The Ternary Relation of Relevant Logic Semantics

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1 Introduction

A major problem with the world-semantics of relevant logics concerns the ternary relation employed in stating the truth conditions of conditionals. What, exactly, does it mean, and why is it reasonable to employ it in this fashion? In this note, I will give an account of the ternary relation which answers these questions. The answer builds on thoughts that are familiar in relevant logic, but never seem to have been spelled out explicitly in the way that follows.[1]

The idea that a proposition is a function is a familiar one in modern logic. For example, in intensional logics one can think of a proposition as a function from worlds to truth values. The idea that the propositional content of a conditional is a particular sort of function is also familiar. In intuitionist logic, the semantic content of a conditional, $\alpha \to \beta$, is a construction that applies to any proof of α to give a proof of β. This construction is obviously a function. I want to suggest that the conditional in relevant logic is also best thought of as a function. In a nutshell, it is a function which, when applied to the proposition expressed by α gives the proposition at expressed by β.

There is another contentious feature of the standard Routley/Meyer semantics for relevant logics: the * function deployed in stating the truth conditions for negation. I shall have nothing here to say concerning that. I will therefore restrict myself to positive relevant logic. It will be useful (though

[1] For example, the thought that fusion is something like functional application is found in the motiviating remarks of Slaney (1990). The definition of R given below is to be found in Read (1988), though he takes fusion to be a sort of intensional conjunction. I'm grateful to Stephen Read for helpful discussions on the subject.

not essential) to suppose the language to contain the constant T, meaning something like 'Something is true', true at all worlds, and so satisfying the axiom $\alpha \rightarrow T$.

2 The Semantics and its Meaning

A relational structure for the semantics of such a logic[2] is a tuple $\langle N, W, R, \nu \rangle$, where W is a set of worlds, $N \subseteq W$ are the normal worlds and R is a ternary relation on W. ν is a function which assignes a truth value (1 or 0), $\nu_w(p)$ to every parameter at each world, w.

If $x, y \in W$, define the relation $x \leq y$ by: $\exists n \in N$, $Rnxy$. A structure must satisfy the following conditions:[3]

R1 $x \leq x$

R2 If $x \leq y$ and $Ryzw$ then $Rxzw$

R3 $x \leq y$ and $\nu_x(p) = 1 \Rightarrow \nu_y(p) = 1$

RS is called the heredity condition, and, employing R2 can be shown to extend to all all formulas.

The truth conditions for the logical constants of the language are as follows.

$\nu_w(T) = 1$

$\nu_w(\alpha \wedge \beta) = 1$ iff $\nu_w(\alpha) = 1$ and $\nu_w(\beta) = 1$

$\nu_w(\alpha \vee \beta) = 1$ iff $\nu_w(\alpha) = 1$ or $\nu_w(\beta) = 1$

$\nu_w(\alpha \rightarrow \beta) = 1$ iff $\forall y, z \in W$ (if $Rwyz$ and $\nu_y(\alpha) = 1$ then $\nu_z(\beta) = 1$)

Validity is defined in terms of truth preservation at normal worlds.

Turning to the meaning of the components of the structure, we think of a world as a set of propositions (the propositions true at that world), which is closed under entailment, under conjunction, and is prime (that is, whenever a disjunction is a member, so is at least one disjunct). It then follows that for all $w \in W$:

[2]As given in Routley *et al.* (1982), section *.*.

[3]p is an arbitrary propositional parameter. Lower case greek letters are arbitrary formulas.

$\mathbf{P}_\vee$ $a \vee b \in w$ iff $a \in w$ or $b \in w$

$\mathbf{P}_\wedge$ $a \wedge b \in w$ iff $a \in w$ and $b \in w$

If a and b are propositions, define $a[b]$ to be the object obtained by applying the function a to the argument b. If a is not a function, or b is not in its domain, let $a[b]$ be the proposition expressed by T. If $x, y \in W$, let $x[y]$ be:

$\{a[b]\mathbf{;}\ a \in x,\ b \in y\}$

Note that $x[y]$ may not be a world. For example, there is no reason to suppose it to be prime. However, we can use it to define the relationsip R on worlds as follows:

$Rxyz$ is $x[y] \subseteq z$

In other words, $Rxyz$ iff whenever the result of of applying any function (proposition) in x to any proposition in y is all contained in z.[4] Thinking of conditionals in this way makes it absolutely clear why a *ternary* relation is appropriate. One place for the function one for its argument; one for its value.

To complete intended informal explanation of the semantics, we need to say what N is. If a and b are propositions, let $a \to b$ be the proposition that a entails b. Let the members of N be exactly those worlds, n such that for any proposition $a \to b$:

$a \to b \in n$ iff a does entail b.

In other words, the normal worlds are exactly those words where $\to$ marks the genuine entailments: the worlds where the logical laws are the correct ones.

3 Justifying the Conditions

Given the explanations of the semantic notions just given, both the constraints on them and their deployment in stating truth conditions makes perfectly good sense.

