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First-Order da Costa Logic

Abstract. Priest (2009) formulates a propositional logic which, by employing the world-semantics for intuitionist logic, has the same positive part but dualises the negation, to produce a paraconsistent logic which it calls 'Da Costa Logic'. This paper extends matters to the first-order case. The paper establishes various connections between first order da Costa logic, da Costa's own C_ω, and classical logic. Tableau and natural deductions systems are provided and proved sound and complete.

Keywords: Da Costa Logic, quantified intuitionist logic, paraconsistent logic.

1. Introduction

'Dualising Intuitionist Negation'[1] was a paper written to celebrate the outstanding contribution of Newton da Costa to logic and many other spheres of intellectual life. It is a pleasure to add to the celebration in this paper.

Da Costa's C systems of paraconsistent logic, and especially C_ω, can be thought of as taking intuitionist logic, preserving its positive part, but dualising its negation to give a logic with truth value "gluts", rather than "gaps". There are several ways in which one might do something which realises this motivation.[2] DIN gives a very natural way, different from da Costa's, based on the Kripke semantics for intuitionist logic. Essentially, the idea is to replace the truth conditions of negation in a Kripke model:[3]

$$\nu_w(\neg\alpha) = 1 \text{ iff for all } w' \text{ such that } wRw',\ \nu_{w'}(\alpha) = 0$$

by their dual:

$$\nu_w(\neg\alpha) = 1 \text{ iff for some } w' \text{ such that } w'Rw,\ \nu_{w'}(\alpha) = 0$$

DIN calls the result *da Costa logic.*

DIN deals only with propositional da Costa logic, but in Section 3 of that paper I noted that the technique employed there could be extended

[1] Priest (2009). Hereafter, DIN.

[2] See, e.g., Brunner and Carnielli (2005), Querioz (1998), Urbas (1996).

[3] The notation is that of DIN.

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to quantifiers. This note spells out the details. A first-order version of the propositional semantics is, in fact, given by Rauszer (1977). However, unlike Kripke interpretations for intuitionist logic, this uses constant domains, and hence verifies principles that are not valid in positive intuitionist logic, notably the confinement principle: $\forall x(Px \vee Qa) \vdash \forall x Px \vee Qa$.[4]

First, I will specify a Kripke semantics for first-order da Costa logic. Next, I will give a tableau system which is sound and complete with respect to the semantics. This is then used to establish various facts about the logic, its relationship to classical logic, C_ω, and intuitionist logic. Finally, I give a natural deduction system which is sound and complete with respect to the semantics.[5]

2. Kripke Semantics

Let us start with the Kripke semantics for intuitionist logic. These are as in 20.3 of *Introduction to Non-Classical Logic*[6]—though I have made two necessary changes to the presentation there. First, I have included a designated world, @. The semantics (and definition of validity) of INCL could have been formulated equivalently in this way. In INCL every constant of the language is required to denote something in the domain of every world. This is no longer appropriate once we move to da Costa logic, since constants that denote something at the "base world" may denote things that do not exist once we move backwards ("in time") down the relation R. This makes the alternative approach mandatory. Secondly, I have required that $\delta_w(P) \subseteq D_w$, not D. The constraint is standard in formulations of intuitionistic logic. (If one has established that an object, a, has a property, then a must have already been constructed.) However, it makes no difference to the valid inferences.[7] It does in the present case, as we shall have occasion to note.

A Kripke semantics for first-order intuitionist logic is as follows. An interpretation is a structure $\langle W, @, R, D, \delta\rangle$, where W is a non-empty set of worlds, and @ is a distinguished member of W. R is a reflexive and tran-

[4]Rauszers' logic also contains the dual of the intuitionist $\rightarrow$. This can be added to the logic of the present paper as well, as in DIN, Section 7.

[5]DIN also investigates the algebraic semantics for propositional da Costa logic. These extend naturally to a semantics for first-order da Costa logic in terms of complete da Costa algebras (the duals of complete Heyting algebras). I leave the details of this for another occasion.

[6]Priest (2006), hereafter INCL.

[7]See INCL, 20.13, Problem 12.

sitive relation on W. For every $w \in W$, D_w is a collection of objects, and $D_@ \neq \phi$. $D = \bigcup\{D_w : w \in W\}$. δ is a function such that for every constant in the language, a, $\delta(a) \in D_@$;[8] and for every world, w, and n-place predicate, P, $\delta_w(P) \subseteq D_w$. (This is the so called *Negativity Constraint*, since it is the hallmark of negative free logics.[9]) The structure satisfies the following conditions, for all w, w', and P:

if wRw' then $D_w \subseteq D_{w'}$

if wRw' then $\nu_w(P) \subseteq \nu_{w'}(P)$

A truth value, $\nu_w(\alpha) \in \{1, 0\}$, is assigned to every formula, α, at every world, w, by the following truth conditions:[10]

$\nu_w(Pa_1 \dots a_n) = 1$ iff $\langle \delta(a_1), \dots, \delta(a_n) \rangle \in \delta_w(P)$

