Graham Priest

Indefinite Extensibility—Dialetheic Style

Abstract. In recent years, many people writing on set theory have invoked the notion of an indefinitely extensible concept. The notion, it is usually claimed, plays an important role in solving the paradoxes of absolute infinity. It is not clear, however, how the notion should be formulated in a coherent way, since it appears to run into a number of problems concerning, for example, unrestricted quantification. In fact, the notion makes perfectly good sense if one endorses a dialetheic solution to the paradoxes. It then morphs from a supposed solution to the paradoxes into a diagnosis of their structure. In this paper I show how.

Keywords: Indefinite extensibility, Inclosure schema, Dialetheism, Paraconsistency, Absolute infinity, Absolute generality.

1. Introduction

In recent years, many people writing on set theory have invoked the notion of an indefinitely extensible concept. The notion, it is usually claimed, plays an important role in solving the paradoxes of absolute infinity. It is not clear, however, how the notion should be formulated in a coherent way, since it appears to run into a number of problems concerning, for example, unrestricted quantification. In fact, the notion makes perfectly good sense if one endorses a dialetheic solution to the paradoxes. It then morphs from a supposed solution to the paradoxes into a diagnosis of their structure. In this paper I will show how. Much of what I say here is covered at greater length in *Beyond the Limits of Thought*,[1] but here I shall bring the machinery to bear explicitly on the notion of indefinite extensibility in a relatively self-contained way. Concerning the issue of unrestricted quantification, I've had my say in a review of the collection of essays edited by Rayo and Uzquiano (2006).[2] A small lament in that review is the absence of an essay in the collection which explores a dialetheic approach. One might take the present paper to fill that gap.

---

[1] Priest (2002).

[2] Priest (2007).

Special Issue: Advances in Philosophical Logic
Edited by **Heinrich Wansing, Roberto Ciuni and Caroline Willkommen**

Studia Logica (2013) 101: 1263–1275
DOI: 10.1007/s11225-013-9532-1

2. On to Greater Things

Let us go back and start at the beginning. A number of philosophers concerned with the foundations of set theory have found it useful to articulate the notion of an *indefinitely extensible* (IE) concept. Michael Dummett is one such. He says that an:[3]

> *indefinitely extensible* concept is one such that, if we can form a definite conception of a totality all of whose members fall under the concept, we can, by reference to that totality, characterise a larger totality all of whose members fall under it.

Sometimes the collections determined by such concepts are themselves called IE. Thus Russell, perhaps the first person to put his finger on the idea—though he calls it something different—says the following:[4]

> there are what we might call *self-reproductive* processes or classes. That is, there are some properties such that, given a class of terms all having such a property, we can always define a new term also having the property in question.

Everyone's favourite example of the notion is that of being an ordinal. For any bunch of ordinals, there is a bigger. The idea is inherent in the very way that Cantor characterised the ordinals. According to him there were two principles which generate ordinals:[5]

1. If α is an ordinal number then there is a next number $\alpha + 1$ which is the immediate successor of α.
2. If any definite succession of defined ... [ordinal] numbers exists, for which there is no largest, then a new number is created ... which is thought of as the *limit* of those numbers, i.e., it is defined as the next number larger than all of them.

In other words, for any bunch of ordinals, x, there is **l**east **o**rdinal **g**reater than all of them, $log(x)$. One can throw this in, to make a bigger collection.

One way to articulate the concept rigorously was provided by Russell himself.[6] φ is an IE concept iff there is some operation, δ, such that for every set, x, such that $\forall y(y \in x \rightarrow \varphi(y))$:

---

[3]Dummett (1993), p. 441.

[4]Russell (1905), p. 144 of reprint.

[5]See Hallett (1984), p. 49.

[6]Russell (1905), p. 142 of reprint.

1. $\delta(x) \notin x$
2. $\varphi(\delta(x))$

The notation comes from *Beyond the Limits of Thought*,[7] where 1 is called *Transcendence*, and 2 is called *Closure*. The 'δ' stands for 'diagonaliser'.

Being an ordinal is not the only notion that is IE in this sense. The concept of being a set itself is an IE one. Given any set, x, we may take $\delta(x)$ to be $\{y \in x : y \notin y\}$. Similarly, the notion of being a well-founded set is IE. For this, we may take $\delta(x)$ to be x itself. One can easily check that Transcendence and Closure are forthcoming in these cases.

