Contradiction and the Structure of Unity

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Unless an object is utterly simple, it has parts. The parts are not a congeries, but are structured so as to produce a whole. But how do they do so? Answers to this question soon lead to difficulties and contradictions. I will argue that they are best handled by an appropriate conception of identity, and construct a formal model of this notion.

In the first part of the paper, we will look at the notion of unity, and the problems to which it gives rise. The key concept here will be that of a *gluon*, and, in particular, the way that this behaves with respect to identity. In the second part of the paper, I will give a formal model of gluons and identity, deploying a second-order paraconsistent logic. The discussion is neutral as to the range of second-order quantifiers, but this is an important issue with a number of ramifications. It is explored in the third and final part of the paper.

1 Unity and its Problems

1.1 Frege on the Unity of the Proposition

Let us start with what might at first appear to be rather a trivial issue: Frege on the unity of the proposition.[1] Consider the sentence 'Sortes homo est'. The sentence is constituted by a noun-phrase, 'Sortes', and a verb phrase, 'homo est'. According to Frege, the sentence expresses a proposition (thought), that Socrates is a person, and the proposition is constituted by the referents of the two parts of the sentence. 'Sortes', like all noun phrases, refers to an object, the person Socrates, s. 'Homo est', like all verb phrases,

[1]On which, see Priest (1995a), 12.2.

refers to a concept, c, that of being a man. But the proposition is not just a congeries of its two parts, s and c. Somehow these cooperate to form a unity. But how?[2]

Frege's answer was that concepts are radically different from objects. Unlike objects, they are "unsaturated", radically incomplete. The concept referred to by 'homo est' has a "gap" in it, which is plugged by the object referred to by 'Sortes' to produce a single object. Note the form of words here:

(*) The concept referred to by 'homo est' has a "gap" in it.

The notions of being unsaturated, of having a gap, and so on, are of course metaphorical. This is not in itself a problem: literal explanation must give out somewhere. But what is a problem is that Frege's account drives him straight into a contradiction. The expression 'the concept referred to by "homo est"' would, for all the world, appear to refer to a concept. But it is a noun-phrase. It therefore refers to an object, not a concept. And objects and concepts are *in toto mundo* different: the latter are unsaturated; the former are not.

Frege was well aware of the matter. His solution was to insist that, despite appearances, the description in question does refer to an object, not a concept; but he was aware that this put him in a difficult situation. He says:[3]

> I admit that there is a quite peculiar obstacle in the way of an understanding with my reader. By a kind of necessity of language, my expressions, taken literally, sometimes miss my thoughts; I mention an object when what I intend is a concept. I fully realize that in such cases I was relying on the reader who would be ready to meet me half-way—who does not begrudge me a pinch of salt.

But Frege underestimated the problem. If he is right in his insistence that the description refers to an object, this undercuts his whole explanation of the unity of the proposition. Merely reflect for a moment on (*). This is now simply false.

[2]Strictly speaking,. the concept and the object cooperate to produce a truth value. It is the senses of the subject and predicate that cooperate to produce a proposition. But this does not affect the point being made here.

[3]Geach and Black (1953), p. 54.

1.2 Unity

This is all well known—and frequently ignored as a minor puzzle. Moreover, the problem can be avoided entirely by rejecting Frege's theory of meaning. But the problem is just, in fact, a special case of a much more general one concerning unities, which cannot be avoided in such a simple way. Let me explain the general problem.

Things have parts. A computer has components, a country has regions, a history has epochs, a piece of music has notes, an argument has statements, and so on. What is the relationship between a thing and its parts? For a start, the thing is more than the simple sum of its parts. Thus, one can have the materials to build a house, but until the house is built it does not exist. The parts of the house are not sufficient: they have to be arranged in a certain way. Similarly, a piece of music has to have its notes related to each other in the right way. And an argument has to be structured into premises and conclusion. Thus, an object is more than the sum of its parts.

What is this more? We might call it the structure, form, or arrangement of the parts. Exactly how to understand what this is, is a sensitive matter. Conceivably, it may be a different sort of thing in different cases: what constitutes the unity of a house is likely to be different from what constitutes the unity of an argument. And what constitutes the unity in any of these cases is, very likely, itself a contentious issue. In the case of a house, for example, is it the geometric shape; is it the causal interactions between the bricks; or is it the design in the mind of the architect? Never mind. Whatever structure is, it is something that binds the parts into a whole. But now we have a contradiction. The structure is, after all, *something*, an object. We can refer to it, think about it, quantify over it. On the other hand it cannot be an object. If it were, the collection of the parts *plus* the structure would be just as much a congeries as the original collection of parts. So the problem of binding would not be solved. In Frege, note, the role of binding is played by the concept. It is therefore that which occupies this contradictory role.

Let me put the problem in abstract terms. Take any thing, object, entity, with parts, p_1, ..., p_n. (Suppose that there is a finite number of these; nothing hangs on this.) A thing is not merely a congeries of parts: it is a unity. There must, therefore, be something which constitutes them as a single thing, a unity. Let us call it, neutrally, the *gluon* of the object, g. Now what of this gluon? Ask whether it itself is a thing, object, entity? It both is and it isn't. It is, since we have just talked about it, referred to it, thought about it. But it isn't, since, if it is, p_1, ..., p_n, g, constitute a congeries, just as much as the original one, and we still have no account of what constitutes the unity of the object.

We have, then, a aporia. Whatever it is that constitutes the unity of an entity must itself both be and not be an entity. It *is* an entity since we are talking about *it*; it is *not* an entity since it is then part of the problem of a unity, not its solution.

1.3 The Aporia

Faced with this aporia, we have essentially four options:

1. We can say that there are no gluons.
2. We can reject the arguments to the effect that a gluon is an object.
3. We can reject the arguments to the effect that it is not an object.
4. We can accept the contradictory nature of gluons.

Whilst, no doubt, there is much to be said about the matter, the prospects in cases 1-3 look bleak.