$\nu_w(\alpha \wedge \beta) = 1$ iff $\nu_w(\alpha) = 1$ and $\nu_w(\beta) = 1$

$\nu_w(\alpha \vee \beta) = 1$ iff $\nu_w(\alpha) = 1$ or $\nu_w(\beta) = 1$

$\nu_w(\alpha \rightarrow \beta) = 1$ iff for all w' such that wRw', if $\nu_{w'}(\alpha) = 1$ then $\nu_{w'}(\beta) = 1$

$\nu_w(\neg\alpha) = 1$ iff for all w' such that wRw', $\nu_{w'}(\alpha) = 0$

$\nu_w(\exists x\alpha) = 1$ iff for some $d \in D_w$, $\nu_w(\alpha_x(k_d)) = 1$

$\nu_w(\forall x\alpha) = 1$ iff for all w' such that wRw', and all $d \in D_{w'}$, $\nu_{w'}(\alpha_x(k_d)) = 1$

The truth conditions and the constraints on the structure ensure the *Heredity Condition*:

if wRw' and $\nu_w(\alpha) = 1$ then $\nu_{w'}(\alpha) = 1$

for all w and α. (The proof is by a simple induction over the construction of formulas.)

Validity is defined in terms of truth preservation at @ in all interpretations:

$\Sigma \models \alpha$ iff, for all interpretations, if $\nu_@(\beta) = 1$ for all $\beta \in \Sigma$, $\nu_@(\alpha) = 1$

---

[8] The truth conditions of quantifiers below require further constants. The denotations of these constants have to be in D, but not $D_@$.

[9] INCL, 13.4.2.

[10] k_d is a constant such that $\delta(k_d) = d$, and $\alpha_x(a)$ is α with all occurrence of x replaced by a.

The semantics for da Costa logic are exactly the same as those for intuitionist logic, except that the truth conditions for negation are replaced by their dual:

$\nu_w(\neg\alpha) = 1$ iff for some w' such that $w'Rw$, $\nu_{w'}(\alpha) = 0$

It is easy to establish that the Heredity Condition is still forthcoming, and hence that the logic is closed under uniform substitution (since any formula then behaves as does a propositional parameter).[11]

Da Costa logic and intuitionist logic have the same positive part; and the negation of each conservatively extends this (since any positive interpretation can be extended trivially to an interpretation for either kind of negation—or both). The propositional part of da Costa logic is as in DIN. The novelty of first-order da Costa logic is therefore in the interplay between negation and the quantifiers. We will consider this in a moment. First, it will help to have a tableau system for the logic and its associated machinery.

3. Tableaux

A tableau system for intuitionist logic is as follows.[12] There is a designated existence predicate, $\mathfrak{E}$, thought of as applying at world w, to the things in D_w. Lines are of the form $\alpha, +i$, $\alpha, -i$ or irj. A tableau for the inference $\beta_1, \ldots, \beta_n/\alpha$ starts with lines of the form $\beta_1, +0, \ldots, \beta_n, +0, \alpha, -0$, as well as $\mathfrak{E}a, +0$, for every constant, a, occurring in the premises and conclusion—or one of the form $\mathfrak{E}c, +0$ if there are none such.

A branch of a tableau closes if it contains lines of the form $\alpha, +i$ and $\alpha, -i$. The tableau itself closes if all branches do. The rules for the connectives are as follows:

$$\begin{array}{c} \alpha \wedge \beta, +i \\ \downarrow \\ \alpha, +i \\ \beta, +i \end{array} \qquad\qquad \begin{array}{ccc} & \alpha \wedge \beta, -i & \\ \swarrow & & \searrow \\ \alpha, -i & & \beta, -i \end{array}$$

[11]The proof is by induction over the construction of formulas. The cases for the connectives are as in DIN. The case for the particular quantifier is left as an exercise. Here is the case for the universal quantifier. Suppose that wRw' and $\nu_{w'}(\forall x\alpha) \neq 1$. Then for some w'' such that $w'Rw''$, and some $d \in D_{w''}$, $\nu_{w''}(\alpha_x(k_d) = 0$. Since R is transitive, $\nu_w(\forall x\alpha) = 0$.

[12]This corresponds to the tableaux of type 1, INCL, 20.4. 20.5 gives an alternative tableau system, type 2, which does not employ an existence predicate. It is harder to formulate type 2 tableaux in the present case, since, because of the Negativity Constraint and the "backward looking" nature of negation, D_w is determined by more than just the quantifier rules.