If φ is any IE notion, the ordinals can be embedded in the collection of entities that satisfy it. The function, f, defined by following transfinite recursion will do the job: $f(\alpha) = \delta\{f(\beta) : \beta < \alpha\}$. IE collections are, then, very large. Assuming an appropriate version of the axiom of choice, which allows all collections to be well-ordered, any collection can be embedded into the ordinals. All IE totalities, then, have the same size: absolute infinity.

3. The Existence of Ω

So far, not too much to argue about. Not so from here on in. Let φ be IE, and let $\Omega = \{y : \varphi(y)\}$. Call the principle that:

- Ω exists

Existence. Is existence true? If it is, we quickly obtain a contradiction, since $\delta(\Omega) \notin \Omega$; but $\varphi(\delta(\Omega))$, that is, $\delta(\Omega) \in \Omega$.

If we add Existence to Transcendence and Closure, we obtain what *Beyond the Limits of Thought* calls 'Russell's Scheme'.[8] According to Russell, this was the Schema which lay behind all the set-theoretic paradoxes. In his thinking, it morphed into the Vicious Circle Principle which lay behind all the paradoxes of self reference. His solution to these was simply to reject Existence. Indeed, the above quote continues:

> Hence, we can never collect *all* the terms having the said property into a whole; because, whenever we hope we have them all, the collection which we have immediately proceeds to generate a new term also having the said property.

[7] Priest (2002), p. 129 ff.

[8] Priest (2002), 9.2.

And orthodoxy has concurred. According to ZF, absolutely infinite sets do not exist.

This is not the only possibility, however. There is the dialetheic one. Existence is true, and so is the contradiction; the logic of sets is paraconsistent. This option was hardly open to Russell and those who had never heard of paraconsistent logic. But now that paraconsistent logic does exist (and cannot be uninvented), the option can be ignored only by sticking one's head in the sand, ostrich-like.

One may think that dialetheism has its problems. Good arguments against the view are, however, much harder to come by than orthodoxy would like to suppose.[9] Indeed, many—perhaps most—who reject the view, do so on the basis of a knee-jerk reaction rather than a rational one. A defence of dialetheism is a task too big to take on here, though. I will content myself with noting why the orthodox view concerning Existence is in some trouble.

First, note that denying Existence cannot be all there is to the matter, since the Russell Schema can be formulated without reference to sets at all, as follows: there is some operation on concepts, δ, such that for any concept, χ, such that $\forall y(\chi(y) \to \varphi(y))$:

1. $\neg\chi(\delta(\chi))$
2. $\varphi(\delta(\chi))$

The notion of a diagonaliser makes just as good sense in this context. Thus, in the Burali-Forti paradox, where $\varphi(y)$ is the condition that y is an ordinal, $\delta(\chi)$ is the least ordinal greater than all y such that $\chi(y)$. And now, it would seem, one cannot simply deny that there is such a concept as being an ordinal. That, after all, is what we are talking about.

What, then, is one to do if one wishes to avoid the paradox? Here Russell and orthodoxy part company. Russell's line in *Principia* is, in fact, to deny that there is such a concept as being an ordinal. The only concepts that make sense are those of being an ordinal of a certain type (in the type-theoretic hierarchy). Trans-type conditions are meaningless. The problems with this are clear. Not only does it render his earlier account of *self-reproductive* processes, and so his diagnosis of the paradoxes, meaningless ('There is some property such that...'); it makes any explanation of his theory, the theory of types, meaningless. One cannot say that ordinals are of different types, let alone that sets are hierarchically organised into types.[10]

[9] As argued in many places, notably Priest (1998) and (2006b).

[10] For a full discussion of the self-referential inconsistency of type theory, see Priest (2002), ch. 9.

Orthodoxy, in the shape of Zermelo-Fraenkel set theory, does better. According to this, $\delta(x)$, in the various cases, is defined only when x is a set, and Ω is not a set. That set does not exist. The denial of Existence is still functioning, just in a more covert way.

