Natural Deduction Systems for Logics in the FDE Family

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Abstract

There is a small family of many-valued logics associated with the logic of First Degree Entailment. These may be called the FDE family. The purpose of the present paper is to provide natural deduction systems for these logics. This can be done in a quite systematic fashion. An appendix to the paper deals with a closely related system which is not in the family, "Paraconsistent Weak Kleene".

1 Introduction

There is a small family of many-valued logics associated with the logic of First Degree Entailment, FDE. Elsewhere,[1] I have called them the FDE family. The purpose of the present paper is to provide natural deduction systems for these logics. This can be done, as we shall see, in a quite systematic way. I will start by defining the family of logics. I will then provide a natural deduction system for the weakest of the logics in the family, $FDEe$. Soundness and Completeness will be proved. It will then shown how to modify matters to accommodate the stronger systems in the family. An appendix deals with a closely related system that is not in the family, "Paraconsistent Weak Kleene", PWK.

[1] Priest (2014b).

2 The FDE Family

We will be dealing with propositional logics with connectives $\wedge$, $\vee$, and $\neg$. $\supset$ may be defined in the usual way. The logics in our family are all many-valued logics, and the weakest member, $FDEe$, has five values: t, f, b, n, e. The first four of these are naturally thought of as: *true only*, *false only*, *both true and false*, and *neither true nor false*. Their behaviour can be captured in the familiar Hasse diagram:

$$\begin{array}{ccccc} & & t & & \\ & \nearrow & & \nwarrow & \\ b & & & & n \\ & \nwarrow & & \nearrow & \\ & & f & & \end{array}$$

Negation maps t to f and vice versa; and b and n are fixed points for negation. Conjunction is the greatest lower bound; and disjunction is the least upper bound.

The fifth value has can be interpreted in various ways. One is as *meaningless* or defectiveness of some other kind.[2] Another is as ineffability.[3] The salient formal feature of e is that any truth function has e as an output iff some input is e.

In a many-valued logic of this kind, there is a set of designated values, D, and an inference is valid if any evaluation in which all the premises are designated, the conclusion is designated. In $FDEe$, $D = \{t, b\}$. So $\Sigma \models A$ iff for every evaluation: if $\nu(B) = t$ or b for all $B \in \Sigma$, then $\nu(A) = t$ or b.

The semantic conditions for the connectives can be depicted in a more usual form by the following matrices:

$\neg$	
t	f
b	b
n	n
f	t
e	e

$\vee$	t	b	n	f	e
t	t	t	t	t	e
b	t	b	t	b	e
n	t	t	n	n	e
f	t	b	n	f	e
e	e	e	e	e	e

$\wedge$	t	b	n	f	e
t	t	b	n	f	e
b	b	b	f	f	e
n	n	f	n	f	e
f	f	f	f	f	e
e	e	e	e	e	e

[2]See Goddard and Routley (1973), and the discussion of Bochvar logic in Haack (1996).
[3]See Priest (2014a).

If we write the value of A as $|A|$ then, as is not difficult to check, the tables can be expressed in a linear form as follows:

Negation

- $|\neg A| = e$ iff $|A| = e$
- $|\neg A| = t$ iff $|A| = f$
- $|\neg A| = b$ iff $|A| = b$
- $|\neg A| = n$ iff $|A| = n$
- $|\neg A| = f$ iff $|A| = t$

Disjunction

- $|A \vee B| = e$ iff $|A| = e$ or $|B| = e$
- $|A \vee B| = t$ iff ($|A| = t$ and $|B| \neq e$) or ($|B| = t$ and $|A| \neq e$) or ($|A| = b$ and $|B| = n$) or ($|A| = n$ and $|B| = b$)
- $|A \vee B| = b$ iff ($|A| = b$ and $|B| = b$) or ($|A| = b$ and $|B| = f$) or ($|A| = f$ and $|B| = b$)
- $|A \vee B| = n$ iff ($|A| = n$ and $|B| = n$) or ($|A| = n$ and $|B| = f$) or ($|A| = f$ and $|B| = n$)
- $|A \vee B| = f$ iff $|A| = f$ and $|B| = f$

Conjunction

- $|A \wedge B| = e$ iff $|A| = e$ or $|B| = e$
- $|A \wedge B| = t$ iff ($|A| = t$ and $|B| = t$)
- $|A \wedge B| = n$ iff ($|A| = n$ and $|B| = n$) or ($|A| = t$ and $|B| = n$) or ($|A| = n$ and $|B| = t$)
- $|A \wedge B| = b$ iff ($|A| = b$ and $|B| = b$) or ($|A| = b$ and $|B| = t$) or ($|A| = t$ and $|B| = b$)
- $|A \wedge B| = f$ iff ($|A| = f$ and $|B| \neq e$) or ($|B| = f$ and $|A| \neq e$) or ($|A| = b$ and $|B| = n$) or ($|A| = n$ and $|B| = f$)

Let us call t and f the *classical values*. Then note that deleting the row and column for any non-classical value in the matrices deletes all mention of that value. In other words, the matrices define perfectly good many-valued logics when the row and column for any number of the non-classical values are deleted—and if b is deleted, D is simply $\{t\}$. All the logics in our family have the classical values; and we may characterise the logics in terms of which *other* values they have, as follows:[4]

- $\emptyset$: classical logic, CL.
- e: Bochvar logic (also known as weak Kleene 3-valued logic), B_3.
- n: strong Kleene 3-valued logic, K_3.
- b: logic of paradox, LP.
- en: a logic not previously formulated.
- eb: AL.
- bn: first degree entailment, FDE.
- bne: $FDEe$.