$$
\begin{array}{ccc}
& \alpha \vee \beta, +i & \\
\swarrow & & \searrow \\
\alpha, +i & & \beta, +i
\end{array}
\qquad\qquad
\begin{array}{c}
\alpha \vee \beta, -i \\
\downarrow \\
\alpha, -i \\
\beta, -i
\end{array}
$$

$$
\begin{array}{ccc}
& \alpha \to \beta, +i & \\
& irj & \\
\swarrow & & \searrow \\
\alpha, -j & & \beta, +i
\end{array}
\qquad\qquad
\begin{array}{c}
\alpha \to \beta, -i \\
\downarrow \\
irk \\
\alpha, +k \\
\beta, -k
\end{array}
$$

$$
\begin{array}{c}
\neg\alpha, +i \\
irj \\
\downarrow \\
\alpha, -j
\end{array}
\qquad\qquad
\begin{array}{c}
\neg\alpha, -i \\
\downarrow \\
irk \\
\alpha, +k
\end{array}
$$

where k, in any of these rules, is a number new to the branch. We also have rules for r and the Heredity Condition:

$$
\begin{array}{c}
. \\
\downarrow \\
iri
\end{array}
\qquad
\begin{array}{c}
irj \\
jrk \\
\downarrow \\
irk
\end{array}
\qquad
\begin{array}{c}
irj \\
Pa_1 \ldots a_n, +i \\
\downarrow \\
Pa_1 \ldots a_n, +j
\end{array}
$$

where P is any predicate (including $\mathfrak{E}$). For quantifiers, the rules are as follows:

$$
\begin{array}{c}
\exists x A, +i \\
\downarrow \\
\mathfrak{E}c, +i \\
A_x(c), +i
\end{array}
\qquad\qquad
\begin{array}{c}
\forall x A, -i \\
\downarrow \\
irk \\
\mathfrak{E}c, +k \\
A_x(c), -k
\end{array}
$$

$$
\begin{array}{ccc}
& \exists x A, -i & \\
\swarrow & & \searrow \\
\mathfrak{E}a, -i & & A_x(a), -i
\end{array}
\qquad\qquad
\begin{array}{ccc}
& \forall x A, +i & \\
& irj & \\
\swarrow & & \searrow \\
\mathfrak{E}a, -j & & A_x(a), +j
\end{array}
$$

c and k are new to their branches. a is any constant on the branch.

For da Costa logic, we simply replace the rules for negation with:

$$\begin{array}{ccc} \neg\alpha, +i & \qquad & \neg\alpha, -i \\ \downarrow & & jri \\ kri & & \downarrow \\ \alpha, -k & & \alpha, +j \end{array}$$

where k is new to the branch; and add the Negativity Rule:

$$\begin{array}{c} Pa_1 \ldots a_n, +i \\ \downarrow \\ \mathfrak{E}a_1, +i \\ \vdots \\ \mathfrak{E}a_n, +i \end{array}$$

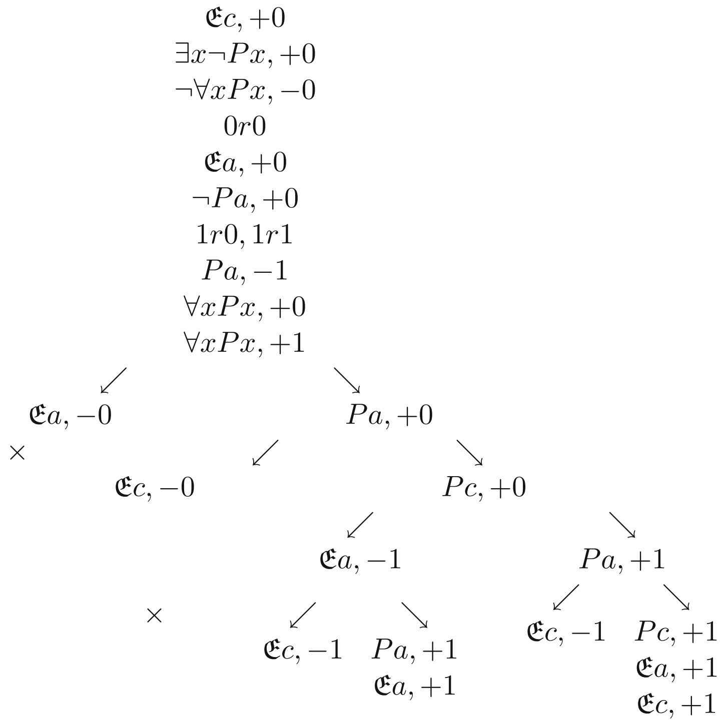
where P is any predicate (except $\mathfrak{E}$, for which the rule is redundant).

Example 1. Here is a tableau to show that $\neg\exists xPx \vdash \forall x\neg Px$

$$\begin{array}{c} \mathfrak{E}c, +0 \\ \neg\exists xPx, +0 \\ \forall x\neg Px, -0 \\ 0r0 \\ 1r0, 1r1 \\ \exists xPx, -1 \\ 0r2, 2r2, 1r2 \\ \mathfrak{E}a, +2 \\ \neg Pa, -2 \\ Pa, +2 \\ Pa, +0 \\ Pa, +1 \\ \mathfrak{E}a, +1 \\ \mathfrak{E}a, +0 \\ \swarrow \qquad \searrow \\ \mathfrak{E}a, -1 \quad Pa, -1 \\ \times \qquad\quad \times \end{array}$$

Lines 5 and 6 follow from line 2. Lines 7, 8, and 9 follow from line 3. The next three lines follow from line 9 and the facts about r. The next two lines follow from the Negativity Rule. The lines after the split follow from line 6. Note that without the Negativity Rule, the left branch of the tableau would not close. An interpretation can be read off from the open branch, thus modified, demonstrating that the inference is invalid without the Negativity Constraint, which shows that the constraint affects validity in a way that it does not in intuitionist logic. Details are left as an exercise.