4. The Lure of Existence

So why is this line problematic? First, Existence is *prima facie* true. The naively correct view is that *any* condition specifies a collection. In particular, then, $\varphi(x)$ does. Of course, this is only a *prima facie* consideration, but it has a bite. No one would ever have thought to deny Existence, had it not turned out to be a salient principle in the generation of contradiction. If the rejection of Existence is to be well-motivated, though, there must be an *independent* reason for supposing that it fails. I am not aware of any that withstand much inspection. In particular, simply citing Transcendence as a reason for supposing Ω not to exist (as Russell does above) will not do. All, dialetheist or not, can agree that we have Transcendence. That is what generates half of the contradiction. Those who would give an independent reason for denying Existence need something else.

Next, a sustained denial of Existence is hard to maintain. Set theorists, indeed, often find themselves prone to talk in ways that appear to presuppose it. Merely consider, for example, what I said about the cardinality of absolute infinities at the end of Section 1 above. This refers to IE totalities, including the totality of all ordinals, in a perfectly natural way.

Worse: a denial of Existence gets in the way of important contemporary mathematics. The project of reducing mathematics to ZF set theory was fine for what we might call local mathematics (where a structure less than the totality of all sets is being investigated). But modern mathematics wants to do global mathematics. Thus, the ordinals are supposed to provide a canonical form for well-orderings, but there are clearly well-orderings greater than the totality of all ordinals. These are studied in the delightfully named 'mouse theory'.[11] The theory of well-ordered order types is in disarray. Worse still, category theorists want to apply the benefits of category theory to the category of all sets (and of all categories) and even to categories constructed out of these, like functor categories. These structures make no sense at all if there is no set of all sets.[12]

---

[11]For a survey of mouse-theory, see Schimmerling (2001). For a discussion of its conceptual problems, see Shapiro and Wright (2006), pp. 289 ff.

[12]The matter is discussed at greater length in 2.3 of Priest (2006a).

Of course, set theorists are aware of the problem. A standard device in the face of it is to draw a distinction. Classes are of two kinds, sets and proper classes. Absolutely infinite totalities are of the latter kind. This is merely a nominal change, however. The original problem then still arises, since we are just as wont to talk about the collection of all classes as we are to talk about the collection of all sets. And the functor category of the category of all sets still makes no sense in this framework.

Another standard ploy is to suppose that there is an inaccessible cardinal, ϑ, and that when we talk of sets we are talking of the members of the cumulative hierarchy of rank less than V_ϑ; when we talk of proper classes, we are talking of sets of rank $V_{\vartheta+1}$. There can now be collections of proper classes (of rank $V_{\vartheta+2}$). Ultimately, though, we still have the same problem: there is no collection of all the sets in the hierarchy. So the problem has not been solved, merely hidden.

5. The Domain Principle

The third reason why it is hard to reject Existence is that there is good independent reason for it. Following Hallett, let us call it the *Domain Principle*.[13] It was first articulated by Cantor as follows:[14]

> In order for there to be a variable quantity in some mathematical study, the 'domain' of its variability must strictly speaking be known beforehand through definition. However, this domain cannot itself be something variable, since otherwise each fixed support for the study would collapse. Thus, this 'domain' is a definite, actually infinite series of values.

Cantor is interested in justifying the thought that there are actually infinite (transfinite) quantities, not just potentially infinite ones. Thus, for example, if we are committed to the natural numbers as a potential infinity, we are committed to them as an actual (completed) totality.

What he is pointing out has a much more general application, however. Suppose that we make a claim about a variably quantity. If this is to have a determinate sense, there must be a determinate totality of variability. In the context of modern logic, we can put the point this way. Suppose we have a statement containing a variable, such as:

S Every quadratic equation with real coefficients has two solutions.

---

[13] Hallett (1984), p. 7.

[14] Hallett (1984), p. 25.

This has no determinate truth value unless there is a determinate collection over which the variable ranges. (Cantor says 'known collection', but that's too much. Maybe it's necessary to know it if the truth value is to be known, but simple existence is good enough for the determinacy of the truth value.) Thus, for example, **S** is true if the quantifier ranges over the complex numbers, false if it ranges over the real numbers.[15]

To bring the point home to the case at hand: assuming, as seems to be the case, that one can make determinately true or false statements about the ordinals, such as 'There is [is not] an inaccessible ordinal', then there must be a determinate totality of all ordinals. Similarly for every other IE concept.