Note, finally, and again as is easy to check, in any of these logics, the matrix for $A \wedge B$ is the same as that for $\neg(\neg A \vee \neg B)$, and the matrix for $A \vee B$ is the same as that for $\neg(\neg A \wedge \neg B)$. For theoretical purposes, we may therefore take one of these to be defined in terms of the other. In what follows, we take conjunction to be defined in terms of disjunction.

3 Natural Deduction Rules

The natural deduction systems for our family of logics will be given in the style of Prawitz (1965). A basic deduction is anything of the form: A. A is both an undischarged assumption and the conclusion. A deduction is anything that can be generated from basic deductions by applying rules

[4]For $FDEe$, see Priest (2014a), where it is called $FDE\varphi$. See also Daniels (1990). For FDE, see Priest (2008), ch. 8. For K_3 and LP, see Priest (2008), ch. 7. For B_3, see Haack (1996), pp. 169-70. (Bochvar augments this logic with further connectives). For AL see Oller (1999).

from the appropriate set. $\Sigma \vdash A$ iff there is a deduction whose undischarged assumptions are in Σ, and whose conclusion is A.

The rules for $FDEe$ are as follows. A bar over an assumption means that it is discharged by the rule. A double line means that the rule works in both directions. If A is any formula, we will write $A^\dagger$ for any formula which contains all the propositional parameters in A.

- $\wedge I$

$$\frac{A \quad B}{A \wedge B}$$

- $\wedge E$

$$\frac{A \wedge B}{A} \qquad\qquad \frac{A \wedge B}{B}$$

- $\vee E$

$$\frac{A \vee B \quad \begin{array}{c}\overline{A}\\ \vdots\\ C\end{array} \quad \begin{array}{c}\overline{B}\\ \vdots\\ C\end{array}}{C}$$

- Weak $\vee I$

$$\frac{A \quad B^\dagger}{A \vee B} \qquad\qquad \frac{A^\dagger \quad B}{A \vee B}$$

- DeM

$$\frac{\neg(A \vee B)}{\neg A \wedge \neg B} \qquad\qquad \frac{\neg(A \wedge B)}{\neg A \vee \neg B}$$

DN

$$\frac{A}{\neg\neg A}$$

In due course, and for future reference, four other rules will concern us:

- Excluded Middle (EM)

$$\frac{}{B \vee \neg B}$$

- Explosion

$$\frac{A \wedge \neg A}{B}$$

- $\vee$I

$$\frac{A}{A \vee B} \qquad \frac{B}{A \vee B}$$

- Weak EM

$$\frac{A^\dagger}{A \vee \neg A}$$

4 Soundness and Completeness for $FDEe$

4.1 Soundness for $FDEe$

In this section we show that in $FDEe$ (= *bne*), if $\Sigma \vdash A$ then $\Sigma \models A$.

The proof is by recursion over the construction of deductions. The basis case is trivial. There is a recursion case for each rule. These are straightforward, and mostly left to the reader. I do two casess as examples.

The first is $\vee E$. Suppose that we have proofs, π_1, π_2, and π_3, such that the undischarged assumptions and conclusions of the three deductions are, respectively:

- Π_1, $A \vee B$
- $\Pi_2 \cup \{A\}$, C
- $\Pi_3 \cup \{B\}$, C

The application of $\vee E$ gives us a proof whose undischarged assumptions are $\Pi_1 \cup \Pi_2 \cup \Pi_3$, and whose conclusion is C. Suppose that $\Sigma \supseteq \Pi_1 \cup \Pi_2 \cup \Pi_3$. Then by Recursion Hypothesis (RH), $\Sigma \models A \vee B$. Hence $\Sigma \models A$ or $\Sigma \models B$. By RH, $\Sigma \cup \{A\} \models C$ and $\Sigma \cup \{B\} \models C$. So in either case, $\Sigma \models C$, as required.

The second example is Weak $\vee I$. Suppose that we have a proofs, π_1 and π_2, whose undischarged assumptions and conclusions are, respectively:

- Π_1, A

- Π_2, $B^\dagger$

Suppose that $\Sigma \supseteq \Pi_1 \cup \Pi_2$. Then by RH, $\Sigma \models A$ and $\Sigma \models B^\dagger$. Let ν be any evaluation that designates all members of Σ (i.e., gives them either the value t or the value b). Then ν designates A and $B^\dagger$. Hence it assigns no propositional parameter in B the value e, and so it does not assign B the value e. Hence it designates $A \vee B$. That is, $\Sigma \models A \vee B$.