Example 2. Here is another tableau, to show that: $\exists x \neg Px \nvdash \neg\forall x Px$



Lines 5 and 6 follow from line 2. Lines 7 and 8 follow from line 6. Lines 9 and 10 follow from line 3. The various splits then follow from lines 9 and 10. The final formulas concerning $\mathfrak{E}$ on the right hand branches follow by the Negativity Rule.

Given an open tableau, a countermodel is read off from an open branch as follows. There is a world, w_i, for every i on the branch. @ is w_0. $w_i R w_j$ iff irj is on the branch. $\delta(b) = \delta_b$, and $\langle \delta(a_1), \ldots, \delta(a_n) \rangle \in \delta_{w_i}(P)$ iff $Pa_1 \ldots a_n, i$ occurs on the branch. $D_{w_i} = \delta_{w_i}(\mathfrak{E})$.

In the countermodel determined by the left hand open branch of the above tableau: $W = \{w_0, w_1\}$, $@ = w_0$, $w_0 R w_0$, $w_1 R w_1$, and $w_1 R w_0$. $\delta_{w_0}(\mathfrak{E}) = D_{w_0} = \{\partial_a, \partial_c\}$ and $\delta_{w_1}(\mathfrak{E}) = D_{w_1} = \phi$. $\delta(a) = \partial_a$, $\delta(c) = \partial_c$, $\delta_{w_0}(P) = \{\partial_a, \partial_c\}$ and $\delta_{w_1}(P) = \phi$. We may depict the interpretation as follows.

$\overset{\frown}{w}_1 \quad \rightarrow \quad \overset{\frown}{w}_0\ (= @)$

	∂_a	∂_c
P	$\times$	$\times$
$\mathfrak{E}$	$\times$	$\times$

	∂_a	∂_c
P	$\surd$	$\surd$
$\mathfrak{E}$	$\surd$	$\surd$

Pa fails at w_1, so $\neg Pa$ holds at w_0. Since ∂_a exists at w_0, $\exists x \neg Px$ is true at w_0. $\forall x Px$ is true at w_1 and w_0; hence $\neg \forall x Px$ fails at w_0.

The tableau system is sound and complete with respect to the semantics. The proof is a straightforward modification of that given for intuitionist tableaux of type 1 in INCL. We establish two lemmas.[13]

Locality Lemma Let $I_1 = \langle D, W, R, \nu_1 \rangle$, $I_2 = \langle D, W, R, \nu_2 \rangle$ be two interpretations. Let the language of the interpretations be L. If α is any closed formula of L such that ν_1 and ν_2 agree on the denotations of all the predicates and constants in it, then, for all $w \in W$:

$$\nu_{1w}(\alpha) = \nu_{2w}(\alpha)$$

Denotation Lemma Let $I = \langle D, W, R, \nu \rangle$ be any interpretation. Let its language be L. Let α be any formula of L with at most one free variable, x, and a and b be any two constants such that $\nu(a) = \nu(b)$. Then, for all $w \in W$:

$$\nu_w(\alpha_x(a)) = \nu_w(\alpha_x(b))$$

The proofs of the lemmas are as in INCL 20.9.2 and 20.9.3, except that the cases for $\neg$ are different in the obvious ways. The Soundness and Completeness proofs are then as in 23.9.4-23.9.9, except that (i) the clauses for negation in the Soundness and Completeness Lemmas are as in DIN, Section 3; and (ii) in the induced interpretation, $@ = w_0$, and the interpretation satisfies the Negativity Constraint because of the Negativity Rule.

4. The Properties of da Costa Logic

Let us now turn to the properties of first-order da Costa logic. We start with the relationship between da Costa logic and C_ω.

[13]The language of an interpretation is the language augmented by a constant, k_d, for each $d \in D$.

DIN, Section 2, shows that propositional da Costa logic properly extends propositional C_ω. Axiomatically, first-order C_ω is obtained from propositional C_ω by adding the following axioms and rules:[14]

$\vdash \alpha_x(c) \to \exists x\alpha$

$\vdash \forall x\alpha \to \alpha_x(c)$

If $\vdash \beta \to \alpha_x(c)$ then $\vdash \beta \to \forall x\alpha$

If $\vdash \alpha_x(c) \to \beta$ then $\vdash \exists x\alpha \to \beta$

where c does not occur in β. We may establish that these are valid/validity-preserving in da Costa logic as follows.