One response to the argument is simply to deny that we can quantify over all ordinals, or all entities satisfying some other IE notion. Several philosophers have recently argued this.[16] Those who take this line would appear to start off at a singular disadvantage: they cannot state their own thesis. One cannot quantify over all ordinals. All what?

This is not the place to discuss the ins and outs of the debate. In their paper on the matter, Shapiro and Wright (2006) admirably locate five possible lines one might take with respect to the issue of quantifying over all ordinals and other IE totalities—together with the problems of each. They conclude the summary as follows:[17]

> Frankly, we do not see a satisfactory postion here. It seems that every one of the available theoretical options has difficulties which would be justly treated as decisive against it, were it not for the fact that the others fare no better. Such situations are not unprecedented in philosophy, but this one seems particularly opaque and frustrating.

The fifth of their possibilities is the dialethic one. I quote:[18]

> (v) Allow the quantification and the predicates, allow the associated order-types, allow that they are ordinals as originally understood, . . . and just accept that there are ordinals that come later than all the ordinals. *Cost*: none—unless one demurs from the acceptance of a contradiction.

---

[15] For further discussion, see Priest (2002), 8.7, 8.8.

[16] See, e.g., the papers in Rayo and Uzquiano (2006). The editors' introduction is a helpful overview of the taxonomy of the debate.

[17] Shapiro and Wright (2006), p. 293.

[18] *Ibid.* Ellipsis original.

The statement tells all: a dialethic solution to the problem gives you everything you want. A joke does duty as an argument against it. It is not uncommon, in fact, to see how often dialetheism is dismissed with a flippant remark. Thus, in the same collection, Williamson passes over the view saying simply that dialetheism is a 'fate worse than death'.[19] Jokes about dialethism are fine. I enjoy them as much as anyone else. It is just sad to see good philosophers confusing humour with rational argument.[20]

6. The Inclosure Schema

We will come to a fourth reason why rejecting Existence is problematic in a moment, but first let us turn to another matter. Start by considering König's paradox. To say that a set is *definable* is to say that it is referred to by some (non-indexical English) noun-phrase. There are only a countable number of definable ordinals. But there are lots more ordinals than that. So there must be a least non-definable ordinal. By construction, it's not definable; but we have just defined it with the phrase 'least non-definable ordinal'.

It might be thought that this paradox fits Russell's Schema. $\varphi(y)$ is 'y is a definable ordinal', $\delta(x)$ is $log(x)$. Transcendence and Closure both appear to hold by construction; and Existence holds since Ω is a countable set of ordinals. But there's a rub. Unless the set x is a definable set, there is no reason to suppose that $log(x)$ is definable. It will be, if x is itself definable by some name, 'n'. $log(x)$ will then be defined by the phrase 'the least ordinal greater than n'.

We can make a simple modification to Russell's Schema, however.[21] We now require that there be a condition, $\psi(x)$, such that $\psi(\Omega)$, and $\delta(x)$ satisfies the Transcendence and Closure conditions provided that $\psi(x)$. So we now have the following, where $\Omega = \{y : \varphi(y)\}$:

1. Ω exists and $\psi(\Omega)$
2. If $x \subseteq \Omega$ and $\psi(x)$ then:

[19] Williamson (2006), p. 387.

[20] Actually, I don't begrudge Shapiro and Wright their joke. They have been two of the classically oriented logicians who have been most open-minded about dialetheism. But even they cannot, in the end, prise themselves away from the knee-jerk reaction. On p. 272, they say that dialetheism is 'not for those of a nervous disposition'. Even these two excellent philosophers, then, suffer from a failure of nerve.

[21] I return to the Schema as formulated with sets. It is obvious how to modify the following in such a way as to talk only of conditions.

- $\delta(x) \notin x$
- $\delta(x) \in \Omega$

Beyond the Limits of Thought calls this the Inclosure Schema.[22] The conditions of the Inclosure Scheme generate a contradiction at the limit, Ω, as before. But now König's paradox fits the Schema, as just observed. (Ω is obviously definable.) But the paradoxes of absolute infinity fit the Schema too. For these, the condition $\psi(x)$ is just the vacuous condition $x = x$. Indeed it can be shown that all the standard paradoxes of self-reference fit the Schema. Thus, for example, in the liar paradox, $\varphi(y)$ is 'y is true', $\psi(x)$ is 'x is definable', and $\delta(x)$ is a sentence, a, of the form $\langle a \notin \dot{x} \rangle$ (where $\dot{x}$ is a name for x). In Berry's paradox, $\varphi(y)$ is 'y is a natural number definable by a noun-phrase of less than 100 words', $\psi(x)$ is 'x is a set of natural numbers definable by a noun-phrase of less than 90 words', and $\delta(x)$ is the least y such that $y \notin \dot{x}$. Indeed, *Beyond the Limits of Thought* argues that it is the Inclosure Scheme which characterises the structure of all the paradoxes of self-reference.[23]

