Synthese (2014) 191:3057–3062
DOI 10.1007/s11229-014-0473-7

Chunk and permeate III: the Dirac delta function

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Received: 22 April 2014 / Accepted: 22 April 2014 / Published online: 29 May 2014

Abstract Dirac's treatment of his well known Delta function was apparently inconsistent. We show how to reconstruct his reasoning using the inconsistency-tolerant technique of Chunk and Permeate. In passing we take note of limitations and developments of that technique.

Keywords Dirac delta function · Chunk and permeate · Paraconsistency · Paraconsisent

1 Introduction

It is well known that in developing the calculus, Newton displayed a certain mathematical opportunism in dividing by small numbers and subsequently setting them to zero. This is a *prima facie* inconsistency. In 2004, Brown and Priest introduced a strategy for dealing with reasoning situations involving the use of incompatible premisses. They called their strategy *Chunk and Permeate* (hereafter C&P), and showed how to deal with Newton's calculus by taking the derivatives of typical polynomial expressions. In 2008 they extended the applications of C&P to the Old Quantum Theory, that is Bohr's original theory of the atom, which likewise appeals to incompatible principles to reach its desired conclusion of predicting the energy levels of the hydrogen atom.

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The main idea of the C&P strategy is that we allow reasoning *within* different "chunks", which may be inconsistent with each other. Then some, but not all, information is allowed to flow, or "permeate", from chunk to chunk, which may then be used in the chunk into which information flows. The flow of information is carefully controlled in such a way that each chunk remains internally consistent. (For full details, see the papers just cited.)

In this paper, we provide a further application, in explaining the reasoning displayed by Dirac when he introduced his famous Delta "function", $\delta(x)$. This device was characterised by three main properties:

(1) $\int_{-\infty}^{\infty} \delta(x)dx = 1$

(2) $\delta(x) = 0, \quad \text{for all } x \neq 0$

(3) $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$ (where f is any function continuous around the origin).

It is apparent that the combination of the first two properties is not compatible with δ being a function on the real numbers. Dirac stresses the importance of (3) in the quantum theory, as it follows by a simple translation of the origin that $\int f(x)\delta(x-a) = f(a)$, which is to say that "the process of multiplying a function of x by $\delta(x-a)$ and integrating over all x is equivalent to the process of substituting a for x".

Now it is possible to treat the delta function by using a single inconsistent theory, see Mortensen (1995) Chapter 7. In contrast, C&P breaks up the background assumptions into at least two chunks, which provide incompatible reasoning environments. There can be more than two chunks, but for present purposes we simplify with just two chunks, the Source S and the Target T. Thus we draw conclusions within S, then pass them to T, which is an incompatible mathematical regime, where final conclusions are drawn. Dirac's reasoning will be seen to be a plausible model of this.[1] Brown and Priest describe C&P as a "rational reconstruction". This is appropriate here as an acknowledgement of the informal and heuristic character of Dirac's own discussion, as well as a testimonial to Dirac's brilliant mathematical opportunism, worthy of someone who occupied the same chair as Newton at Cambridge.

The *locus classicus* of Dirac's discussion is his famous *Principles of Quantum Mechanics* (3 ed 1930, pp. 58–61). In these pages, he does two main things. First, he argues that (3) above follows from (1) and (2). Then he shows how to model the delta function as the "derivative" of the step function, or Heaviside function, deducing (3) directly from this definition, and in passing showing that (2) follows from (3). In the next section, we look at the first of these, in the following section, we deal with the second.[2]

[1] To specify things with full rigor, we would need a formal specification of *what* sort of information is allowed to flow between S and T. We forego this here, as appropriate for a more technical paper. In this paper, we are concerned only with the heuristic of the procedure.

[2] At this point, Mortensen wishes to register divergence of opinion from Brown and Priest. The point of C&P is to isolate premisses which are inconsistent with one another, to avoid taking such premisses fully collectively, by not allowing all consequences of S to flow to T. This contrasts with the traditional view of reasoning (deductive, inductive, abductive, consistent or inconsistent), where one believes the conclusion

2 Dirac's first argument

Dirac prepares the way by saying that one does not need to suppose that the delta function is anything other than a sharply-peaked function with a finite peak which is nonzero over a small finite interval around the origin. Assuming (2), Dirac argues that the left hand side of (3): "can depend only on the values of $f(x)$ very close to the origin, so that we may replace $f(x)$ by its value at the origin, $f(0)$, without essential error. Equation (3) then follows from the first of equations".

This cries out for a routine expansion within non-standard analysis, using infinitesimals. In turn we see that such an argument goes uniformly over to a C&P regime. One quick *proviso*: we will simplify by not bothering with integrals over the whole real line, preferring instead to take integrals over finite intervals including the origin. Extending to integrals over the whole real line is a further limiting process, which adds nothing essential to the argument. Indeed, this is a simplification that Dirac himself makes in his second argument.