For the first axiom, suppose that $\nvDash \alpha_x(c) \to \exists x\alpha$. Then there is an interpretation, I, such that $\alpha_x(c) \to \exists x\alpha$ fails at @. Hence, for some w such that $@Rw$, $\alpha_x(c)$ holds at w, and $\exists x\alpha$ does not. Then for every $d \in D_w$, $\alpha_x(k_d)$ fails at w. But $\delta(c) \in D_@$, and so $\delta(c) \in D_w$. Where $d = \delta(c)$, the Denotation Lemma therefore entails that $\alpha_x(c)$ fails at w. Contradiction.[15]

For the first rule, suppose that $\nvDash \beta \to \forall x\alpha$. Then there is an interpretation where $\beta \to \forall x\alpha$ fails at @. So there is a w such that $@Rw$, β is true at w, and $\forall x\alpha$ fails at w. Hence, for some w' such that wRw' and some $d \in D_{w'}$, $\alpha_x(k_d)$ fails at w'. By heredity, β holds at w'. Now take an interpretation which is the same as I, except that $\delta(c) = d$. In this interpretation, by the Denotation Lemma and the Locality Lemma, β holds at w' and $\alpha_x(c)$ fails at w'. Since $@Rw'$, $\beta \to \alpha_x(c)$ fails at @ in this interpretation. Hence it is not logically valid.

The cases for the second axiom and the second rule are similar. Consequently, first-order da Costa logic (properly) extends first-order C_ω.

Next, classical logic. Any inference valid in first-order da Costa logic is valid in classical logic (since a classical countermodel is, effectively, a one-world da Costa interpretation). The converse is not true, since da Costa logic is paraconsistent. Despite this, the logical truths of the $\to$-free fragment of propositional da Costa logic coincide with the $\to$-free logical truths of classical logic, as proved in DIN, Section 2.

However, for the first-order case, not even the logical truths of the $\forall$-$\vee$-$\wedge$-$\neg$ or the $\exists$-$\vee$-$\wedge$-$\neg$ fragments are identical with those of classical logic. The

[14]Da Costa (1974: 503).

[15]Note that the fact that $\alpha_x(a)$ is true at a world does not, in general, entail that $\exists x\alpha$ is true there. Take an interpretation with two worlds, w and @, where $wR@$, wRw, $@R@$, $D_@$ contains just the the denotation of a, and D_w is empty. Then $\neg Pa$ is true at w, but $\exists x\neg Px$ is not true there.

first of these is perhaps not so surprising given the domain-shift in the truth conditions of $\forall$. Indeed, a variation of the confinement principle fails: here is a countermodel to show that the classically valid $\neg\forall x(Pa \vee Qx) \vee (Pa \vee \forall xQx)$ is not valid in da Costa logic:

$\overset{\frown}{w}_0$ (= @) $\rightarrow$ $\overset{\frown}{w}_1$

$\overset{\frown}{w}_0$ (= @):

	∂_a	∂_b
P	$\times$	$\times$
Q	$\surd$	$\times$
$\mathfrak{E}$	$\surd$	$\times$

$\overset{\frown}{w}_1$:

	∂_a	∂_b
P	$\surd$	$\times$
Q	$\surd$	$\times$
$\mathfrak{E}$	$\surd$	$\surd$

Pa fails at w_0, and $\forall xQx$ fails at w_0 (since Qb fails at w_1). Hence, $Pa \vee \forall xQx$ fails at w_0. But $Pa \vee Qa$ holds at w_0 and w_1, and $Pa \vee Qb$ holds at w_1. Hence $\forall x(Pa \vee Qx)$ holds at w_0, so $\neg\forall x(Pa \vee Qx)$ fails at w_0 too.

For the second, here is a countermodel to show that $\nvDash \neg\exists x(Px \wedge \neg Px)$, which holds in classical logic:

$\overset{\frown}{w_0}$(= @)

	∂_{a_0}	∂_{a_1}	∂_{a_2}	$\dots$
P	$\surd$	$\surd$	$\surd$	$\dots$
$\mathfrak{E}$	$\surd$	$\surd$	$\surd$	$\dots$

$\uparrow$

$\overset{\frown}{w_1}$

	∂_{a_0}	∂_{a_1}	∂_{a_2}	$\dots$
P	$\times$	$\surd$	$\surd$	$\dots$
$\mathfrak{E}$	$\surd$	$\surd$	$\surd$	$\dots$

$\uparrow$

$\overset{\frown}{w_2}$

	∂_{a_0}	∂_{a_1}	∂_{a_2}	$\dots$
P	$\times$	$\times$	$\surd$	$\dots$
$\mathfrak{E}$	$\surd$	$\surd$	$\surd$	$\dots$

$\vdots$

$Pa_0 \wedge \neg Pa_0$ holds at w_0; $Pa_1 \wedge \neg Pa_1$ holds at w_1; and so on. Hence, $\exists x(Px \wedge \neg Px)$ holds at every world, and so $\neg\exists x(Px \wedge \neg Px)$ fails at w_0.

