? What formal machinery would we require to formalize the proof? (Always a good guide to the nature of the proof.)

We would need a very weak set theoretic metalanguage for arithmetic. In fact the whole proof can be formalized in a metalanguage based on $\mathcal{Z}$ (Zermelo set theory).

We see then that the argument must be done in a metalanguage; it is impossible to do it in the system itself.

Now English and all other natural languages contain their own metalanguage (i.e. are semantically closed) and it is precisely for this reason that we can prove in English that ϕ is true, i.e. why ϕ is assertable in English.

Now the fact that natural languages contain their own metalanguage has further consequences. The semantic paradoxes (and I think we can show the set theoretic paradoxes also) occur in English precisely because it is semantically closed.

Any reasonably large language that contains its own metalanguage and which can talk about its own truth, denotation, definability, etc., will contain semantic paradoxes.

The paradoxes of a) Epimenides and b) Berry are well enough known not to require further comment.

a) This sentence is false.

b) The least number not definable in less than nineteen syllables is definable in eighteen syllables.

We note that the idea behind a), b) and Gödel's undecidable sentence c) are very similar.

c) This sentence is not provable (- in effect!)

The important point however is that the semantic paradoxes occur in English for exactly the same reason that the truth of ϕ is assertable in English, viz. English is semantically closed.

Look at it another way. We are assuming that English can be turned into a formal axiomatic system and that the truth of its Gödel sentence ϕ can be proved in English. Hence ϕ is assertable (i.e. provable) in English.

Gödel's theorem states that any such system can prove its own Gödel sentence if and only if it is inconsistent.

It follows that English is inconsistent. However, we have been well aware of this fact for a long time: a) and b) are both examples of sentences which are both provable and refutable in English.

It is clear then that any language for arithmetic without paradoxes will have an unprovable statement that is true (i.e. provable in a sufficient metalanguage). Conversely any such system that can prove all its true statements will have paradoxes. Peano arithmetic is an example of the first sort of system. English is an example of the second.

Now there will never be any question of mathematics actually being done in a formal system, but we shall always wish to formalize our mathematics since it improves our understanding of what we are doing.

What then is the best way to formalize natural language mathematics?

If the formal system is to be consistent then there is no questio of it being semantically closed. So perhaps the best we can do is use not one formal system but a hierarchy M_γ of formal systems M_α, $\alpha < \gamma$, each of which is a metalanguage for the language below it. For example M_0 could be Peano Arithmetic, $M_{\alpha+1}$ a metalanguage for M_α and $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$ for limit ordinals λ.

The system would not be properly semantically closed, but we should be able to talk about any sentence of M_λ in M_λ for a limit ordinal λ.

For small α, each $M_{\alpha+1}$ would contain true arithmetic sentences not provable in M_α by Gödel's theorem. But since there are only a countable number of sentences of M_0 this must cease to hold by M_{ω_1}. In fact we can no longer apply Gödel's theorem to M_β where β is constructive or recursive ω_1, since M_β is not recursively enumerable, although each M_α is for all $\alpha < \beta$.

We could hence regard M_β as an approximation to natural language.

Why, however, should we insist that a formalization should be consistent? What is wrong with inconsistency?

Now this brings us to question an assumption we have made tacitly till now. We have assumed that our formal system must be based on a classical or intuitionist type logic, and if such a system is inconsistent then everything is provable in it. Hence the system is useless.

But English is not this sort of system however. English is inconsistent - witness any of the paradoxes, and yet we do not assert everything in English.

It is not, therefore, a system in which a contradiction implies anything. Consider the following argument:

The least number not definable in less than 19 syllables is not definable in less than 19 syllables. (Call this ψ.)

Than ψ or It is not raining. (1)

But the least number not definable in less than 19 syllables is definable in 18 syllables. (We have just defined it thus).

Hence Not - ψ. (2)

So by (1) and (2) It is not raining.

The above argument is classically and insuitionistically valid but would never be accepted by anyone waiting for a bus in the rain.

We see then that English must be formalized in such a way that a contradiction does not imply everything.

However, for this to happen one of the following rules of inference must be invalid:

$$\frac{\phi \wedge \psi}{\phi} \qquad \frac{\phi \wedge \psi}{\psi} \qquad \frac{\phi}{\phi \vee \psi} \qquad \frac{\phi : \ \neg\phi \vee \psi}{\psi}$$

and it is not at all a simple matter to say which and why, and to formalize such a logic.

When this has been done we can have systems of limited inconsiste (i.e. in which some but not all contradictions are provable). For this is precisely what English is: a system of limited inconsistency containing its own metalanguage.

Until this sort of logic has been formalized, however, we will have to be content with a hierarchy of first order metalanguages as the nearest consistent approximation we can get and accept the consequences.

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This also contains extracts from (2), (3) and (7). It gives the original source for the items from it listed above.