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NEIGHBORHOOD SEMANTICS FOR INTENTIONAL OPERATORS

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Abstract. *Towards NonBeing* (Priest, 2005) gives a noneist account of the semantics of intentional operators and predicates. The semantics for intentional operators are modelled on those for the $\Box$ in normal modal logics. In this paper an alternative semantics, modelled on neighborhood semantics for $\Box$, is given and assessed.

§1. Introduction. In *Towards Non-Being*[1] I gave a semantics for intentional operators ('... believes that ...', '... fears that ...', etc.). The semantics is modeled on the semantics of the $\Box$ operator in normal modal logics. In these logics, the modal operator corresponds semantically to a binary relation, R, and:

$w \Vdash \Box A$ iff for all w' such that wRw', $w' \Vdash A$

where $w \Vdash B$ means that B holds at world w. The modal $\Box$ operator has a significantly different sort of semantics, however: neighborhood semantics.[2] In these, at each world, w, there is a set of subsets of worlds, $\Box_w$, and:

$w \Vdash \Box A$ iff $[A] \in \Box_w$

where $[A]$ is the set of worlds where A holds. Moreover, modeling the semantics of intentional operators on these semantics offers the prospect of some significant advantages.[3] The purpose of this paper is to investigate neighborhood semantics for intentional operators.

§2. The semantics of TNB. TNB gives a noneist semantics for intentional operators. The worlds in an interpretation all have the same domain. What exists at each world is determined by the extension of the monadic existence predicate at that world. The domain of each world may therefore contain things that do not exist there.

More relevant for present purposes, the worlds may be not only possible but impossible. This is necessary since the worlds, among other things, realize the contents of our intentional propositional states, and these may be impossible: one may believe that one has squared the circle, or dream that one's mother is one's father. Impossible worlds are obtained using the standard technology of relevant logic.[4] In particular, truth and falsity

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[1] Priest (2005). Hereafter, TNB.

[2] Neighborhood semantics were invented by Montague (1970) and Scott (1970). On these semantics, see Chellas (1989), Waagbø(1992), Arló-Costa and Pacuit (2006), and Sillari (2008).

[3] As observed in Priest (2009, 1.3).

[4] As in Priest (2001, chapters 9 and 10).

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are treated even-handedly. Generalizing the treatment of $\Box$ in normal modal logics: for each object, d, and each intentional operator, Ψ, there is a binary relation, R^d_Ψ, and the truth/falsity conditions for sentences containing the operator are as follows:

$w \Vdash^+ a\Psi A$ iff for all w' such that $wR^{\delta(a)}_\Psi w'$, $w' \Vdash^+ A$
$w \Vdash^- a\Psi A$ iff for some w' such that $wR^{\delta(a)}_\Psi w'$, $w' \Vdash^- A$

where $\delta(a)$ is the denotation of a, $w \Vdash^+ B$ means that B is true at w, and $w \Vdash^- B$ means that B is false there.

If matters were left at this, however, intentional operators would have a variety of properties that they do not, as a matter of fact, have. These properties fall under the sobriquet of *logical omniscience*. Since validity is defined in terms of truth preservation at the actual world, @, which must be possible, we have neither of:

If $\vDash A$ then $\vDash a\Psi A$
If $A \vDash B$ then if $a\Psi A \vDash a\Psi B$

But we do have:

(1) If $\vDash A \rightarrow B$ then $a\Psi A \vDash a\Psi B$
(2) $a\Psi A, a\Psi B \vDash a\Psi(A \wedge B)$

And since the domains are constant, we have the analogues of the Barcan formula and the Converse Barcan formula:[5]

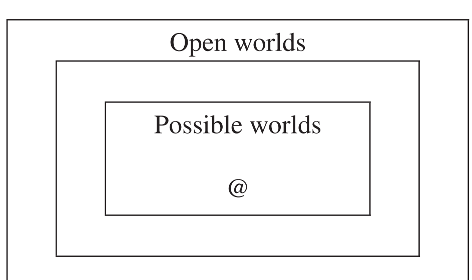
(3) $\mathfrak{A}x\, a\Psi A \vDash a\Psi\mathfrak{A}x A$
(4) $a\Psi\mathfrak{A}x A \vDash \mathfrak{A}x\, a\Psi A$

These inferences fail for most unidealized intentional operators.