Concepts satisfying the restricted form of Transcendence and Closure obviously have the same sort of "self-reproductive" feature that Russell and Dummet had in mind in the quotations in section 1. Which form of the conditions provides the correct definition of being IE, one might ask: the restricted form or the unrestricted form? It seems to me pointless to argue about this. In the end, the matter may just be one of linguistic fiat. The sensible thing to do, it seems to me, is to define two different species of IE concept—or better, a more general and a less general one. I will call any concept that satisfies the unrestricted Transcendence and Closure conditions (that is, what I have till now been calling 'IE') *strongly IE* (SIE), and any that fit the restricted conditions *weakly IE* (WIE).

The extensions of WIE concepts may not have the same properties as those of SIE concepts. There is no reason, for example, why they should be absolutely infinite. Indeed, clearly they may not be: the set of definable ordinals in König's paradox is countable.[24] This fact provides the fourth

[22] Priest (2002), e.g., pp. 133-6.

[23] Priest (2002), Part 3.

[24] Shapiro and Wright (2006), p. 262, argue that Berry's paradox does not deliver an example of an IE notion on the ground that the process cannot be iterated into the transfinite. Roughly, if you start of with some finite set of definable natural numbers and add the least indefinable number again and again, by the time you get to ω you have all the natural numbers. This shows that the relevant φ is not SIE, but not that it is not WIE. One should note, also, that their considerations do not apply to König's paradox.

argument against solving the paradoxes of absolute infinity by denying the existence of absolutely infinite sets. All the paradoxes that fit the Inclosure Schema are clearly in the same family. So they ought to have the same sort of solution.[25] But denying the existence of absolutely infinite sets is completely irrelevant to many of the paradoxes. Of course, you could deny the existence of Ω in all cases. But denying the existence of small, even finite, sets (as in Berry's paradox) seems to have little to recommend it.

7. The Gödel Phenomenon

Dummett himself is alive to the possibility that not all IE sets are absolutely infinite. He argues that the phenomenon of Gödel's first incompleteness theorem generates a certain kind of IE set.[26] The idea is simple: given any set of provable sentences, the sentence that says of itself that it is not provable is not in the set, but it can be proved. Of course, by 'provable' here, Dummett does not mean 'provable in S' where S is some fixed (consistent) axiomatization; the Gödel sentence for Peano Arithmetic cannot (presumably) be proved in Peano Arithmetic. Dummett is talking about provability in the naive sense.[27]

Dummett was right about this. The concept of being provable in this sense is WIE. Indeed, it fits the Inclosure Schema, as follows. $\varphi(y)$ is 'x is provable'; $\psi(x)$ is 'x is definable'; and $\delta(x)$ is a sentence, a, of the form $\langle a \notin \dot{x} \rangle$.[28] (That x be definable is a necessary condition for constructing such a sentence.) Ω ($= \{x : x \text{ is provable}\}$) is obviously definable. Suppose that x is a definable subset of Ω. If $a \in x$ then a is provable, and so true. Hence, $a \notin x$. But we have just proved this, i.e., a. So $a \in \Omega$.

---

[25] These matters are argued at greater length in Priest (2002), e.g., ch.11.

[26] Dummett (1963).

[27] Shapiro and Wright (2006), 10.4, note the existence of a related phenomenon which is 'something like a process of indefinite extension'. Starting with, say, Peano Aritmetic, we may consider a hierarchy of axiomatic systems generated by, at each stage, adding the Gödel sentence of the stage before, collecting up at limits. The process, as they point out, is not SIE, since at some stage the theory concerned ceases to be axiomatic, and so one cannot diagonalise out of it. One might think that it is WIE. Specifically, we may take $\varphi(y)$ to be 'y is a true sentence of arithmetic' and $\psi(x)$ to be 'x is a consistent axiomatization containing PA', Transcendence and Closure are satisfied. Of course, we do not have a paradox. The inclosure Schema is not satisfied, since we do not have $\psi(\Omega)$: the set of true sentences is not axiomatic.