4.2 Henkin Lemma

In this section, in preparation for proving completeness, we prove the *Henkin Lemma*:

If $\Sigma \nvdash A$, there is a $\Pi \supseteq \Sigma$ such that:

- $\Pi \nvdash A$
- Π is deductively closed
- Π is prime (that is, if $A \vee B \in \Pi$, $A \in \Pi$ or $B \in \Pi$)

Proof: Enumerate the formulas of the language: $\langle B_i : i \in \omega \rangle$. Define by recursion:

- $\Pi_0 = \Sigma$
- $\Pi_{n+1} = \Pi_n \cup \{B_n\}$ if $\Pi_n \cup \{B_n\} \nvdash A$; $\Pi_{n+1} = \Pi_n$ otherwise
- $\Pi = \bigcup_{n \in \omega} \Pi_n$

Clearly, $\Pi \supseteq \Sigma$.

We now show that the conditions hold.

- $\Pi \nvdash A$. By induction on n, each Π_n is such that $\Pi_n \nvdash A$. By compactness, $\Pi \nvdash A$.
- Π is deductively closed. Suppose for *reductio* that $\Pi \vdash B_n$ and $B_n \notin \Pi$. Then $\Pi_n \cup \{B_n\} \vdash A$. So $\Pi \vdash A$. Contradiction.
- Π is prime. Suppose for *reductio* that $\Pi \vdash B_n \vee B_m$, but that $B_n \notin \Pi$ and $B_m \notin \Pi$. By construction $\Pi_n \cup \{B_n\} \vdash A$ and $\Pi_m \cup \{B_m\} \vdash A$. So $\Pi \vdash A$ (by $\vee$E). Contradiction.

4.3 Completeness for $FDEe$

In this section we show that in $FDEe$, if $\Sigma \models A$ then $\Sigma \vdash A$. We prove the contrapositive. Suppose that $\Sigma \nvdash A$. Construct Π as in the previous subsection. Define $|B|$ as follows:

- $|B| = t$ if $B \in \Pi$ and $\neg B \notin \Pi$
- $|B| = f$ if $\neg B \in \Pi$ and $B \notin \Pi$
- $|B| = b$ if $B \in \Pi$ and $\neg B \in \Pi$
- $|B| = n$ if $B \notin \Pi$, $\neg B \notin \Pi$, and there is some $B^\dagger \in \Pi$
- $|B| = e$ if $B \notin \Pi$, $\neg B \notin \Pi$, and there is no $B^\dagger \in \Pi$

The conditions are obviously exclusive and exhaustive. Note that the last condition simplifies to:

- $|B| = e$ if there is no $B^\dagger \in \Pi$

If we can show that $|B|$ really is an evaluation, the result follows, since the evaluation makes all members of Π (and so Σ) designated, but not A.

To show this, we show that the definition satisfies the (linear) conditions of Section 2. We may take conjunction to be defined in terms of negation and conjunction. So we need consider only these. For each of them, there are five cases—one for each value.

For negation, here are two of the cases. The others are left to the reader.

For t, we need to show that:

- $|\neg B| = t$ iff $|B| = f$

That is:

- $|\neg B| \in \Pi$ and $|\neg\neg B| \notin \Pi$ iff $|\neg B| \in \Pi$ and $|B| \notin \Pi$

This follows from deductive closure and DN.

For e, we need to show that:

- $|\neg B| = e$ iff $|B| = e$

That is:

- there is no $(\neg B)^{\dagger} \in \Pi$ iff there is no $B^{\dagger} \in \Pi$

This holds since any $B^{\dagger}$ is a $(\neg B)^{\dagger}$, and vice versa.

Before we do the cases for disjunction, here are a couple of useful facts, which follow simply from the rules and deductive closure:

- $\neg(A \vee B) \in \Pi$ iff $\neg A \wedge \neg B \in \Pi$ iff $\neg A \in \Pi$ and $\neg B \in \Pi$

So:

- $\neg(A \vee B) \notin \Pi$ iff $\neg A \notin \Pi$ or $\neg B \notin \Pi$

We now check the five cases for $\vee$.

For e, we need to show that:

- $|A \vee B| = e$ iff $|A| = e$ or $|B| = e$

That is:

- there is no $(A \vee B)^{\dagger} \in \Pi$ iff there is no $A^{\dagger} \in \Pi$ or there is no $B^{\dagger} \in \Pi$

That is, contraposing:

- there is some $(A \vee B)^{\dagger} \in \Pi$ iff there is some $A^{\dagger} \in \Pi$ and there is some $B^{\dagger} \in \Pi$

⇒: This is obvious.

⇐: if $A^{\dagger} \in \Pi$ and $B^{\dagger} \in \Pi$ then $A^{\dagger} \wedge B^{\dagger} \in \Pi$. $A^{\dagger} \wedge B^{\dagger}$ is $(A \vee B)^{\dagger}$.