Finally, the relationship between the quantifiers and negation in da Costa logic. Consider the following:

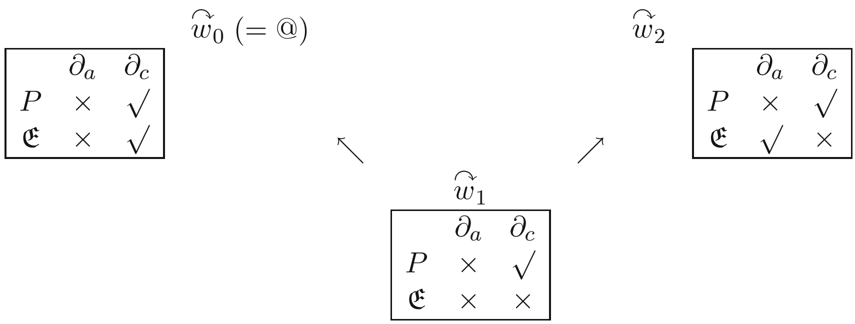
1. $\exists x\neg\alpha \vDash \neg\forall x\alpha$
2. $\neg\forall x\alpha \vDash \exists x\neg\alpha$

3. $\forall x \neg \alpha \vDash \neg \exists x \alpha$

4. $\neg \exists x \alpha \vDash \forall x \neg \alpha$

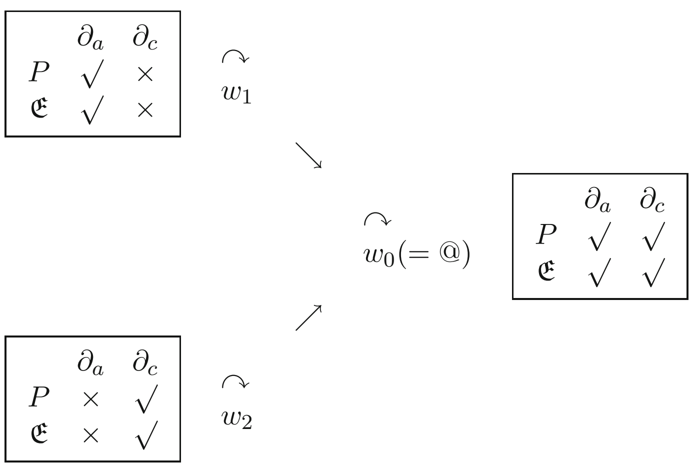
In intuitionist logic, 1, 3, and 4 are true; 2 is not. One might have expected that in da Costa logic, the relationship would be dual: that we would have 3 and 4, and 2 but not 1—or maybe 1 and 2, but only one of 3 and 4. But in fact, all of 1-4 fail (though a special case of 4 holds, as we have seen in Example 1 of the previous section). We have already seen that 1 is invalid in Example 2 of the previous section. Here are countermodels for 2, 3, and 4.

For 2, $\neg \forall x Px \nvDash \exists x \neg Px$:



Because of w_2, $\forall x Px$ fails at w_1; hence $\neg \forall x Px$ holds at w_0. Pc holds at w_0 and w_1; hence $\neg Pc$ fails at w_0, as, therefore, does $\exists x \neg Px$.

For 3, $\forall x \neg Px \nvDash \neg \exists x Px$:



$\neg Pa$ and $\neg Pc$ hold at w_0, as, then, does $\forall x \neg Px$. $\exists x Px$ is true at all worlds; hence, $\neg \exists x Px$ fails at w_0.

Finally, and a bit less obviously, for 4, $\neg\exists x\neg Px \not\models \forall x\neg\neg Px$:

$$\begin{array}{ccc} \curvearrowright & & \curvearrowright \\ w_1 & \longrightarrow & w_0(=@) \\ \boxed{\begin{array}{cc} & \partial_a \\ P & \times \\ \mathfrak{E} & \times \end{array}} & & \boxed{\begin{array}{cc} & \partial_a \\ P & \times \\ \mathfrak{E} & \surd \end{array}} \end{array}$$

$\exists x\neg Px$ fails at w_1 (since nothing exists there). Hence $\neg\exists x\neg Px$ holds at w_0. But Pa fails at w_1. Hence $\neg Pa$ holds at w_1 and w_0; and so $\neg\neg Pa$ fails at w_0. Consequently, $\forall x\neg\neg Px$ fails at w_0.

5. Natural Deduction

Tableau proof systems are very good for determining validity and invalidity, but for many purposes a natural deduction system is better—especially since the tableau system given above makes use of an existence predicate, which is alien to most presentations of intuitionist logic. So let us, finally, see how a natural-deduction system for first-order da Costa logic may be obtained from one for positive intuitionist logic. The system is in the style of Prawitz. The rules for positive intuitionist logic are as follows:[16]

$$\frac{\alpha\wedge\beta}{\alpha} \qquad \frac{\alpha\wedge\beta}{\beta} \qquad \frac{\alpha \quad \beta}{\alpha\wedge\beta}$$

$$\frac{\alpha}{\alpha\vee\beta} \qquad \frac{\beta}{\alpha\vee\beta} \qquad \frac{\alpha\vee\beta \quad \begin{array}{c}\overline{\alpha}\\ \vdots \\ \gamma\end{array} \quad \begin{array}{c}\overline{\beta}\\ \vdots \\ \gamma\end{array}}{\gamma}$$

$$\frac{\alpha \quad \alpha\to\beta}{\beta} \qquad \frac{\begin{array}{c}\overline{\alpha}\\ \vdots \\ \beta\end{array}}{\alpha\to\beta}$$

$$\frac{\forall x\alpha}{\alpha_x(a)} \qquad \frac{\alpha_x(a)}{\forall x\alpha}$$

[16]See, e.g., Tennant (1978), ch. 4.