[28] The situation is essentially the same as that in the case of the Knower paradox. See Priest (2002), 10.2.

Note that Ω is countable. (We may suppose it to be a subset of some fixed language containing the predicate 'is provable'.) Again, it is implausible to deny that it exists. Dummett is well aware of this. He does not take the phenomenon of indefinite extensibility to demonstrate the non-existence of Ω. For him, it shows that one cannot quantify over IE sets using classical logic; the quantifier principles must be intuitionist. In particular, we cannot assume bivalence:[29]

> what the paradoxes [of absolute infinity] revealed was not the existence of concepts with inconsistent extensions, but of what may be called indefinitely extensible concepts. The concept of an ordinal number is a prototypical example. The Burali-Forti paradox ensures that no definite totality comprises everything intuitively recognisable as an ordinal number, where a definite totality is one quantification over which always yields a statement that is determinately true or false.

It should be noted, though, that merely eschewing classical logic in favour of intuitionist logic is not going to solve the paradoxes. Many can be derived using just intuitionistically valid principles.[30]

8. The Ordinals are Countable

What we have seen is that the concept of Indefinite Extesibility makes perfectly good sense from a dialetheic perspective—indeed, it makes better sense than from a classical perspective. Dialetheism does alter the lie of the land, however; and in ways that might not be expected.[31] Let me finish by demonstrating an interesting example of this. König's paradox can be deployed to show that the set of ordinals is countable.

Suppose that there are indefinable ordinals. This entails that the least indefinable ordinal is indefinable. But that ordinal is definable, since it

[29] Dummett (1991), p. 316.

[30] Thus, many paradoxes, such as the Liar, proceed via a step of the form $A \leftrightarrow \neg A$. One can then infer $A \wedge \neg A$ with the intuitionistically acceptable *consequentia mirabilis*: $(A \rightarrow \neg A) \rightarrow \neg A$.

[31] One very stark way in which this happens can be found in Weber (2010). In effect, what he shows it that when φ is IE (weakly or strongly), and $\Omega = \{y : \varphi(y)\}$, any set can be injected into $\{\Omega\}$, in the following sense. $\delta(\Omega) \in \Omega$ and $\delta(\Omega) \notin \Omega$. By extensionality, $\Omega \neq \Omega$. Now consider the function, f, that maps every member of x to Ω. If $y, z \in x$ and $y \neq z$, then $f(y) \neq f(z)$. Note that there are other senses of injection for which this is not the case. The different senses of injection are equivalent in classical logic, but not in relevant logic.

is defined by 'the least indefinable ordinal'. Hence, by contraposition, all ordinals are definable. By standard cardinality considerations, there are countably many definable ordinals; hence there are countably many ordinals. (A countable infinity, since all the natural numbers are (definable) ordinals.)

The argument needs to treated with some care. The sensitive part is where contraposition is applied. Let us write $D\alpha$ for 'α is definable'. Then we have:

(*) $\exists\alpha\neg D\alpha \models \neg D\mu\alpha\neg D\alpha$

Since '$\mu\alpha\neg D\alpha$' defines $\mu\alpha\neg D\alpha$, we have $D\mu\alpha\neg D\alpha$. Contraposition then gives us $\neg\exists\alpha\neg D\alpha$, that is, $\forall\alpha D\alpha$. But is the contraposition valid? Validity is defined in terms of truth-preservation forward; but, in general, it need not preserve falsity backward.[32] It does in this case provided that we define the semantics of μ suitably. $\exists\alpha A(\alpha)$ is true if some ordinal satisfies $A(\alpha)$; it is false if every ordinal satisfies $\neg A(\alpha)$. Suppose that $\exists\alpha A(\alpha)$ is true. (We do not need to worry about how '$\mu\alpha A(\alpha)$' behaves if $\exists\alpha A(\alpha)$ is not true.) If there is an ordinal that satisfies $A(\alpha)$ but not its negation, let $\mu\alpha A(\alpha)$ denote the least such. (This has a reasonable claim to being the least α such that $A(\alpha)$, since if $\beta < \alpha$, $\neg A(\beta)$.) If not, let $\mu\alpha A(\alpha)$ denote the least ordinal satisfying $A(\alpha)$.