For f, we have to show that :

- $|A \vee B| = f$ iff $|A| = f$ and $|B| = f$

That is:

- $A \vee B \notin \Pi$ and $\neg(A \vee B) \in \Pi$ iff $A \notin \Pi$ and $\neg A \in \Pi$ and $B \notin \Pi$ and $\neg B \in \Pi$

⇒: $\neg(A \vee B) \in \Pi$, so $\neg A \in \Pi$ and $\neg B \in \Pi$. Suppose that $A \in \Pi$. Then since $\neg(A \vee B) = B^{\dagger} \in \Pi$ we have $A \vee B \in \Pi$. Contradiction. The argument for $B \in \Pi$ is similar.

$\Leftarrow$: $\neg A \in \Pi$ and $\neg B \in \Pi$. So $\neg(A \vee B) \in \Pi$. Suppose that $A \vee B \in \Pi$. Then $A \in \Pi$ or $B \in \Pi$. Contradiction.

For t we need to show that:

- $|A \vee B| = t$ iff ($|A| = t$ and $|B| \neq e$) or ($|B| = t$ and $|A| \neq e$) or ($|A| = b$ and $|B| = n$) or ($|A| = n$ and $|B| = b$)

That is:

- $A \vee B \in \Pi$ and $\neg(A \vee B) \notin \Pi$ iff the disjunction of:
  - $[\alpha]$ $A \in \Pi$ and $\neg A \notin \Pi$ and some $B^\dagger \in \Pi$
  - $[\beta]$ $B \in \Pi$ and $\neg B \notin \Pi$ and some $A^\dagger \in \Pi$
  - $[\gamma]$ $A \in \Pi$ and $\neg A \in \Pi$ and $B \notin \Pi$ and $\neg B \notin \Pi$ and some $B^\dagger \in \Pi$
  - $[\delta]$ $B \in \Pi$ and $\neg B \in \Pi$ and $A \notin \Pi$ and $\neg A \notin \Pi$ and some $A^\dagger \in \Pi$

$\Leftarrow$: In case $[\alpha]$, $A \in \Pi$ and some $B^\dagger \in \Pi$, so $A \vee B \in \Pi$ and $\neg A \notin \Pi$, so $\neg(A \vee B) \notin \Pi$. The other three cases are similar.

$\Rightarrow$: Since $A \vee B \in \Pi$, some $A^\dagger \in \Pi$ and some $B^\dagger \in \Pi$. So we need only consider the other clauses. Since $A \vee B \in \Pi$, $A \in \Pi$ or $B \in \Pi$. Since $\neg(A \vee B) \notin \Pi$ $\neg A \notin \Pi$ or $\neg B \notin \Pi$. This gives us four cases. If $A \in \Pi$ and $\neg A \notin \Pi$, then we are in case $[\alpha]$. If $B \in \Pi$ and $\neg B \notin \Pi$, we are in case $[\beta]$. So suppose that $A \in \Pi$ and $\neg B \notin \Pi$. (If A and B are reversed, the case is similar.) Either $\neg A \in \Pi$ or $\neg A \notin \Pi$. In the second case, we are in case $[\alpha]$. In the first case, we again argue by cases. Either $B \in \Pi$ or $B \notin \Pi$. In the first case, we are in case $[\beta]$. In the second case, we are in case $[\gamma]$.

For b we have to show that:

- $|A \vee B| = b$ iff ($|A| = b$ and $|B| = b$) or ($|A| = b$ and $|B| = f$) or ($|A| = f$ and $|B| = b$)

That is:

- $A \vee B \in \Pi$ and $\neg(A \vee B) \in \Pi$ iff the disjunction of:

- $[\alpha]$ $A \in \Pi$ and $\neg A \in \Pi$ and $B \in \Pi$ and $\neg B \in \Pi$
- $[\beta]$ $A \in \Pi$ and $\neg A \in \Pi$ and $B \notin \Pi$ and $\neg B \in \Pi$
- $[\gamma] B \in \Pi$ and $\neg B \in \Pi$ and $A \notin \Pi$ and $\neg A \in \Pi$

$\Leftarrow$: In case $[\alpha]$, $A \in \Pi$, so $A \vee B \in \Pi$, and $\neg A \in \Pi$ and $\neg B \in \Pi$, so $\neg(A \vee B) \in \Pi$. The other two cases are similar.

$\Rightarrow$: Since $\neg(A \vee B) \in \Pi$, $\neg A \in \Pi$ and $\neg B \in \Pi$. So we need address only the situation with respect to A and B. Since $A \vee B \in \Pi$, $A \in \Pi$ or $B \in \Pi$. Take the first case. The other is similar. Either $B \in \Pi$ or $B \notin \Pi$. In the first case, we are in case $[\alpha]$. in the second case, we are in case $[\beta]$.