In the first case, there are no complex unities, which seems quite false: I am such a unity. And even if we suppose that there are no such unities, there certainly appear to be; that is, there are unities in thought. This means that the mind constitutes unities—as, perhaps, for Kant. But in this case, there are gluons. These are mental entities, but they fall foul of the aporia in the usual way. At the other extreme, one might suppose that there are unities, but that they have no parts, and hence that there are no gluons. This is a desperate move. It runs in the face of the common-sense observation that, if someone steals the keyboard of my computer, it is then missing an essential part (etc.).[4]

In the second case, we must insist that the gluon is simply not an object. But this seems wildly implausible: we can refer to it, quantify over it, talk about *it*. Anything we can think about is an object, a unity, a single thing (whether or not it exists). There seems little scope here.

In the third case, we can suppose that the gluon is just a plain old object. But then we are bereft of an explanation of the unity of an object. How

[4]Bradley, who we will meet in a moment, had an extreme case of this sort of position. There is just one thing, the Absolute, with no parts; all appearance to the contrary is simply illusion. Even if we take everything we experience to be illusory, however, the position does not really avoid the problem. If my motor bike is not a *real* object, it certainly appears to me to be such. There are, therefore, objects, unities, in thought, and we are back with the problem of what makes these thought-objects unities.

could we even have had the impression that any object *could* constitute the unity of another bunch of objects? Only because of the habit of taking the unity for granted. We write 'Sortes homo est' and the rest is obvious. But putting 'Sortes'and 'homo est' next to each other does not do the job; it is just produces a list of two things. When we think of the two as cooperating, the magic has already occurred.

Let us assume, then, that gluons are contradictory objects.

1.4 The Regress

So far so good. But we still have the question of how a gluon binds the parts (including itself) into a whole. Its being inconsistent does not immediately address this question—though, one might now suspect, inconsistency is going to play some role. To move towards an answer to the question, which will also bring us to the notion of identity, let us ask why, if the gluon is simply an object, it cannot bind together the parts. One consideration is a regress argument as old as Plato's *Parmenides*.

At one stage in his career, Russell was much concerned with the question of the unity of the proposition, and one possibility he considered was that it was the copula, 'is' that binds the constituents together. So, in Fregean terms, there is just one concept, which is the copula.[5] He then explains why the copula cannot be on a footing with the other constituents:[6]

> It might be thought that 'is', here, is a constant constituent. But this would be a mistake: 'x is α' is obtained from 'Socrates is human', which is to be regarded as a subject-predicate proposition, and such propositions, we said, have only two constituents [Socrates and humanity]. Thus 'is' represents merely the way in which the constituents are put together. This cannot be a new constituent, for if it were there would have to be a new way in which it and the two other constituents are put together, and if we take this way as again a constituent, we find ourselves embarked on an infinite regress.

Russell is using an argument used earlier to great effect by Bradley. Again addressing the problem of the unity of the proposition, Bradley starts by supposing that a proposition has components A and B. What constitutes them into a unity? A natural thought is that it is some relation between them, C. But, he continues:[7]

[5] A discussion of this view, in the context of its regress, is given in Gaskin (1995).
[6] Eames and Blackwell (1973), p. 98.
[7] Allard and Stock (1994), p. 120.

> [we] have made no progress. The relation C has been admitted different from A and B... Something, however, seems to be said of this relation C, and said, again, of A and B ... [This] would appear to be another relation, D, in which C, on one side, and, on the other side, A and B, stand. But such a makeshift leads at once to the infinite process... [W]e must have recourse to a fresh relation, E, which comes between D and whatever we had before. But this must lead to another, F; and so on indefinitely... [The situation] either demands a new relation, and so on without end, or it leaves us where we were, entangled in difficulties.

And Bradley is, in fact, aware that this is not just a problem concerning the unity of the proposition. It is much more general. Thus, in discussing the unity of the mind, Bradley writes:[8]

> When we ask 'What is the composition of Mind,' we break up that state, which comes to us as a whole, into units of feeling. But since it is clear that these units, by themselves, are not all the 'composition', we are forced to recognise the existence of the relations... If units have to exist together, they must stand in relation to one another; and, if these relations are also units, it would seem that the second class must also stand in relation to the first. If A and B are feelings, and if C their relation is another feeling, you must either suppose that component parts can exist without standing in relation to one another, or else that there is a *fresh* relation between C and AB. Let this be D, and once more we are launched off on the infinite process of finding a relation between D and C–AB; and so on forever. If relations are facts that exist *between* facts, then what comes between the relations and the other facts?

We can state the regress problem generally in terms of gluons, thus: Suppose that we have a congeries of parts, a, b, c, ..., and that one is puzzled as to what constitutes their unity. Suppose one attempts to explain this by the postulation another object, the gluon, g. Then invoking g simply adds an extra element to the melange. If one is puzzled by the unity in the first case, one should be equally puzzled by the supposed unity of the extended collection in the second. Thus, e.g., instead of the congeries of the physical parts of a house, we now have the congeries of [parts plus configuration]. More generally, we have the parts of an object plus the relationship between

[8] Allard and stock (1994), pp. 78-9.

them—or the *action* of the relation, or the *fact* that they are so related. How is this any better? To use a metaphor (suggested to me by Stewart Candlish): if one has to join two links of a chain together, it helps not one whit to say that one does this by inserting a connecting link.