[4]The semantics given here are the non-simplified semantics. In the simplified semantics, to give truth conditions for $\to$ uniformly in terms of R, we need the condition $Rnyz$ iff $y = z$, where n is a normal world. For this condition to hold on the present account, we would need: $n[y] \subseteq z$ iff $y = z$. This condition clearly fails since we can have distinct z_1 and z_2 for which $n[y] \subseteq z_1$ and $n[y] \subseteq z_2$.

For R1: Suppose that $c \in n[x]$. Then for some $a \to b \in n$, $a \in x$, $c = (a \to b)[a] = b$. But that means that a entails b, and x is closed under entailment. So, $b = c \in x$. That is, $n[x] \subseteq x$, i.e., $Rnxx$.

For R3: Suppose that $x \leq y$. Then for some $n \in N$, $Rnxy$, i.e., $n[x] \subseteq y$. If $a \in x$ then, since $a \to a \in n$, $a = (a \to a)[a] \in n[x]$. So $a \in y$. That is, $x \subseteq y$. R3 follows as a special case.

For R2: Suppose that $x \leq y$. Then $x \subseteq y$. It follows that $x[z] \subseteq y[z]$. For if $a \in x[z]$, then for some $b \in z$, $b \to a \in x$. But then $b \to a \in y$; $a \in y[z]$. Thus, if $Ryzw$, i.e., $y[z] \subseteq w$, it follows that $x[z] \subseteq w$, i.e., $Rxzw$.

For the truth conditions: Since T is a proposition that holds at all worlds, the first it trivial. $P_\wedge$ and $P_\vee$ obviously deliver the truth conditions for conjunction and disjunction. For $\to$: Suppose that $a \to b \in w$, and $Rwxy$, i.e., $w[x] \subseteq y$. Then if $a \in x$, $(a \to b)[a] = b \in y$. Conversely, suppose that $a \to b \notin w$. Then we can find worlds, x and y, such that $a \in x$, $b \notin y$, and $w[x] \subseteq y$. The construction of the canonical model shows how to do this.[5] Starting with $\{a\}$ and the empty set, we extend each of these to worlds, x and y, with the relevant properties. In particular, we can keep b out of the second, since $a \to b \notin w$.

Note that the same sort of argument as the one for $\to$ applies to an understanding of the truth conditions of modal operators in terms of worlds and the binary relative-possibility relation. The truth conditions tell us that:

$\Box a \in w$ iff for all w' such that wRw', $a \in w'$

If we take wRw' to mean that for all $\Box a \in w'$, $a \in w$, the truth of the left to right direction is clear. For the right to left, suppose that $\Box a \notin w$; we need to know that there is a possible world, w', such that wRw', and $a \notin w'$. This is exactly what the consturction in the canonical model delivers for us.

4 Extensions

Extensions of the logic B are obtained semantically by adding further constraints on R. Thus, to get the relevant logic R, we add:

If $\exists w(Rxyw$ and $Rwuv)$ then $\exists w(Rxuw$ and $Rywv)$

If $Rxyz$ then $Ryxz$

[5]See, e.g., ****.

If $Rxyz$ then $\exists w(Rxyw$ and $Rwyz)$

The functional interpretation of the arrow makes these conditions rather implausible. Consider the second, for example. It says that for any x, y, and z, if $x[y] \subseteq z$ then $y[x] \subseteq z$; so if $x[y] \subseteq y[x]$. But generally speaking, there is no connection between $a[b]$ and $b[a]$. Functional application is hardly commutative. Similarly, the third tells us that for all x, y, and z, if $x[y] \subseteq z$ then, for some w, $x[y] \subseteq w$ and $w[y] \subseteq z$. So for some w such that $x[y] \subseteq w$ and $w[y] \subseteq x[y]$ But if $a[b] = c$, there is no reason to suppose that $c[b] = a[b]$.

5 Conclusion

To summarise, what we have seen are the following:

1. It is perfectly natural to understand the meaning of a conditional as a function.

2. If one does this, then an intelligible meaning for the semantic ternary relation is straightforward. The reason one needs a ternary relation is clear: one place for the function; one place for the argument; one place for the value.

3. Understanding the meaning of the conditional in this way motivates the relevant logic B.

4. Further constraints on R would appear to be unmotivated. Hence, extensions of B do not appear to be justified by this understanding of the conditional.

References

[1] G.Priest (2001), ‘Paraconsistent Logic’, in D.Gabbay and F.Guenthner (eds.), *Handbook of Philosophical Logic*, 2nd edition, Dordrecht: Kluwer Academic Publishers.

[2] S.Read (1988), *Relevant Logic: a Philosophical Examination of Inference*, Oxford: Basil Blackwell.

[3] R.Routley, V.Plumwood, R.K.Meyer and R.Brady (1982), *Relevant Logics and their Rivals*, Atascadero, Ca: Ridgeview.

[4] J.Slaney (1990), 'A General Logic', *Australasian Journal of Philosophy*, 68, 74-88.