For n, we have to show that:

- $|A \vee B| = n$ iff $(|A| = n$ and $|B| = n)$ or $(|A| = n$ and $|B| = f)$ or $(|A| = f$ and $|B| = n)$

That is:

- $A \vee B \notin \Pi$ and $\neg(A \vee B) \notin \Pi$ and some $(A \vee B)^\dagger \in \Pi$ iff the disjunction of:
  - $[\alpha]$ $A \notin \Pi$ and $\neg A \notin \Pi$ and some $A^\dagger \in \Pi$ and $B \notin \Pi$ and $\neg B \notin \Pi$ and some $B^\dagger \in \Pi$
  - $[\beta]$ $A \notin \Pi$ and $\neg A \notin \Pi$ and some $A^\dagger \in \Pi$ and $B \notin \Pi$ and $\neg B \in \Pi$
  - $[\gamma]$ $B \notin \Pi$ and $\neg B \notin \Pi$ and some $B^\dagger \in \Pi$ and $A \notin \Pi$ and $\neg A \in \Pi$

$\Leftarrow$: In each case, $A \notin \Pi$ and $B \notin \Pi$, so $A \vee B \notin \Pi$. In each case $\neg A \notin \Pi$ or $\neg B \notin \Pi$, so $\neg(A \vee B) \notin \Pi$. And in each case, some $A^\dagger \in \Pi$ and some $B^\dagger \in \Pi$. (For any C, $\neg C$ is a $C^\dagger$.) So $A^\dagger \wedge B^\dagger = (A \vee B)^\dagger \in \Pi$.

$\Rightarrow$: First note that some $(A \vee B)^\dagger \in \Pi$, so some $A^\dagger \in \Pi$ and some $B^\dagger \in \Pi$. We therefore need to consider only the other clauses. Now, suppose that $A \in \Pi$; then since some $B^\dagger \in \Pi$, $A \vee B \in \Pi$. Contradiction. So $A \notin \Pi$. Similarly, $B \notin \Pi$. Now, either $\neg A \notin \Pi$ or $\neg B \notin \Pi$. Suppose $\neg A \notin \Pi$. (The other case is similar.) Then either $\neg B \in \Pi$ or $\neg B \notin \Pi$. In the first possibility, we are in case $[\beta]$. In the second, we are in case $[\alpha]$.

5 Subtracting Values

We next consider removing each of the non-classical values from *bne*. In each case, we add a new rule of inference, which is sound in virtue of the value deleted, and the completeness proof simplifies appropriately.

5.1 *ne*

First, remove b to give *ne*. The rules of *bne* are augmented by Explosion. All the old rules are still sound. (If there was no counter-example before, there is no counter-example now.) So only Explosion needs to be checked, and this is straightforward. For completeness, in the canonical model the case where $B \in \Pi$ and $\neg B \in \Pi$ is ruled out by Explosion. In the argument by cases, the case for b disappears. The only other case that mentions b is that for t. This becomes the following. We need to show that:

- $|A \vee B| = t$ iff ($|A| = t$ and $|B| \neq e$) or ($|B| = t$ and $|A| \neq e$)

That is:

- $A \vee B \in \Pi$ and $\neg(A \vee B) \notin \Pi$ iff the disjunction of:
  - $[\alpha]$ $A \in \Pi$ and $\neg A \notin \Pi$ and some $B^\dagger \in \Pi$
  - $[\beta]$ $B \in \Pi$ and $\neg B \notin \Pi$ and some $A^\dagger \in \Pi$

$\Leftarrow$: In case $[\alpha]$, $A \in \Pi$, so $A \vee B \in \Pi$; and $\neg A \notin \Pi$, so $\neg(A \vee B) \notin \Pi$. The case for $[\beta]$ similar.

$\Rightarrow$: Since $A \vee B \in \Pi$, some $A^\dagger \in \Pi$ and some $B^\dagger \in \Pi$. So we need only consider the other clauses. Since $A \vee B \in \Pi$, $A \in \Pi$ or $B \in \Pi$. Since $\neg(A \vee B) \notin \Pi$, $\neg A \notin \Pi$ or $\neg B \notin \Pi$. This gives us four cases. If $A \in \Pi$ and $\neg A \notin \Pi$, then we are in case $[\alpha]$. If $B \in \Pi$ and $\neg B \notin \Pi$, we are in case $[\beta]$. So suppose that $A \in \Pi$ and $\neg B \notin \Pi$. (If A and B are reversed, the case is similar.) Since $A \in \Pi$,$\neg A \notin \Pi$, by Explosion. So we are in case $[\alpha]$.

5.2 *be*

Next, remove n to give be. In the proof theory, we add Weak EM. Soundness is easily established. In the canonical model, the clause for n is removed, and the clause for e becomes:

- $|B| = e$ if $B \notin \Pi$ and $\neg B \notin \Pi$

The four cases are exclusive and exhaustive.

In checking the cases, the case for e becomes as follows. We need to show that:

- $|A \vee B| = e$ iff $|A| = e$ or $|B| = e$

That is:

- $A \vee B \notin \Pi$ and $\neg(A \vee B) \notin \Pi$ iff the disjunction of:
  - $[\alpha]$ $A \notin \Pi$ and $\neg A \notin \Pi$
  - $[\beta]$ $B \notin \Pi$ and $\neg B \notin \Pi$

⇒: This is obvious.