How to break the regress? The regress is generated by the thought that g is distinct from a, b, c, d, etc. If this is the case, then there is room, as it were, for something to be inserted between g and a, etc. Or to use another metaphor, there is a metaphysical space between g and a, and one requires something in the space to make the join. Thus, the regress will be broken if g is identical to a. There will then be no space for anything to be inserted. Of course, g must be identical with b, c, d, and so on, for exactly the same reason. Thus, g is able to combine the parts into a unity by being identical with each one (including itself). The situation may be depicted thus:

$$\begin{array}{ccccc} & & b & & \\ & & \shortparallel & & \\ a & = & g & = & c \\ & & \shortparallel & & \\ & & d & & \end{array}$$

It should be immediately obvious that the notion of identity in question will not behave in the way that identity is often supposed to behave. In particular, the transitivity of identity will fail. We have $a = g$ and $g = c$, but we will not have $a = c$. It might be doubted that there is any such coherent notion, or that if there is, it is not really one of identity. These concerns cannot be set aside lightly, and the only way assuage them is to provide a mathematical theory of identity that provides what is required. To this I now turn.

2 Gluon Models

2.1 Second-Order *LP*

The theory will be based on the semantics of a formal logic. As we have seen, gluons are to be expected to behave inconsistently. The formal logic must therefore be a paraconsistent one. The logic *LP*,[9] being simple, and having multiple other applications, recommends itself. I will therefore use this. For reasons that will become obvious, we will work with the second-order version of *LP*.[10] The details of the logic are as follows.

[9]See, e.g., Priest (1987), ch. 5.

[10]For second-order *LP*, see section 7.2 of Priest (2002b).

The language of the formal theory has the connectives $\wedge$, $\vee$ and $\neg$, and the first- and second-order quantifiers $\forall$ and $\exists$. The material conditional and biconditional are defined in the usual way: $A \supset B$ is $\neg A \vee B$; $A \equiv B$ is $(A \supset B) \wedge (B \supset A)$. For simplicity, we suppose that all predicates are monadic, and that there are no function symbols. First order variables are lower case, and monadic second-order variables are upper case.

There are various forms that the semantics of second-order LP may take; importantly, there are various possible constraints one may place on the range for the second-order variables. For the moment, we will place no constraints (other than that it be non-empty). An interpretation for the language, I, is a triple $\langle \mathcal{D}_1, \mathcal{D}_2, \partial \rangle$. $\mathcal{D}_1$ is the non-empty domain of first-order quantification. $\mathcal{D}_2$ is the non-empty domain of second-order quantification. Members, D, of $\mathcal{D}_2$ are of the form $\langle D^+, D^- \rangle$, where $D^+, D^- \subseteq \mathcal{D}_1$, and $D^+ \cup D^- = \mathcal{D}_1$. ∂ assigns every individual constant a member of $\mathcal{D}_1$, and every predicate a member of $\mathcal{D}_2$. I will write $\partial(P)$ as $\langle \partial^+(P), \partial^-(P) \rangle$. $\partial^+(P)$ and $\partial^-(P)$ are the *extension* and *antiextension* of P (the set of objects of which P true and false, respectively). Note that we do not assume that for every α, $\beta \subseteq D$, $\langle \alpha, \beta \rangle \in \mathcal{D}_2$. Thus it is natural to think of the second-order variables as ranging over properties of some fairly robust metaphysical kind; an arbitrary extension/antiextension may not be of this kind.

We now define what it is for a (closed) formula to be true, $\Vdash^+$, and false $\Vdash^-$, in an interpretation. To state the truth and falsity conditions for the quantifiers, we augment the language, if necessary, to ensure that each member of $\mathcal{D}_1$ and $\mathcal{D}_2$ has a name. Thus, if $d \in \mathcal{D}_1$, we add an individual constant, k_d, to the language, such that $\partial(k_d) = d$; and if $D \in \mathcal{D}_2$, we add a predicate constant, P_D, to the language, such that $\partial(P_D) = D$. The extended language is called the *language of the interpretation*. If A is any formula, v_i $(1 \leq i \leq n)$ any (first- or second-order) variable, and c_i a corresponding (individual or predicate) constant, then $A_{v_1,\ldots,v_n}(c_1, \ldots, c_n)$ will be the formula A with each free occurrence of v_i replaced by c_i.

We can now state the truth and falsity conditions for every closed sentence in the language of the interpretation as follows:

$\Vdash^+ Pc$ iff $\partial(c) \in \partial^+(P)$

$\Vdash^- Pc$ iff $\partial(c) \in \partial^-(P)$

$\Vdash^+ \neg A$ iff $\Vdash^- A$

$\Vdash^- \neg A$ iff $\Vdash^+ A$

$\Vdash^+ A \wedge B$ iff $\Vdash^+ A$ and $\Vdash^+ B$

$\Vdash^- A \wedge B$ iff $\Vdash^- A$ or $\Vdash^- B$

$\Vdash^+ A \vee B$ iff $\Vdash^+ A$ or $\Vdash^+ B$

$\Vdash^- A \vee B$ iff $\Vdash^- A$ and $\Vdash^- B$

$\Vdash^+ \exists x A$ iff for some $d \in \mathcal{D}_1$, $\Vdash^+ A_x(k_d)$

$\Vdash^- \exists x A$ iff for all $d \in \mathcal{D}_1$, $\Vdash^- A_x(k_d)$

$\Vdash^+ \forall x A$ iff for all $d \in \mathcal{D}_1$, $\Vdash^+ A_x(k_d)$

$\Vdash^- \forall x A$ iff for some $d \in \mathcal{D}_1$, $\Vdash^- A_x(k_d)$

$\Vdash^+ \exists X A$ iff for some $d \in \mathcal{D}_2$, $\Vdash^+ A_X(P_D)$

$\Vdash^- \exists X A$ iff for all $d \in \mathcal{D}_2$, $\Vdash^- A_X(P_D)$

$\Vdash^+ \forall X A$ iff for all $d \in \mathcal{D}_2$, $\Vdash^+ A_X(P_D)$

$\Vdash^- \forall X A$ iff for some $d \in \mathcal{D}_2$, $\Vdash^- A_X(P_D)$

Finally, validity: If the members of $\Sigma \cup \{A\}$ are closed sentences of the language, $\Sigma \vDash A$ iff in every interpretation in which every member of Σ is true, so is A.