It is not obviously true what Dirac says, that the same results can be obtained by taking delta to be a function with height and width of real size. The most one gets is an approximation (of small finite size) to the desired result. However, if we take the source theory S with the theory of nonstandard analysis, we can achieve our result, because approximations can be defined as at most infinitesimally different. So we let $\delta(x)$ be a function with property (1) and instead of (2) take:

(4) $\delta(x) = 0$, for all x with $x \leqslant -h < 0$ or $0 < h \leqslant x$, for some positive infinitesimal h.

There are such functions in classical non-standard analysis, and furthermore the area under them can be made to be unity. For example, if we draw a tall thin triangle with base $2h$ which is infinitesimal, and height $1/h$ which is infinite, then its area = base$\times$ height$/2 = 1$. This is property (1) above, adjusted for finitude.

Now, to obtain (3), we let $f(x)$ be any function continuous at $x = 0$. That is, by definition in non-standard analysis:

(5) $x \approx 0$ implies $f(x) \approx f(0)$
That is:
(6) for any infinitesimal θ, $x = \theta$ implies $f(x) = f(0) + \eta(x)$ for some infinitesimal $\eta(x)$.
Now, consider the integral $\int_{-a}^{a} f(x)\delta(x)dx$ (where a is a standard real). Outside of the interval $(-h, h)$ the $\delta(x)$ term is zero. Hence we need only consider x inside

Footnote 2 continued
on the basis of believing the premisses used to get there. C&P is something less than full agreement with all the premisses and therefore the conclusion. It thus appears more like a useful heuristic device than a rigorous mathematical proof technique, part of the context of discovery rather than the context of justification, at least for the *apriori* justifications of pure mathematics. In the case of the delta function, this is apparent from the fact that following theorists strove to discover a (consistent) set of definitions and proofs. It took 30 years before this was found in Laurent Schwartz ' theory of distributions. This is a rigorous and apparently consistent mathematical theory, which has the main drawback of considerably increased complexity, deriving from its treatment of delta not as a function but as a functional. Of course, heuristics are a useful part of empirical science too (CM).

that interval, $x = \theta \in (-h, h)$. Thus from (6), $f(x) = f(0) + \eta(x)$. This is what Dirac means (or should mean) when he speaks of replacing $f(x)$ without essential error, namely at most infinitesimal error. Hence:

(7)
$$\begin{aligned}\int_{-h}^{h} f(x)\delta(x)dx &= \int_{-h}^{h} (f(0) + \eta(x))\delta(x)dx \\ &= f(0)\int_{-h}^{h} \delta(x)dx + \int_{-h}^{h} \eta(x)\delta(x)dx \\ &= f(0) + \int_{-h}^{h} \eta(x)\delta(x)dx \\ &= f(0) + \alpha\end{aligned}$$

where α is infinitesimal. The penultimate step is by property (1). For the last step, for any real r, $|\eta(x)| < r$. So $|\eta(x)|.\delta(x) < r.\delta(x)$, and $\int_{-h}^{h} |\eta(x)|.\delta(x)dx < \int_{-h}^{h} r.\delta(x)dx = r$. Now, $\left|\int_{-h}^{h} \eta(x).\delta(x)dx\right| \leq \int_{-h}^{h} |\eta(x)|.\delta(x)dx$, since subtractions are changed to additions. Hence, $\left|\int_{-h}^{h} \eta(x).\delta(x)dx\right| < r$, for any real r; and so $\int_{-h}^{h} \eta(x).\delta(x)dx$ is infinitesimal, α.

So far, all is in accordance with nonstandard analysis. At this point, in nonstandard analysis we ”take standard parts”, since in nonstandard analysis derivatives and integrals are defined to the nearest real number. In contrast, C&P can pass the result of (7) to the target chunk T, in which it is stipulated that all infinitesimals including α are zero. This yields the desired conclusion (3). Moreover, instead of (4), we now have (2), since h is set to zero.

It is clear that there is a straightforward comparison between *any* proof in nonstandard analysis, and the corresponding proof in C&P. The former defines the answer to be the standard part; and wherever this happens, the latter passes to a different C&P deductive regime, in which any infinitesimal α is declared to be zero. This is how the C&P argument is for Newton's derivatives, see Brown and Priest (2004). Of course, it does not follow that C&P is *merely* consistent nonstandard analysis under another name. After all, C&P is envisaged as a general technique not restricted to NSA; and even within NSA, there may be other ways to conduct such arguments. There is also the point that equivalence leaves open the question of whether one or the other has explanatory priority.

Now we turn to Dirac's second argument.