⇐: Here is the case for $[\alpha]$. The case for $[\beta]$ is similar. Since $\neg A \notin \Pi$, $\neg(A \vee B) \notin \Pi$. Suppose that $A \vee B \in \Pi$, then $A \in \Pi$ or $\neg A \in \Pi$, by Weak EM. Contradiction.

There is no case for n to consider. The only other case that mentions e or n is that for t, which is now as follows. We need to show that:

- $|A \vee B| = t$ iff ($|A| = t$ and $|B| \neq e$) or ($|B| = t$ and $|A| \neq e$)

that is:

- $A \vee B \in \Pi$ and $\neg(A \vee B) \notin \Pi$ iff the disjunction of:
  - $[\alpha]$ $A \in \Pi$ and $\neg A \notin \Pi$ and ($B \in \Pi$ or $\neg B \in \Pi$)
  - $[\beta]$ $B \in \Pi$ and $\neg B \notin \Pi$ and ($A \in \Pi$ or $\neg A \in \Pi$)

$\Leftarrow$: In case $[\alpha]$, $A \in \Pi$ so $A \vee B \in \Pi$; and $\neg A \notin \Pi$, so $\neg(A \vee B) \notin \Pi$. Case $[\beta]$ is similar.

$\Rightarrow$: $A \vee B \in \Pi$, so $A \in \Pi$ or $B \in \Pi$. Suppose the first. The second is similar. Since $\neg(A \vee B) \notin \Pi$, $\neg A \notin \Pi$ or $\neg B \notin \Pi$. Suppose the first. Since $A \vee B \in \Pi$ then $B \vee \neg B$, by Weak EM. So either $B \in \Pi$ or $\neg B \in \Pi$. In either case we are in case $[\alpha]$. In the second case $B \in \Pi$, again by Weak EM. So we are in case $[\beta]$.

5.3 *bn* (FDE)

Finally, remove e to give *be* (that is, FDE). For this, Weak $\vee I$ is replaced by $\vee I$. Soundness is easily checked. In the canonical model, the clause for n becomes:

- $|B| = n$ if $B \notin \Pi$ and $\neg B \notin \Pi$

The four cases are exclusive and exhaustive.

There is no case for e to check. The clauses that mention n are those for t and n. The case for n is as follows. We must show that:

- $|A \vee B| = n$ iff ($|A| = n$ and $|B| = n$) or ($|A| = n$ and $|B| = f$) or ($|A| = f$ and $|B| = n$)

That is:

- $A \vee B \notin \Pi$ and $\neg(A \vee B) \notin \Pi$ iff the disjunction of:
  - $[\alpha]$ $A \notin \Pi$ and $\neg A \notin \Pi$ and $B \notin \Pi$ and $\neg B \notin \Pi$
  - $[\beta]$ $A \notin \Pi$ and $\neg A \notin \Pi$ and $B \notin \Pi$ and $\neg B \in \Pi$
  - $[\gamma]$ $B \notin \Pi$ and $\neg B \notin \Pi$ and $A \notin \Pi$ and $\neg A \in \Pi$

$\Leftarrow$: In each case, $A \notin \Pi$ and $B \notin \Pi$, so $A \vee B \notin \Pi$. In each case $\neg A \notin \Pi$ or $\neg B \notin \Pi$, so $\neg(A \vee B) \notin \Pi$.

$\Rightarrow$: Since $A \vee B \notin \Pi$, $A \notin \Pi$ and $B \notin \Pi$, by $\vee$I. Now, either $\neg A \notin \Pi$ or $\neg \mathrm{B} \notin \Pi$. Suppose that $\neg A \notin \Pi$. (The other case is similar.) Then either $\neg B \in \Pi$ or $\neg B \notin \Pi$. In the first possibility, we are in case $[\beta]$. In the second, we are in case $[\alpha]$.

In the case for t, we must now show that:

- $|A \vee B| = t$ iff $|A| = t$ or $|B| = t$ or ($|A| = b$ and $|B| = n$) or ($|A| = n$ and $|B| = b$)

That is:

- $A \vee B \in \Pi$ and $\neg(A \vee B) \notin \Pi$ iff the disjunction of:
  - [α] $A \in \Pi$ and $\neg A \notin \Pi$
  - [β] $B \in \Pi$ and $\neg B \notin \Pi$
  - [γ] $A \in \Pi$ and $\neg A \in \Pi$ and $B \notin \Pi$ and $\neg B \notin \Pi$
  - [δ] $B \in \Pi$ and $\neg B \in \Pi$ and $A \notin \Pi$ and $\neg A \notin \Pi$

$\Leftarrow$: In case [α] $A \in \Pi$ so $A \vee B \in \Pi$; and $\neg A \notin \Pi$, so $\neg(A \vee B) \notin \Pi$. The other three cases are similar.

$\Rightarrow$: Since $A \vee B \in \Pi$, $A \in \Pi$ or $B \in \Pi$. Since $\neg(A \vee B) \notin \Pi$, $\neg A \notin \Pi$ or $\neg B \notin \Pi$. This gives us four cases. If $A \in \Pi$ and $\neg A \notin \Pi$, then we are in case [α]. If $B \in \Pi$ and $\neg B \notin \Pi$, we are in case [β]. So suppose that $A \in \Pi$ and $\neg B \notin \Pi$. (If A and B are reversed, the case is similar.) Either $\neg A \in \Pi$ or $\neg A \notin \Pi$. In the second case, we are in case [α]. In the first case, we again argue by cases. Either $B \in \Pi$ or $B \notin \Pi$. In the first case, we are in case [β]. In the second case, we are in case [γ].