The first-order part of LP in the above semantics is entirely standard. The second-order part is a natural extrapolation. I merely pause, therefore, to note a few of the properties of the material biconditional that will feature in what follows. In particular, is easy to check the following. (I omit set braces in the premises.)

$\vDash A \equiv A$

$A \equiv B \vDash B \equiv A$

$A, B \vDash A \equiv B$

$\neg A, \neg B \vDash A \equiv B$

$A, \neg B \vDash \neg(A \equiv B)$

$B, \neg B \vDash A \equiv B$

$\neg A \equiv \neg B \vDash A \equiv B$

$A \equiv B, B \equiv C \nvDash A \equiv C$ (Make B both true and false.)

2.2 Defining Identity

With this background, we can now come to identity. Taking our cue from a version of Leibniz' Law, identity may be defined in second-order logic in a standard fashion. Thus, let us define $a_1 = a_2$ as:

$Def_=$**:** $\forall X(Xa_1 \equiv Xa_2)$

Because the material biconditional is reflexive and symmetric, it follows that identity is too: $\vDash a = a$ and $a_1 = a_2 \vDash a_2 = a_1$. The material biconditional is not, however, transitive; identity inherits this property. Thus, consider an interpretation where:

- $\mathcal{D}_1 = \{1, 2, 3\}$
- $\partial(a_i) = i$ $(i = 1, 2, 3)$
- $\partial(P) = \langle\{1, 2\}, \{2, 3\}\rangle \in \mathcal{D}_2$
- For every other $D \in \mathcal{D}_2$, $D^- = \mathcal{D}_1$

Since $Pa_2 \wedge \neg Pa_2$ is true, so is $Pa_1 \equiv Pa_2$; and for every other predicate, Q, in the language of the interpretation Q, $\neg Qa_1 \wedge \neg Qa_2$ is true, so $Qa_1 \equiv Qa_2$. Hence, $\forall X(Xa_1 \equiv Xa_2)$, that is $a_1 = a_2$, is true. Similarly, $a_2 = a_3$. But $Pa_1 \equiv Pa_3$ is not true; hence, neither is $\forall X(Xa_1 \equiv Xa_3)$, that is, $a_1 = a_3$ is not true. Thus, $a_1 = a_2, a_2 = a_3 \nvDash a_1 = a_3$.[11] Since transitivity of identity is a special case of substitutivity of identicals, this, too, fails. For another counter-example, note that $a_2 = a_3, Pa_2 \nvDash Pa_3$. In the above interpretation, the premises are true, but the conclusion is not. Finally, note that identity statements may not be consistent. Thus, in the above interpretation, since $Pa_2 \wedge \neg Pa_2$ is true, so is $\neg(Pa_2 \equiv Pa_2)$. It follows that $\exists X \neg(Xa_2 \equiv Xa_2)$, so $\neg\forall X(Xa_2 \equiv Xa_2)$, i.e., $a_2 \neq a_2$.

It might be objected that the account of identity just given is inadequate since what is required in $Def_=$ is not a material biconditional, but a genuine (and detachable) conditional, such as the conditional of an appropriate relevant logic. We would then have transitivity and substitutivity of identity (though maybe not consistency). However, this would be too fast. It is not at all clear that what is required. For example it is not clear that

[11] It is worth noting that this is not the only account of identity in which transitivity fails. It fails in the logic of multiple denotation (Priest (1995)). It fails if there is a trivial object (Priest (1998a)). Indeed, the trivial object is a *sort* of global gluon. It fails for fuzzy identity, too (Priest (1989b)). In all these logics, substitutivity of identicals holds under certain well-defined conditions.

there is a relevant implication between, e.g., ‘Mary Ann Evans was a woman’and ‘George Elliot was a woman’—at least, not without the suppressed information that Mary Ann Evans was George Elliot. What is required is that for every predicate, P (in the language of the interpretation), Pa_1 and Pa_2 have the same truth value; and this is what the material biconditional delivers.

It might still be objected that this is not the case in LP, since $A \equiv B$ is true (and false) if A is true only but B is both true and false. But again, this is too fast. There are only two truth values, *true* and *false*. It is just that sentences may have various combinations thereof. In particular, $A \equiv B$ is true iff A and B are both true, or both false. It is easy enough to check that $A \equiv B$ is logically equivalent to $(A \wedge B) \vee (\neg A \wedge \neg B)$. If A is true only and B is both true and false, *both* are true, hence one should expect the material biconditional to be true—and since one is true and the other is false, one should expect it to be false as well.

2.3 Gluon Models

The theory of identity allows us to construct interpretations in which there are objects that behave in a manner appropriate for gluons. Take any interpretation. If $\gamma \subseteq \mathcal{D}_1$, we will say that the interpretation satisfies the *Gluon Condition* with respect to γ just if there is a $g \in \gamma$ such that for any $D \in \mathcal{D}_2$:

if for some $d \in \gamma$, $d \in D^+$, then $g \in D^+$

if for some $d \in \gamma$, $d \in D^-$, then $g \in D^-$

γ may be thought to comprise the parts of some object, and g glues them together. To achieve the identity of the gluon with each of its parts, we require that it have all the properties of each of them.

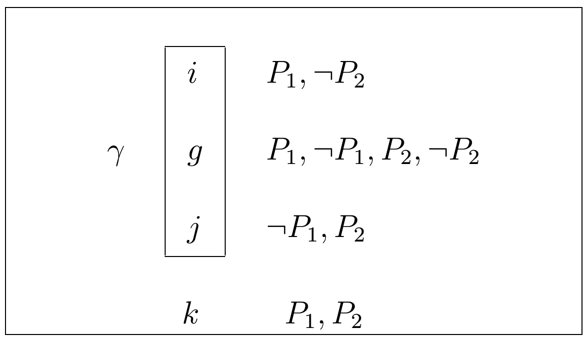
To illustrate the Condition, consider the following interpretation. There are two predicates, P_1 and P_2.