To destroy these inferences, TNB introduces a new class of impossible worlds, open worlds. These are not closed under entailment. (If our intentional states are not closed under entailment, then neither should be the worlds that realize them.) Essentially, at open worlds every sentence with free variables, $x_1, \ldots, x_n$, behaves as an atomic sentence.[6] Thus, if w is such a world:

$w \Vdash^+ A(a_1, \ldots, a_n)$ iff $\langle \delta(a_1), \ldots, \delta(a_n)\rangle \in \delta^+_w(A(x_1, \ldots, x_n))$
$w \Vdash^- A(a_1, \ldots, a_n)$ iff $\langle \delta(a_1), \ldots, \delta(a_n)\rangle \in \delta^-_w(A(x_1, \ldots, x_n))$

where $\delta^+_w(A(x_1, \ldots, x_n))$ and $\delta^-_w(A(x_1, \ldots, x_n))$ are the extension and anti-extension of $A(x_1, \ldots, x_n)$ at w, respectively. Thus, the overall structure of the worlds may be depicted as follows:



[5] Following TNB, I write the particular and universal quantifiers as $\mathfrak{S}$ and $\mathfrak{A}$, respectively.

[6] The exact details are a little bit more complicated, since they employ the notion of a matrix, which I explain later.

§3. Neighborhood semantics. To generalize the neighborhood semantics for $\Box$ to intentional operators, we take an interpretation to be a structure $\langle @, W, N, R, D, \delta \rangle$.[7] W is a set of worlds; $N \subseteq W$ is the set of possible worlds; $@ \in N$. R is a ternary relation on W, subject to the constraint that if $w \in N$:[8]

(*) $Rwxy$ iff $x = y$

D is the nonempty domain of quantification. For every constant in the language, c, $\delta(c) \in D$, and for each n-place predicate, P, and $w \in W$, $\delta_w(P)$ is a pair comprising the extension and anti-extension of P at w. Let us write this as $\langle P^+_w, P^-_w \rangle$; each member of the pair is a subset of D^n. Finally, for each intentional operator, Ψ, and each $d \in D$ and $w \in W$, $\delta_{\langle w,d \rangle}(\Psi)$ is a pair of the form $\left\langle \Psi^+_{\langle w,d \rangle}, \Psi^-_{\langle w,d \rangle} \right\rangle$.[9] Each of the sets in the pair is a set of subsets of W. In standard fashion, call a set of worlds a *proposition*; then, essentially the first member of the pair is the set of propositions that, at w, d Ψs to be true, and the second is the set of propositions that, at w, d Ψs to be false.

To give the truth/falsity conditions for the quantifiers, we suppose the language augmented to contain a name, k_d, for each $d \in D$. So for every $d \in D$, $\delta(k_d) = d$.[10] The truth/falsity conditions are now given for the closed formulas of the extended language.

Then the truth conditions for atomic sentences are as follows:

$w \Vdash^+ Pt_1 \dots t_n$ iff $\langle \delta(t_1), \dots, \delta(t_n) \rangle \in P^+_w$
$w \Vdash^- Pt_1 \dots t_n$ iff $\langle \delta(t_1), \dots, \delta(t_n) \rangle \in P^-_w$

For the extensional connectives:

$w \Vdash^+ A \wedge B$ iff $w \Vdash^+ A$ and $w \Vdash^+ B$
$w \Vdash^- A \wedge B$ iff $w \Vdash^- A$ or $w \Vdash^- B$