6 Subtracting Multiple Values

Modifying matters to omit multiple values is straightforward. Essentially, we just combine the modifications for each of the values—though there is a small wrinkle in the case for b, as I shall note. Detailed checking is left to the reader.

If we remove b and e, we get n (i.e., K_3). The proof theory adds Explosion and replaces Weak $\vee$I with $\vee$I. Soundness is easy to check, and the canonical model simplifies appropriately. (The only cases to check are for t, f, and n. We simply remove all mention of b from the argument for bn.)

If we remove b and n, we get e (i.e., B_3). The proof theory adds Explosion and Weak EM.[5] Soundness is easy to check, and in the canonical model simplifies appropriately. The only cases to check are for t, f, and e. We simply remove all mention of b from the argument for be.

If we remove n and e, we get b (i.e., LP). One might think that the proof theory would replace Weak $\vee$I with $\vee$I, and add Weak EM; but that doesn't quite work. The rules are certainly sound, but they are not complete. The reason is that LP has logical truths (such as $A \vee \neg A$), but as a simple proof establishes, any deduction with just these rules has undischarged assumptions, so no logical truth is established. The trick is to add not Weak EM, but EM. Soundness is easily established, and the canonical model simplifies appropriately. (There only cases to check are those for t, f, and b—EM implies that they are exhaustive. We simply remove all mention of n from the argument for bn.)

Finally, if we remove b, n, and e, we get $\emptyset$, that is CL. For this, we replace Weak $\vee$I with $\vee$I, and add Explosion and EM. Soundness is easily verified, and the canonical model argument reduces to the standard Henkin completeness proof for classical logic.[6]

[5] I note that these additions make Weak $\vee$I redundant, since this can now be derived as follows:

$$\dfrac{\dfrac{A \quad B^\dagger}{A \wedge B^\dagger [= (B \vee A)^\dagger]}}{(A \vee B) \vee \neg(A \vee B)}$$

We now reason by $\vee$E. In the second case, we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(A \vee B)}{\neg A \wedge \neg B}}{\neg A} \qquad A}{A \wedge \neg A}}{A \vee B}$$

[6] Thanks go to Robrto Ciuni and two anonymous referees of this journal for helpful comments.

7 Appendix: PWK

The system "Paraconsistent Weak Kleene (PWK)" is an interesting, but relatively little known system invented by Sören Halldén and Arthur Prior.[7] The system is the same as B_3, except that e is designated, so it is not in the FDE family as I have characterised it.[8] But the techniques of this paper extend to it naturally. In this appendix, I provide a natural deduction system for PWK.

7.1 Semantics

The matrices for PWK are as follows:

$\neg$	
t	f
f	t
e	e

$\vee$	t	f	e
t	t	t	e
f	t	f	e
e	e	e	e

$\wedge$	t	f	e
t	t	f	e
f	f	f	e
e	e	e	e

t and e being designated. In linear form, the truth conditions are:

Negation

- $|\neg A| = e$ iff $|A| = e$
- $|\neg A| = f$ iff $|A| = t$
- $|\neg A| = t$ iff $|A| = f$

Disjunction

- $|A \vee B| = e$ iff $|A| = e$ or $|B| = e$
- $|A \vee B| = f$ iff $|A| = f$ and $|B| = f$
- $|A \vee B| = t$ iff ($|A| = t$ and $|B| \neq e$) or ($|B| = t$ and $|A| \neq e$)

Conjunction

[7] Halldén (1949) and Prior (1957). For discussion, see Ciuni and Carrara (2016), and Bonzo, Gil-F'erez and Paoli (2017).

[8] An obvious question is why one might designate a value that behaves in this way. I confess that I know of no very plausible reason.

- $|A \wedge B| = e$ iff $|A| = e$ or $|B| = e$
- $|A \wedge B| = f$ iff ($|A| = f$ and $|B| \neq e$) or ($|B| = f$ and $|A| \neq e$)
- $|A \wedge B| = t$ iff ($|A| = t$ and $|B| = t$)

As before, we may take $A \wedge B$ to be defined as $\neg(\neg A \vee \neg B)$.

7.2 Natural Deduction Rules

The natural deduction rules for PWK are: $\vee$I, $\vee$E, DeM, DN, EM, $\wedge$I, plus:

- Weak Explosion

$$\frac{A \quad \neg A}{A^{\dagger}}$$

- Weak $\wedge$E

$$\frac{A \wedge B}{A \vee B^{\dagger}} \qquad \frac{A \wedge B}{A^{\dagger} \vee B}$$

The proof of soundness is left as an exercise.