- $\mathcal{D}_1 = \{i, j, g, k\}$.
- $\partial^+(P_1) = \{i, g, k\}$
- $\partial^-(P_1) = \{j, g\}$
- $\partial^+(P_2) = \{j, g, k\}$
- $\partial^+(P_2) = \{i, g\}$

- $\mathcal{D}_2 = \{\partial(P_1), \partial(P_2)\}$.

The interpretation can be depicted by the following diagram:

$\mathcal{D}_1$



It is easy to see that the interpretation satisfies the gluon condition for the set γ.

I will now establish the consequences of the Gluon Condition, showing that gluons have the required properties. Suppose that we have an interpretation satisfying the Condition for some set γ, and let $\partial(\hbar) = g$. For a start, gluons are identical to the other parts:

Identity Lemma: If $\partial(a) \in \gamma$ then $\Vdash^+ a = \hbar$.

Proof: Let $D \in \mathcal{D}_2$. Either $a \in D^+$ or $a \in D^-$. In the first case, $g \in D^+$. So $\Vdash^+ P_D a \wedge P_D \hbar$, and $\Vdash^+ P_D a \equiv P_D \hbar$. In the second case, by similar reasoning, $\Vdash^+ \neg P_D a \equiv \neg P_D \hbar$, so $\Vdash^+ P_D a \equiv P_D \hbar$. Hence, $\Vdash^+ \forall X(Xa \equiv X\hbar)$, i.e., $\Vdash^+ a = \hbar$.

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Next, anything true of a part is true of a gluon: gluons mimics the other parts. Specifically:

Mimicing Lemma: If $\partial(a) \in \gamma$ then for any formula, A, in the language of the interpretation with at most one free variable, x:

$$\text{if } \Vdash^+ A_x(a) \text{ then } \Vdash^+ A_x(\hbar)$$
$$\text{if } \Vdash^- A_x(a) \text{ then } \Vdash^- A_x(\hbar)$$

Proof: This is proved by joint induction on the formation of A. If A is atomic, the result holds by definition. For the logical operators, there is a case for $+$, and a case for $-$. Here are the cases for $+$; those for $-$ are similar.

$$\begin{array}{rcll} \Vdash^+ (B \wedge C)_x(a) & \Rightarrow & \Vdash^+ B_x(a) \text{ and } \Vdash^+ C_x(a) & \\ & \Rightarrow & \Vdash^+ B_x(\hbar) \text{ and } \Vdash^+ C_x(\hbar) & \text{IH} \\ & \Rightarrow & \Vdash^+ (B \wedge C)_x(\hbar) & \end{array}$$

The argument for $\vee$ is similar. For negation:

$$\begin{array}{rcll} \Vdash^+ (\neg B)_x(a) & \Rightarrow & \Vdash^- B_x(a) & \\ & \Rightarrow & \Vdash^- B_x(\hbar) & \text{IH} \\ & \Rightarrow & \Vdash^+ (\neg B)_x(\hbar) & \end{array}$$

For the second-order universal quantifier:

$$\begin{array}{rcll} \Vdash^+ (\forall X B)_x(a) & \Rightarrow & \text{for all } D \in \mathcal{D}_2,\ \Vdash^+ B_{x,X}(a,P_D) & \\ & \Rightarrow & \text{for all } D \in \mathcal{D}_2,\ \Vdash^+ B_{x,X}(\hbar,P_D) & \text{IH} \\ & \Rightarrow & \Vdash^+ (\forall X B)_x(\hbar) & \end{array}$$

The case for the particular quantifier is similar. For the first-order universal quantifier, let A be $\forall y B$. If x is y, there are no free occurrences of x in A, and hence the result is trivial. So suppose x and y are distinct.

$$\begin{array}{rcll} \Vdash^+ (\forall y B)_x(a) & \Rightarrow & \text{for all } d \in \mathcal{D}_2,\ \Vdash^+ B_{x,y}(a,k_d) & \\ & \Rightarrow & \text{for all } d \in \mathcal{D}_2,\ \Vdash^+ B_{x,y}(\hbar,k_d) & \text{IH} \\ & \Rightarrow & \Vdash^+ (\forall y B)_x(\hbar) & \end{array}$$

The case for the particular quantifier is similar.

■

Finally, let us observe some corollaries which concern inconsistency. We have seen that a gluon is identical with each part. If there is only one part, there is nothing, as it were, to be glued together, and so gluons are not required. But if there are at least two distinct parts, the gluon is distinct from both of them. Specifically:

Distinctness Corollary: If γ has two members, $\partial(a)$, and $\partial(b)$, such that $\Vdash^+ a \neq b$ then $\Vdash^+ a \neq \hbar$ (and $\Vdash^+ b \neq \hbar$).

Proof: The result follows by the Mimicing Lemma.

■

Finally, we may reasonably understand being an entity as being self-identical. Hence, let us define Ox, 'x is an object', as $x = x$. Then gluons are both objects and not objects, as was to be expected. Specifically:

Gluon Corollary: If γ has two members, $\partial(a)$, and $\partial(b)$, such that $\Vdash^{+} a \neq b$ then $\Vdash^{+} O\hbar \wedge \neg O\hbar$.

Proof: The $O\hbar$ is trivial. $\neg O\hbar$ follows by two applications of the Mimicing Lemma.

■

In the light of these results, a question about the Bradley regress naturally arises. As we have seen, given that an object has distinct parts, the gluon, g, is distinct from each part, p. Why, then, does the Bradley regress not arise again? Why does there not need to be something between p and g which holds them together? There is!—g itself: $p = g$ and $g = g$. The regress terminates after one iteration.

3 Second Order Quantifiers and the Substitutivity of Identicals

3.1 Comprehension

So far so good. But for a fuller story, we have to pay more attention to the range of the second-order quantifiers in interpretations. The matter is closely connected with the principle of the substitutivity of identicals, SI: $a = b, A(a) \vdash A(b)$. As we have already noted, generally speaking, SI is going to fail. This does not, of course, prevent it from holding in special circumstances.