In the second rule for $\forall$, a does not occur in α or in any undischarged assumption on which $\alpha_x(a)$ depends.

$$\frac{\alpha_x(a)}{\exists x\alpha} \qquad \frac{\exists x\alpha \qquad \begin{array}{c}\overline{\alpha_x(a)}\\ \vdots \\ \beta\end{array}}{\beta}$$

In the second rule for $\exists$, a does not occur in α, β, or in any undischarged assumption on which β depends (other than $\alpha_x(a)$).

For da Costa logic, we add to these the following rules for negation:[17]

$$\frac{.}{\alpha \vee \neg\alpha} \qquad \frac{\alpha \vee \beta \qquad \neg\alpha}{\beta}$$

In the second of these, the deduction of $\alpha \vee \beta$ has no undischarged assumptions.

The deduction system is sound and complete with respect to the semantics. For soundness, we establish that if we have a deduction of α with undischarged assumptions Σ then $\Sigma \vDash \alpha$. This is established by a recursion over the structure of proofs. The cases for the propositional operators are essentially as in DIN, Section 5. The cases for the first of each pair of quantifier rules are straightforward. Here are the cases for the second of each pair.

Suppose that we have a proof of $\alpha_x(a)$ with undischarged assumptions Σ. By induction hypothesis, $\Sigma \vDash \alpha_x(a)$. Consider any interpretation, I, in which all members of Σ are true at @. Let $d \in D_@$, and let I' be the interpretation of the language which is the same as I, except that $\delta(a) = d$. By the Locality Lemma, all the members of Σ are true at @ in this interpretation. Hence $\alpha_x(a)$ is true at @ in I'. By the Denotation Lemma, $\alpha_x(k_d)$ is true at @. Hence, $\forall x\alpha$ is true at @ in I'. By the Locality Lemma, $\forall x\alpha$ is true at @ in I, as required.

Suppose that we have a proof of $\exists x\alpha$ with undischarged assumptions Π_1, and a proof of β with undischarged assumptions $\Pi_2 \cup \{\alpha_x(a)\}$ (where Π_2 does not contain $\alpha_x(a)$). By induction hypothesis, $\Pi_1 \vDash \exists x\alpha$ and $\Pi_2 \cup \{\alpha_x(a)\} \vDash \beta$. Now suppose that all members of $\Pi_1 \cup \Pi_2$ are all true at @ in some interpretation, I. Then $\exists x\alpha$ is true at @, and so for some $d \in D_@$, $\alpha_x(k_d)$ is true at @. Let I' be the same as I, except that $\delta(a) = k_d$. Then by the Locality Lemma, all members of Π_2 are true at @ in I', and by

[17] DIN, Section 5.

the Denotation Lemma $\alpha_x(a)$ is too. Hence β is true at @ in I'. By the Locality Lemma, β is true at @ in I, as required.

The Completeness proof is an extension of that for the propositional case in DIN, Section 5. Call a set of sentences, Σ, *saturated* in a set of constants, C, iff whenever $\exists x\alpha \in \Sigma$, $\alpha_x(c) \in \Sigma$, for some $c \in C$. Call Σ *prime deductively closed and saturated* (*pdcs*) *in* C iff it is prime (that is, if $\alpha \vee \beta \in \Sigma$ then $\alpha \in \Sigma$ or $\beta \in \Sigma$), deductively closed, and saturated in C.

Fix some uncountable set of constants, K. We now have the following fact:[18]

LEMMA 1 (Fundamental Lemma). *If* $\Sigma \nvdash \Pi$ *there is set* Δ*, pdcs in a countable subset of* K*, such that* $\Sigma \subseteq \Delta$ *and* $\Delta \nvdash \Pi$.

PROOF. Let C be a countable subset of K disjoint from the constants in $\Sigma \cup \Pi$. Enumerate the formulas of the language of $\Sigma \cup \Pi$ augmented by C: $\alpha_0, \alpha_1, \ldots$ Define the following sequence of sets by recursion:

$\Delta_0 = \Sigma$

$\Delta_{n+1} = \Delta_n \cup \{\alpha_n\}$ if $\Delta_n \cup \{\alpha_n\} \nvdash \Pi$, and Δ_n otherwise.

$\Delta = i < \omega \bigcup \Delta_i$

As in the propositional case, Δ is prime, deductively closed, and $\Delta \nvdash \Pi$. Δ is also saturated in C. For suppose otherwise. Then for some $\exists x\beta \in \Delta$, there is no $c \in C$ such that such that $\beta_x(c) \in \Delta$. Let $\exists x\beta$ be α_i. Only a finite number of members of C occur in Δ_{i+1}. Let c be one that does not. Then $\beta_x(c) \notin \Delta$. $\beta_x(c)$ is α_j for some $j > i$. Hence $\Delta_j \cup \{\beta_x(c)\} \vdash \Pi$. So $\Delta_j \cup \{\exists x\beta\} \vdash \Pi$. Since $\exists x\beta \in \Delta_j$, $\Delta_j \vdash \Pi$, which is not the case. ■

Now suppose that $\Theta \nvdash \alpha$. We define a countermodel as follows. $W = \{\Delta :$ for some countable $C \subseteq K$, Δ is pdcs in $C\}$. $\Pi R \Gamma$ iff $\Pi \subseteq \Gamma$. If $\Pi \in W$, let C_Π be the set of constants occurring in Π. (Clearly, Π is saturated in C_Π.) $D_\Pi = C_\Pi$. $\delta(a) = a$, $\delta_\Pi(P) = \{\langle a_1, ..., a_n\rangle : Pa_1 \ldots a_n \in \Pi\}$. Finally, by the Fundamental Lemma, there is some Δ such that $\Theta \subseteq \Delta$ and $\Delta \nvdash \alpha$ (so that $\alpha \notin \Delta$). $@ = \Delta$.