7.3 Completeness

The proof of the Henkin Lemma of 4.2 is unaffected. The proof of completeness is the same as in 4.3, except that canonical model is defined as follows:

- $|B| = t$ if $B \in \Pi$ and $\neg B \notin \Pi$
- $|B| = f$ if $\neg B \in \Pi$ and $B \notin \Pi$
- $|B| = e$ if $B \in \Pi$ and $\neg B \in \Pi$

EM ensures that we are in one of these cases.

We have to prove that this is a PWK evaluation. The cases for negation are straightforward, and left as an exercise. The cases for disjunction are as follows.

For e, we have to prove that:

- $|A \vee B| = e$ iff $|A| = e$ or $|B| = e$

That is:

- $A \vee B \in \Pi$ and $\neg(A \vee B) \in \Pi$ iff the disjunction of:
  - $[\alpha]$ $A \in \Pi$ and $\neg A \in \Pi$
  - $[\beta]$ $B \in \Pi$ and $\neg B \in \Pi$

$\Leftarrow$: In each case, the result follows from Weak Explosion.

$\Rightarrow$: Since $A \vee B \in \Pi$, $A \in \Pi$ or $B \in \Pi$. Suppose the first of these. The second is similar. Since $\neg(A \vee B) \in \Pi$, $\neg A \wedge \neg B \in \Pi$. By weak $\wedge$E, $\neg A \in \Pi$ or for any $B^\dagger$, $\neg B^\dagger \in \Pi$. In the first case, we are in case $[\alpha]$. In the second case, we are in case $[\beta]$ (since both B and $\neg B$ are of the form).

For f we have to prove that:

- $|A \vee B| = f$ iff $|A| = f$ and $|B| = f$

That is:

- $A \vee B \notin \Pi$ and $\neg(A \vee B) \in \Pi$ iff $A \notin \Pi$ and $\neg A \in \Pi$ and $B \notin \Pi$ and $\neg B \in \Pi$.

$\Rightarrow$: Since $A \vee B \notin \Pi$, $A \notin \Pi$ and $B \notin \Pi$. Since $\neg(A \vee B) \in \Pi$, $\neg A \wedge \neg B \in \Pi$. By Weak $\wedge$E, for any $B^\dagger$, $\neg A \vee B^\dagger \in \Pi$; so $\neg A \in \Pi$ or $B^\dagger \in \Pi$. The second case is impossible, since $B \notin \Pi$. Hence $\neg A \in \Pi$. The case for $\neg B$ is similar.

$\Leftarrow$: Since $A \notin \Pi$ and $B \notin \Pi$, $A \vee B \notin \Pi$. Since $\neg A \in \Pi$ and $\neg B \in \Pi$, $\neg A \wedge \neg B \in \Pi$, so $\neg(A \vee B) \in \Pi$.

For t, we have to prove that:

- $|A \vee B| = t$ iff ($|A| = t$ and $|B| \neq e$) or ($|B| = t$ and $|A| \neq e$)

That is:

- $A \vee B \in \Pi$ and $\neg(A \vee B) \notin \Pi$ iff the disjunction of:

– $[\alpha]$ ($A \in \Pi$ and $\neg A \notin \Pi$) and ($B \notin \Pi$ or $\neg B \notin \Pi$)

– $[\beta]$ ($B \in \Pi$ and $\neg B \notin \Pi$) and ($A \notin \Pi$ or $\neg A \notin \Pi$)

$\Leftarrow$: Here is the case for $[\alpha]$ the case for $[\beta]$ is similar. Since $A \in \Pi$, $A \vee B \in \Pi$. Suppose $\neg(A \vee B) \in \Pi$ then $\neg A \wedge \neg B \in \Pi$. So for any $B^\dagger$, $\neg A \vee (\neg B)^\dagger \in \Pi$, and so either $\neg A \in \Pi$ or every $(\neg B)^\dagger \in \Pi$. The first case contradicts the first conjunct; the second case contradicts the second.

$\Rightarrow$: Since $A \vee B \in \Pi$, $A \in \Pi$ or $B \in \Pi$. Since $\neg(A \vee B) \notin \Pi$, $\neg A \notin \Pi$ or $\neg B \notin \Pi$. This gives us four cases. Suppose $A \in \Pi$ and $\neg A \notin \Pi$. (The case where $B \in \Pi$ and $\neg B \notin \Pi$ is the same.) If we could show that $B \notin \Pi$ or $\neg B \notin \Pi$, we would be in case $[\alpha]$. So suppose that $B \in \Pi$ and $\neg B \in \Pi$. Then, by Weak Explosion, every $B^\dagger \in \Pi$. Hence $\neg(A \vee B) \in \Pi$. Contradiction. Suppose that $A \in \Pi$ and $\neg B \notin \Pi$. (The case where $A \in \Pi$ and $\neg B \notin \Pi$ is the same.) Either $\neg A \notin \Pi$ or $\neg A \in \Pi$. In the first case, we are in case $[\alpha]$. In the second case, $A \in \Pi$ and $\neg A \in \Pi$. Hence, every $A^\dagger \in \Pi$. So $\neg(A \vee B) \in \Pi$, which is impossible.

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