Let us begin the discussion with some necessary technical lemmas.

Denotation Lemma: Let $\langle \mathcal{D}_1, \mathcal{D}_2, \partial \rangle$ be an interpretation. Let A be any formula of the language of the interpretation with at most one free variable, x, and let a and b be any individual constants such that $\partial(a) = \partial(b)$. Then:

$w \Vdash^{+} A_x(a)$ iff $w \Vdash^{+} A_x(b)$

$w \Vdash^{-} A_x(a)$ iff $w \Vdash^{-} A_x(b)$

Proof: The proof is by a joint recursion on the structure of A. I will give the cases for $+$. The cases for $-$ are similar. Suppose that A is atomic. If A does not contain x, the matter is trivial, so suppose that A is Px:

$$\begin{array}{lll} \Vdash^{+} Pa & \text{iff} & \partial(a) \in \partial^{+}(P) \\ & \text{iff} & \partial(b) \in \partial^{+}(P) \quad \text{SI} \\ & \text{iff} & \Vdash^{+} Pb \end{array}$$

Note the application of SI, for future reference. The proofs for the extensional connectives are all similar. Here is the one for $\neg$:

$$\begin{array}{rll} \Vdash^{+} \neg B_x(a) & \text{iff} & \Vdash^{-} B_x(a) \\ & \text{iff} & \Vdash^{-} B_x(b) \quad \text{IH} \\ & \text{iff} & \Vdash^{+} \neg B_x(b) \end{array}$$

Finally, here are the cases for $\forall$. The cases for $\exists$ are similar. Let A be $\forall yB$. If y is x, then there are no free occurrences of x, and the result is trivial. So suppose that x and y are distinct.

$$\begin{array}{rll} \Vdash^{+} (\forall yB)_x(a) & \text{iff} & \text{for all } d \in \mathcal{D}_1,\ \Vdash^{+} B_{y,x}(k_d, a) \\ & \text{iff} & \text{for all } d \in \mathcal{D}_1,\ \Vdash^{+} B_{y,x}(k_d, b) \quad \text{IH} \\ & \text{iff} & \Vdash^{+} (\forall yB)_x(b) \end{array}$$

Let A be $\forall YB$.

$$\begin{array}{rll} \Vdash^{+} (\forall YB)_x(a) & \text{iff} & \text{for all } D \in \mathcal{D}_2,\ \Vdash^{+} B_{Y,x}(P_D, a) \\ & \text{iff} & \text{for all } D \in \mathcal{D}_2,\ \Vdash^{+} B_{Y,x}(P_D, b) \quad \text{IH} \\ & \text{iff} & \Vdash^{+} (\forall YB)_x(b) \end{array}$$

■

Next, two definitions. Essentially, we want to be able to treat formulas with one free first-order variable as monadic predicates. To this end, if A is any formula, X is any predicate variable, and B is any formula with one free variable, x, then let $A_X(B)$ be A with any occurrence of the form Xc replaced by $B_x(c)$. And given an interpretation, and a formula, A, with one first-order free variable, x, $\partial(A) = \langle \partial^{+}(A), \partial^{-}A) \rangle$, where:

$$\partial^{+}(A) = \{d \in \mathcal{D}_1 : \Vdash^{+} A_x(k_d)\}$$

$$\partial^{-}(A) = \{d \in \mathcal{D}_1 : \Vdash^{-} A_x(k_d)\}$$

Substitution Lemma: Let $\langle \mathcal{D}_1, \mathcal{D}_2, \partial \rangle$ be an interpretation. Let A be any sentence of the language of the interpretation with at most one free variable, X. Let B be any formula in the language of the interpretation with one free variable, x, and P any predicate in the language of the interpretation such that $\partial(P) = \partial(B)$, then:[12]

$$\Vdash^{+} A_X(P) \text{ iff } \Vdash^{+} A_X(B)$$

$$\Vdash^{-} A_X(P) \text{ iff } \Vdash^{-} A_X(B)$$

[12]Here, $A_X(P)$ is $A_X(Px)$.

Proof: The proof is by a joint recursion on the structure of A. I will give the cases for $+$. The cases for $-$ are similar. Suppose that A is atomic. If A is not of the form Xc then $A_X(P)$ and $A_X(B)$ are the same, and so the result it trivial. So suppose that A is Xc:

$$\begin{array}{rll} \Vdash^+ A_X(P) & \text{iff} & \Vdash^+ Pc \\ & \text{iff} & \partial(c) \in \partial^+(P) \\ & \text{iff} & \partial(c) \in \partial^+(B) \\ & \text{iff} & \partial(c) \in \{d \in \mathcal{D}_1 : \Vdash^+ A_x(k_d)\} \\ & \text{iff} & \Vdash^+ A_x(c) \\ & \text{iff} & \Vdash^+ A_X(B) \end{array}$$

The penultimate step follows by the Denotation Lemma, since $\partial(c) = \partial(k_{\partial(c)})$. The proofs for the extensional connectives are all similar. Here is the one for $\neg$:

$$\begin{array}{rlll} \Vdash^+ (\neg C)_X(P) & \text{iff} & \Vdash^- C_X(P) & \\ & \text{iff} & \Vdash^- C_X(B) & \text{IH} \\ & \text{iff} & \Vdash^+ (\neg C)_X(B) & \end{array}$$

Finally, here are the cases for $\forall$. The cases for $\exists$ are similar. Let A be $\forall YC$. If Y is X then it is not free in $\forall YC$, and so the result is trivial. So suppose that Y and X are distinct.

$$\begin{array}{rlll} \Vdash^+ (\forall YC)_X(P) & \text{iff} & \text{for all } D \in \mathcal{D}_2,\ \Vdash^+ C_{Y,X}(K_D, P) & \\ & \text{iff} & \text{for all } D \in \mathcal{D}_2,\ \Vdash^+ C_{Y,X}(K_D, B) & \text{IH} \\ & \text{iff} & \Vdash^+ \forall Y(C_X(B)) & \\ & \text{iff} & \Vdash^+ (\forall YC)_X(B) & \end{array}$$