3 Dirac's second argument

In his second argument, Dirac wants to show a kind of existence proof, by providing a function and then defining the delta function as its derivative. The step function (or Heaviside function) $\epsilon(x)$ is given by:

(8) $\epsilon(x) = 0$ (for all $x < 0$), and $\epsilon(x) = 1$ (for all $x > 0$)

We note that this function is not defined at $x = 0$. This seems not to matter, the following argument works even if $\epsilon(0)$ is given a value. What does matter is that $\epsilon(x)$ is discontinuous at 0, so that its derivative function $\epsilon\prime(x)$ is not defined at $x = 0$, which is in accordance with using that to model $\delta(x)$. Dirac wants to show that $\epsilon\prime(x)$ so defined satisfies (3). It also follows that (1) and (2) hold, as we will see.

Dirac uses *integration by parts*. To remind the reader, this is the rule that: $\int udv = uv - \int vdu$, or:

(9) $\int uv\prime dx = uv - \int vu\prime dx.$

Substituting f for u and ϵ for v, we have, for any real a:

(10)
$$\int_{-a}^{a} f(x)\epsilon\prime(x)dx = f(x)\epsilon(x)|_{-a}^{a} - \int_{-a}^{a} \epsilon(x) f\prime(x)dx$$
$$= f(x)\epsilon(x)|_{-a}^{a} - \int_{-a}^{a} \epsilon(x)df$$

Now applying (8) in the form $\epsilon(a) = 1$, $\epsilon(-a) = 0$, the first term becomes $f(a)$. As for the second term, $\int_{-a}^{a} \epsilon(x)df$ breaks up into the sum $\int_{-a}^{0} \epsilon(x)df + \int_{0}^{a} \epsilon(x)df$. Since by (8) $\epsilon(x)$ vanishes for negative x, the first of these summands equals 0. Since by (8) $\epsilon(x)$ is 1 for positive x, the second of these summands equals $\int_{0}^{a} df$, that is $f(a) - f(0)$. Assembling summands:

(11) $\int_{-a}^{a} f(x)\epsilon\prime(x)dx = f(a) - (f(a) - f(0)) = f(0)$

This is (3) as required. Clearly (1) follows by setting f to be the constant function 1, and (2) follows from (8).

To get to (10), we operated in a regime where integration by parts is possible. Now for example Hardy's *Pure Mathematics* (1908, 10 (ed.) 1960, pp. 216–258) requires for integration by parts that the derivatives $u\prime(x)$, $v\prime(x)$ be defined for the relevant values of the variables. That is, $f\prime(0)$ and $\epsilon\prime(0)$ exist. There is no problem with this for (10), many functions behave thus. However, to go from (10) to (11), we must apply (8), which ensures that the derivative of ϵ does not exist at $x = 0$. Dirac's second argument therefore depends on inconsistent premisses.

The C&P version of this is straightforward. It suffices to take it that the Source theory S contains functions, notably $\epsilon(x)$, taken to be a differentiable function in the appropriate range, which is 0 for $x < -h$, 1 for $x > h$, and ascends monotonically from 0 to 1 between $-h$ and h , for some real positive h. The resulting (10) is then passed to the target theory T, in which $h = 0$ holds, that is where ϵ is the step function, which is then applied to obtain (3).

Notice that the above argument is not particularly non-standard, it appeals to standard analysis alone. This shows that C&P arguments do not all take the nonstandard form identified in the previous section, and that the sources and targets are generally different for different applications of C&P. On the other hand, it would seem that it provides a template for translation of infinitesimal arguments into C&P for real number theory alone. For example, take any theory in nonstandard analysis, replace infinitesimals by small real numbers. All calculations take place the same, by construction of the hyperreals. So the story remains the same, except for a small real difference. Permeate to a regime where this is zero, and the answer is as before.

4 Conclusion

We see that Chunk and Permeate provides a simple and straightforward model of Dirac's thinking on the delta "function". We also see that this rational reconstruction is thoroughly implicated in scientific reasoning from incompatible premisses. Along the way, we have also seen that there is a close relationship, amounting to intertranslatability, between nonstandard arguments and at least some C&P arguments.[3]

References

Brown, B., & Priest, G. (2004). Chunk and permeate: A paraconsistent inference strategy. part 1: The infinitesimal calculus. *The Journal of Philosophical Logic*, *33*, 379–388.

Brown, B., & Priest, G. (2008). Chunk and permeate II: Weak aggregation, permeation and old quantum theory, Fourth World Congress on Paraconsistency, Melbourne.

Dirac, P. A. M. (1930). *Principles of quantum mechanics*, 3rd (ed) 1947. Oxford: Oxford University Press.

Hardy, G. H. (1908). *Pure mathematics*, 10 (ed) 1960. Cambridge: Cambridge University Press.

Mortensen, C. (1995). *Inconsistent mathematics*. Dordrecht: Kluwer.

[3] Mortensen and Priest wish to record with sadness the recent death of their co-author, the talented logician and wit Richard Benham.