Now consider (*). If the premise is true, so is the conclusion. Truth is preserved forwards. If it is not false, then it is true only, and so is some instance. Hence, the conclusion is true only (not false). Falsity is preserved backwards.

The ordinals, then, are countable[33]—with whatever consequences this may have. Of course the ordinals are uncountable as well.[34] The standard proof establishes that. In a dialetheic context, it is to be expected that IE totalities behave inconsistently. This is just a particularly striking example.

Acknowledgements. Versions of this talk were given at the conference *Beyond the Possible*, University of Melbourne, July 2011; the University of Waterloo, October 2011; the conference *Indefinite Extendability*, University of St Andrews, December 2011; and the conference *Trends in Logic XI*, Ruhr University of Bochum, June 2012. I am grateful to those present on

---

[32]Consider the inference $p \models q \vee \neg q$, where q is both true and false, and p is just true.

[33]This presupposes that the definition of μ given is a viable one. If the addition of an operator of this kind to naive set theory gives rise to triviality, then it is a notion like Boolean negation, which is senseless. Whether or not it gives rise to triviality, is not currently known.

[34]For the construction of non-trivial models of ZF in which all ordinals are countable (and uncountable), see Priest (2012a) and (2012b).

these occasions for their comments, and especially to Greg Restall and Zach Weber. Finally, thanks go do an anonymous referee for helpful comments about the framing of this essay.

References

[1] Dummett, M. (1973), The Philosophical Significance of Gödel's Theorem, *Ratio* 5: 140–55.

[2] Dummett, M. (1993), *The Seas of Language* (Oxford: Oxford University Press).

[3] Hallett, M. (1984), *Cantorian Set Theory and Limitation of Size* (Oxford: Oxford University Press).

[4] Priest, G. (1998), What's so Bad about Contradictions?, *Journal of Philosophy* 95:410–26. Reprinted as ch. 1 of G. Priest, J. Beall, and B. Armour-Garb (eds.), *New Essays on the Law of Non-Contradiction* (Oxford: Oxford University Press, 2004).

[5] Priest, G. (2002), *Beyond the Limits of Thought*, 2nd ed. (Oxford: Oxford University Press).

[6] Priest, G. (2006a), *In Contradiction*, 2nd ed. (Oxford: Oxford University Press).

[7] Priest, G. (2006b), *Doubt Truth to be a Liar* (Oxford: Oxford University Press).

[8] Priest, G. (2007), Review of Rayo and Uzquiano (2006), *Notre Dame Philosophical Reviews*, 2007.09.17, http://ndpr.nd.edu/review.cfm?id=11144.

[9] Priest, G. (2012a), A Note on the Axiom of Countability, *Al-Mukhatabat* 1:23–32; http://mukhatabat.unblog.fr/.

[10] Priest, G. (2012b), The Axiom of Countability, pp. 164-9 of *Philosophy, Mathematics, Linguistics: Aspects of Interaction* (St Petersburg: Euler International Mathematical Institute).

[11] Rayo, A., and G. Uzquiano (eds.) (2006), *Absolute Generality* (Oxford: Oxford Unversity Press).

[12] Russell, B. (1905), On Some Difficulties in the Theory of Transfinite Numbers and Order Types, *Proceedings of the London Mathematical Society* 4:29–53. Reprinted as pp. 135–64 of D. Lackey (ed.), *Essays in Analysis* (London: Allen and Unwin, 1973).

[13] Schimmerling, E. (2001), The ABC's of Mice, *Bulletin of Symbolic Logic* 7:485-503.

[14] Shapiro, S., and C. Wright (2006), All Things Indefinitely Extensible, ch. 10 of Rayo and Uzquiano (2006).

[15] Weber, Z. (2010), Transfinite Numbers in Paraconsistent Set Theory, *Review of Symbolic Logic* 3:71–92.

[16] Williamson, T. (2006), Absolute Identity and Absolute Generality, ch. 13 of Rayo and Uzquiano (2006).

Graham Priest
Departments of Philosophy
Universities of Melbourne, St Andrews
and
the Graduate Center, City University of New York
g.priest@unimelb.edu.au