Let A be $\forall yC$. Note that the argument is the same, whether or not y is x.

$$\begin{array}{rlll} \Vdash^+ (\forall yC)_X(P) & \text{iff} & \text{for all } d \in \mathcal{D}_1,\ \Vdash^+ C_{y,X}(k_d, P) & \\ & \text{iff} & \text{for all } d \in \mathcal{D}_1,\ \Vdash^+ C_{y,X}(k_d, B) & \text{IH} \\ & \text{iff} & \Vdash^+ \forall y(C_X(B)) & \\ & \text{iff} & \Vdash^+ (\forall yC)_X(B) & \end{array}$$

■

Another definition. Say that an interpretation is *comprehensive* iff for every formula in the language of the interpretation, B, with one free first-order variable, there is a $D \in \mathcal{D}_2$ such that $D = \partial(B)$.

Henceforth, add a new constraint on interpretations: that they be comprehensive.

Second-Order Comprehension: Whenever B is a formula with one free first-order variable, $\forall X A \vDash A_X(B)$ and $A_X(B) \vDash \exists X A$.

■

Proof: Suppose that in an interpretation $\Vdash^+ \forall X A$. Then for all $D \in \mathcal{D}_2$, $\Vdash^+ A_X(P_D)$. Since the interpretation is comprehensive, there is a $D \in \mathcal{D}_2$ such that $D = \partial(B)$. Hence, by the Substitution Lemma, $\Vdash^+ (A_X(P_D))_{P_D}(B)$, i.e., $\Vdash^+ A_X(B)$. The proof for the existential case is similar.

■

3.2 Comprehensive Gluon Models

The only example of a gluon model which we have met so far was in 2.3, and this was not comprehensive. (For example, $\partial(P_1x \vee P_2x) \notin \mathcal{D}_2$.) In this section, we will show that there are comprehensive gluon models. If an interpretation is such that for all α, $\beta \subseteq \mathcal{D}_1$, $\langle\alpha, \beta\rangle \in \mathcal{D}_2$, then clearly it is comprehensive, but it will not, generally speaking, be a gluon model.

Take any interpretation for first-order LP, $I = \langle\mathcal{D}_1^*, \partial^*\rangle$. Let $\gamma \subseteq \mathcal{D}_1^*$, and let g be some new object. If $\beta \subseteq \mathcal{D}_1^*$, define β^g as follows:

if for some $d \in \gamma$, $d \in \beta$ then $\beta^g = \beta \cup \{g\}$

otherwise, $\beta^g = \beta$

Construct a new first-order interpretation. $J = \langle\mathcal{D}_1, \partial\rangle$ as follows.

- $\mathcal{D}_1 = \mathcal{D}_1^* \cup \{g\}$.
- For constants, ∂ is the same as ∂^*.
- For every predicate, P, $\partial(P) = \{\langle\alpha^g, \beta^g\rangle \colon \langle\alpha, \beta\rangle \in \partial^*(P)\}$.

Let J_0 be the (non-comprehensive) second-order interpretation that is the same as J, except that its second-order domain is $\{\partial(P) : P$ is a predicate in the language$\}$. We now construct a sequence of interpretations, J_i, by transfinite induction. The only thing that is going to change is the extent of the second-order domain, so we will write the second-order domain of J_i as $\mathcal{D}_i$. If B is a formula in the language of J_i, with one free variable, x, let $\partial_i(B) = \left\langle \partial_i^+(B), \partial_i^-(B) \right\rangle$, where:

$\partial_i^+(B) = \{d \in \mathcal{D}_1 : \ B_x(k_d)$ is true in $J_i\}$

$\partial_i^-(B) = \{d \in \mathcal{D}_1 : \ B_x(k_d)$ is false in $J_i\}$

- For successor ordinals:

  $\mathcal{D}_i = \{\partial_i(B) : B$ is a formula with one free variable (first-order) in the language of $J_i\}$.

- For limit ordinals, l, $\mathcal{D}_l = \bigcup_{i<l} \mathcal{D}_i$.

Note that, in the construction, the second-order domain is non-decreasing. For limit ordinals, this is obvious. For successor ordinals: suppose that $D \in \mathcal{D}_i$; then in the language of J_i, there is a predicate, P_D, such that $D = \partial(P_D) = \partial_i(P_D x)$. Hence, $D \in \mathcal{D}_{i+1}$. The second-order domain cannot continue growing indefinitely. If κ is the cardinality of the first-order domain, its size is bounded by $2^\kappa \times 2^\kappa$ $(= 2^\kappa)$. Hence there must be a fixed point, k, where $\mathcal{D}_k = \mathcal{D}_{k+1}$. J_k is our interpretation.

To see that J_k is comprehensive, suppose that B is a formula with one free first-order variable, x. By construction, $\partial_i(B) \in \mathcal{D}_{k+1} = \mathcal{D}_k$.

To show that J_k satisfies the Gluon Condition for γ, we establish by induction on i that every J_i does.

If $i = 0$, the Gluon Condition holds by construction.

For successor ordinals: Suppose that it holds for i. Let $D \in \mathcal{D}_{i+1}$, and suppose that for some $d \in \gamma$, $d \in D^+$. Then for some formula, B, with one free variable, x, $D^+ = \partial_i^+(B)$. So in J_i, $B_x(k_d)$ is true. By the Mimicing Lemma and IH, if $\partial(g) = \hbar$, then $B_x(\hbar)$ is true in J_i. By the Denotation Lemma, $g \in \partial_i^+(B) = D^+$. The same is true for ∂^-, giving the result.