It is easy to check that this structure is an interpretation for da Costa logic. The only point that is not immediate concerns the domain-increasing condition. Suppose that $c \in C_\Pi$ and $\Pi R \Delta$. Let $\forall x\alpha$ be any logical truth without constants. Then $\alpha_x(c) \in \Pi$. Since $\Pi \subseteq \Gamma$, $\alpha_x(c) \in \Gamma$. Hence, $D_\Pi \subseteq D_\Gamma$.

[18] $\Sigma \vdash \Pi$ means that for some $n \geq 1$, and $\pi_1, \ldots, \pi_n \in \Pi$, $\Sigma \vdash \pi_1 \vee \ldots \vee \pi_n$.

We now need one more lemma:

LEMMA 2 ($\forall$-*Lemma*). *If* $\Gamma \in W$ *and* $\forall x\pi \notin \Gamma$, *then there is a* $\Delta \in W$ *such that* $\Gamma R\Delta$, *and for some* $c \in D_\Delta$, $\pi_x(c) \notin \Delta$.

PROOF. Let $\Sigma = \Gamma$. Let c be some constant not in $\Sigma \cup \{\pi\}$. Without loss of generality, we can take it to be the first constant in the set C used in the proof of the Fundamental Lemma. Let $\Pi = \{\pi_x(c)\}$. Now, $\Sigma \nvdash \Pi$. For suppose that $\Sigma \vdash \pi_x(c)$. Then since c has no occurrences in $\Sigma \cup \{\pi\}$, $\Sigma \vdash \forall x\pi$, and so $\forall x\pi \in \Sigma$, which it is not. So by the Fundamental Lemma, there is a $\Delta \in W$, with $c \in D_\Delta$, such that $\Sigma \subseteq \Delta$ and $\Delta \nvdash \alpha_x(c)$. This has the required properties. ■

To finish the proof, we need to show that for any $\Pi \in W$, and any β in the language of Π:

$\nu_\Pi(\beta) = 1$ iff $\beta \in \Pi$

The theorem then follows, since the interpretation defined is a countermodel.

The proof is by recursion. The atomic case is true by definition. The cases for the propositional connectives are as in DIN, Section 5. This leave the quantifiers.

- If $\nu_\Pi(\exists x\pi) = 1$ then for some $c \in C_\Pi$, $\nu_\Pi(\pi_x(k_c)) = 1$. But $\delta(k_c) = c$. So by the Denotation Lemma, $\nu_\Pi(\pi_x(c)) = 1$. By IH, $\pi_x(c) \in \Pi$. Hence $\exists x\pi \in \Pi$, by deductive closure.
- Conversely, suppose that $\exists x\pi \in \Pi$. Then for some $c \in C_\Pi$, $\pi_x(c) \in \Pi$. By IH, $\nu_\Pi(\pi_x(c)) = 1$. By the Denotation Lemma, $\nu_\Pi(\pi_x(k_c)) = 1$; so $\nu_\Pi(\exists x\pi) = 1$.
- Suppose that $\forall x\pi \in \Pi$. Then for all Γ such that $\Pi R\Gamma$, $\forall x\pi \in \Gamma$. By deductive closure, for all $c \in C_\Gamma$, $\pi_x(c) \in \Gamma$. By IH and the Denotation Lemma, for every $c \in D_\Gamma$, $\nu_\Gamma(\gamma_x(k_c)) = 1$. Hence, $\nu_\Pi(\forall x\pi) = 1$.
- Conversely, suppose that $\forall x\pi \notin \Pi$. The by the $\forall$-Lemma, there is a $\Delta \in W$ such that $\Pi R\Delta$ and for some $c \in D_\Delta$ $\pi_x(c) \notin \Delta$. By IH and the Denotation Lemma, $\nu_\Delta(\pi_x(k_c)) \neq 1$. So $\nu(\forall x\pi) \neq 1$.

6. Conclusion

Da Costa logic is a smooth logic which extends positive intuitionist logic in a natural way, both semantically and proof-theoretically. The first-order version extends the propositional version exactly as one would expect. Perhaps the most unexpected fact about it is provided by the interaction between negation and the quantifiers. One might have expected the duals of the relations that obtain in intuitionist logic to obtain. However, the "backward looking" nature of negation interacts with the "forward-looking" quantifier truth-conditions and the domain-increasing feature of the semantics to ensure that this is not the case.

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