Mathematical Knowledge. By MARK STEINER. (Ithaca & London: Cornell University Press. 1975. Pp. 164. Price £5.20.)

In recent years the philosophy of mathematics has centred around the nature of mathematical truth. The fact that people actually know such truths has been largely ignored. However, any account of mathematical truth that can be shown to imply that no one can come to know such truths obviously cannot be correct. The purpose of the book is to scrutinize various standard positions in the philosophy of mathematics to see whether they can stand up to this test. In fact the book is concerned mainly with two such theories, logicism and Platonism (the division coming, somewhat uncomfortably, in the middle of ch. 2). I shall take these in turn.

Logicism, the claim that mathematics is reducible to logic, is a logical or ontological claim. Epistemological considerations are virtually absent from the writings of Frege and the early writings of Russell. In fact in the only paper where Russell does talk about knowledge ("The Regressive Method of Discovering the Premises of Mathematics") he quite explicitly divorces the logical priority in mathematics from the epistemological. How then can logicism be subjected to epistemological criticism? In order to do this Steiner lands it with the following claims (p. 25): (1) there is some formal system of logic such that mathematics can be effectively generated from it; (2) it is sufficient to understand proofs written in the system in order to know all the truths of mathematics we know; (3) it is possible for us, with our limited abilities, actually to come to know mathematical truths in the way suggested by (2), that is, by constructing logical proofs of them.

Nothing remotely like (2) or (3) ever occurred in the writings of logicism. It is true that Hempel (Steiner's example of a latter day logicist) in his paper "On the Nature of Mathematical Truth" starts by asking what grounds sanction the acceptance of mathematics. However Hempel's answer is not, as Steiner says (p. 24), that "the reduction of mathematics to logic provides the necessary grounds" but that the reduction shows that the grounds, whatever they are, are the same as those for logical truths. (And Hempel accepts the standard positivist account of analytic truths.)

Nonetheless arguments are to be found attacking (2) and (3) and in ch. 1 Steiner sets out to examine them. Poincaré's attack against (2) (as presented by Parsons in "Frege's Theory of Number") is this: using Frege's definitions of arithmetical concepts we can obtain a predicate 'x is a natural number' and a term (numeral) 'n' for each number n. We can now ask whether the properties *object satisfying 'x is a number'*, and *object denoted by a numeral* are extensionally equivalent. By a meta-linguistic induction, we can prove that they are. However, the question cannot even be stated, let alone answered, in the formal system itself. Steiner's reply, that this is not a truth of mathematics, but a fact about our notation, seems adequate.

Wittgenstein's attack on (3) in *Remarks on the Foundations of Mathematics* is that to know certain logical truths we need some mathematical knowledge. This point is processed by Steiner and finally comes out as the fact that most mathematical truths, when expressed in the primitive notation of, e.g., *Principia Mathematica*, would not be humanly "surveyable". Their proofs would be even less so. This causes Steiner to reject the standard conception of definition as meta-linguistic abbreviation in favour of Leśniewski's view that we may actually increase the vocabulary of our system by adding new defined symbols. (Unfortunately Leśniewski does not get a mention.)

"Logicism" is saved in ch. 1, only to meet its end in the first paragraph of ch. 2. Russell & Co. may have shown that mathematics is reducible to set theory, but set theory is not logic. However, we still have Quine's view that the reduction relieves us of "ontological commitment" to numbers, but Steiner rejects this. Epistemologically, set theory is shakier than arithmetic (its principles are less clear, the danger of contradiction is greater, etc.). Steiner rejects Quine's position, since it presupposes that (p. 75) "one can achieve ontological without epistemological gain, indeed at epistemological loss. This seems to me to be absurd". His main argument for thinking this an absurdity is a somewhat dubious analogy. Number theory is compared to a limited but reliable computer, set theory to a powerful but unreliable one. We would be crazy to throw away the small one unless we had to. The weakness of the analogy is that scientific reduction is not obviously like throwing a computer away. Much more would have to be said about reduction to make this a good analogy. However all Steiner says is that a reduction to an epistemologically weaker theory is only "*bona fide*" if the reduction "effects changes that improve the original theory" (p. 86). In what way was Euclidean geometry improved by its Cartesian reduction to analysis? Or was this not a "*bona fide*" reduction either?

The second part of the book sets out to defend Gödel's form of Platonism, which is that numbers and so on are real objects (though of a different kind from physical objects) which we come to know about by a faculty (intuition) analogous to sensory perception.

Benacerraf's view that numbers are not objects at all (in "What Numbers Could Not Be") is briefly considered and rejected. Steiner then attacks the standard view that the only way we can come to know a non-trivial mathematical truth is by finding a (deductive) proof of it, from previously established truths. Euler established that

$$\sum_{n=0}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$$

Steiner presents the evidence that Euler possessed (which did not include a proof) and then says that in such circumstances a claim to knowledge is quite justified. This conclusion is indeed very tempting, but it is not clear how one would defend this claim against someone who held that Euler had excellent grounds for belief, but not knowledge.

In the last chapter of the book Steiner defends Gödel against an attack by Putnam (in "Mathematical Truth"). The attack rests on a causal theory of knowledge to the effect that for A to know about y, there must be some causal chain beginning with y and ending with A. Since abstract objects are not the sort of things that can enter into causal chains, we cannot know about numbers in the way Gödel suggests. Steiner has no trouble in showing that there is no satisfactory formulation of the causal theory which supports this conclusion.

Even if there are no *a priori* objections to Gödel's view, it is difficult to take this "sixth sense" (mathematical intuition) seriously unless we have a positive account of it. In the last part of the chapter Steiner produces a few unsatisfactory speculations about this. The amazing suggestion that "we may become familiar with the standard model of ZF set theory by abstracting from dots on a blackboard arranged in a certain way" (pp. 134-5) is mooted. The important Wittgensteinian problem of how one could possibly distinguish between veridical and non-veridical intuition is raised and dropped immediately.

The book ends with three arguments for the existence of mathematical intuition: (i) the agreement of mathematicians about what is known; (ii) the fact that mathematicians themselves talk of intuition; (iii) the inevitable Ramanujan. At the risk of belabouring the inconclusiveness of these points, one could use exactly similar arguments to show that chess players have some kind of direct perception of the abstract objects *strategy*, *positional value*, etc.

I did find Steiner's book thought-provoking and it certainly draws attention to an area of the philosophy of mathematics that has largely been neglected. However, it will be clear that I have my reservations about it.

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