For limit ordinals, suppose that the result holds for all $i < l$. Let $D \in \mathcal{D}_l$, and suppose that for some $d \in \gamma$, $d \in D^+$. Then for some $i < l$, $D \in \mathcal{D}_i$. By IH, $g \in D^+$. The argument for $-$ is the same.

3.3 Substitutivity of Identicals

Finally, let us turn to SI and some related issues. By Second Order Comprehension, $\forall X(Xa \equiv Xb) \vDash A_x(a) \equiv A_x(b)$, and so:

$$a = b \vDash A_x(a) \supset A_x(b)$$

Let us call this *material substitutivity.* We do not (as we already know) have $a = b, A_x(a) \vDash A_x(b)$, since the material conditional does not detach. But counter-models to detachment arise only when the antecedent is both true and false. Hence, provided that $A_x(a)$ is consistent, we can, given $a = b$, move from $A_x(a)$ to $A_x(b)$. In the language of Priest (1987), ch. 8, SI is quasi-valid.

We can now address an important objection to the effect that the notion of identity we have been investigating—call it *gluon identity*—is not *real* identity. The objection goes as follows. The meaning of gluon identity is spelled out by the semantics of the language. In that semantics, the domain of objects is furnished with a notion of identity which is the classical notion, and which satisfies full substitutivity. *This*, therefore, is the real notion of identity, and gluon identity is an impostor. In particular, gluon identity can hold between the gluon of an object, g, and another of its part, a. But in the semantics, g and a are simply distinct objects. Thus, in a gluon model, there can be no classical predicate, P, applying to one, but not the other; for such a predicate, $Pg \equiv Pa$ would fail. That is why an interpretation in which $\langle \alpha, \beta \rangle \in \mathcal{D}_2$, for every α, β such that $\alpha \cup \beta = \mathcal{D}_1$, cannot be a gluon model. Gluon identity is, then, not real identity, just a certain kind of incomplete indiscriminability.

There is something wrong about this objection, and something right. For a start, what is wrong: It is certainly the case that the identity relation of the object language and the identity relation of the metalanguage, in which the semantics are expressed, are different. It does not follow that it is the relation of the object language that is not the real notion. I claim that it is gluon identity that is the real notion; simply to claim otherwise is to beg the question.

If one looks at the way that the vernacular notion of identity behaves, it appears to fail to satisfy substitutivity, including transitivity, in numerous ways. For example, substitutivity fails in intentional contexts. Someone can believe that Routley is Routley, without believing that Routley is Sylvan—even though Routley is Sylvan.[13] Transitivity fails in vague contexts. If I replace a part of my bike, say one of the exhaust pipes, it is still the same bike. Now suppose that each day I change one of the parts, until, on day n, not a single old part remains. Let us call the bike on day i, a_i. Then $a_0 = a_1$, $a_1 = a_2$, ..., $a_{n-1} = a_n$. But it is not the case that $a_0 = a_n$. I can, after all, reassemble all the old parts and stand a_0 next to a_n.[14] Or again, suppose that we have an amoeba, a. At a certain time, a divides into two amoebas, b and c. After the split, $a = b$. (Suppose, for example, that c just died.) Similarly, $a = c$. But it is not the case that $b = c$.[15] One might, of course, contest these examples, but they show that those who would claim that identity satisfies substitutivity, or even transitivity, across the board, cannot claim to have common sense appearances on their side.

[13]See Priests (2005), ch. 2.
[14]See Priest (1998).
[15]See Priest (1995b).

In each of these cases, substitutivity arguably holds in some restricted form: outwith intensional contexts, for sharply defined objects, in consistent contexts, or whatever. Exactly the same is true of gluon identity, as we have seen: we have material substitutivity. The classical account of identity, we may suppose, simply over-generalises substitutivity to make it a universal principle, forgetting inconvenient counter-examples.

So much for what is wrong with the objection. Now to what is right. Notwithstanding any of the above, it remains the case that the identity relations of the object language and the metalanguage are out of kilter. There is therefore something *prima facie* awry in the situation. Someone who holds that it is the object-language notion that is the correct notion of identity would, it might seem, be better off specifying the semantics of the object language using *that* notion, homophonically: '$a = b$'is true iff $a = b$. This is too fast, however. Whilst a homophonic semantics is always an option, it is not always the best option. The standard semantics of modal languages, for example, do not specify the semantics of modal operators using modal operators: the specification is given in terms of quantification over worlds.

But in non-homophonic cases such as this, we should at least be clear about what the notions of the metalanguage are, and why we employ them in framing the semantics. In the modal case, the meaning of quantification over worlds is clear enough, and the machinery serves to explicate the properties of the modal operators in a transparent and well-understood way, something that a simple homophonic semantics would not do (at least before the advent of world semantics). In the case of identity, what is the metalinguistic relation, standardly written as '$=$', if it is not identity? The answer is simple: it is the relationship of inter-substitutability: $a = b$ if, in the context of the semantics, $A_x(a)$ iff $A_x(b)$. And it is useful to deploy this notion in giving the semantics of identity for much the same reason that possible world semantics is useful: because it explicates the properties of identity in a transparent and well understood way.

In particular, then, suppose that we have the gluon of an object, g, and another its parts, a. g is identical to a; but the fact that something is true of g in the semantics does not guarantee that it is true of a—and vice versa—any more than the fact that something is true of Routley (that someone believes him to be Routley) guarantees that it is true of Sylvan (that the person believes him to be Routley).

The thought that identity delivers indiscriminability, "Leibniz' Law", is certainly a well entrenched view in logic. However, like the view that contradictions cannot be true, it would appear that it cannot be defended without begging the question. And like that view, both common sense and metaphysics may force us to reject it as an overhasty generalisation. Indeed, the two views are closely connected.[16] The definition of identity in 2.2 implemented a certain version of Leibniz' Law, a version with a material (bi)conditional. Substitutivity fails because the material conditional fails to detach; but it fails to detach precisely because some contradictions are true. Given the definition of identity, then, the failure of substitutivity and dialetheism come to the same thing.

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