$w \Vdash^+ A \vee B$ iff $w \Vdash^+ A$ or $w \Vdash^+ B$
$w \Vdash^- A \vee B$ iff $w \Vdash^- A$ and $w \Vdash^- B$

$w \Vdash^+ \neg A$ iff $w \Vdash^- A$
$w \Vdash^- \neg A$ iff $w \Vdash^+ A$

For the conditional operator, we have:[11]

$w \Vdash^+ A \rightarrow B$ iff for all x, y, such that $Rwxy$, when $x \Vdash^+ A$, $y \Vdash^+ B$
$w \Vdash^- A \rightarrow B$ iff for some x, y, such that $Rwxy$, $x \Vdash^+ A$ and $y \Vdash^- B$

The effect of the condition (*) is to collapse the truth/falsity conditions at *possible* worlds to the more familiar looking conditions for an $S5$ strict conditional:

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7 I take the object language to be the same as in TNB, except that I ignore function symbols to keep things simple. As there, I do not address the issue of an appropriate proof theory for the semantics.

8 Further constraints can be added, generating stronger relevant logics. See Priest (2001, chapter 10).

9 In neighborhood semantics, the operator dual to Ψ is best taken as defined. Thus if $a\Psi A$ is 'a knows that A', then 'for all a knows it is possible that A', is $\neg a \Psi \neg A$.

10 In TNB, I gave the semantics of the quantifiers in terms of evaluations of the free variables. This is equivalent. I follow the present approach because it is simpler.

11 See Priest (2006, 19.8). In TNB, I used a different semantics for $\rightarrow$ at impossible worlds; I use the ternary relation semantics here because, in the present context, they are simpler.

$w \Vdash^+ A \rightarrow B$ iff for all $x \in W$, when $x \Vdash^+ A$, $x \Vdash^+ B$
$w \Vdash^- A \rightarrow B$ iff for some $x \in W$, $x \Vdash^+ A$ and $x \Vdash^- B$

If A is any formula, $A_x(t)$ is the formula obtained from A by substituting t for each free occurrence of x. For the quantifiers, we then have:

$w \Vdash^+ \mathfrak{S}xA$ iff for some $d \in D$, $w \Vdash^+ A_x(k_d)$
$w \Vdash^- \mathfrak{S}xA$ iff for all $d \in D$, $w \Vdash^- A_x(k_d)$

$w \Vdash^+ \mathfrak{A}xA$ iff for all $d \in D$, $w \Vdash^+ A_x(k_d)$
$w \Vdash^- \mathfrak{A}xA$ iff for some $d \in D$, $w \Vdash^- A_x(k_d)$

For the truth/falsity conditions of the intentional operators, one more piece of notation is required. Let $[A]^+$ be the set of worlds where A is true, and $[A]^-$ be the set of worlds where it is false. Then:

$w \Vdash^+ a\Psi A$ iff $[A]^+ \in \Psi^+_{\langle w,\delta(a)\rangle}$
$w \Vdash^- a\Psi A$ iff $[A]^- \in \Psi^-_{\langle w,\delta(a)\rangle}$

To complete the job, we need a definition of logical consequence. This is the standard:

$\Sigma \models A$ iff for every interpretation $\mathcal{I}$, if @ is the base world of $\mathcal{I}$, then if @ $\Vdash^+ B$ for all $B \in \Sigma$, @ $\Vdash^+ A$

It is not now difficult to show that all the versions of logical omniscience cited in the last section fail. Here are counter-models for (1) and (3). (2) and (4) are left as exercises. For (1):

$W = N = \{@, w\}$
$D = \{d\}$
$\delta(a) = \delta(b) = d$
$P^+_@ = \{d\}$, $P^+_w = \phi$
$Q^+_@ = Q^+_w = \{d\}$
$\Psi^+_{\langle @,d\rangle} = \{\{@\}\}$

Other information is irrelevant. $\models Pa \rightarrow (Pa \vee Qa)$, but the interpretation shows that $b\Psi Pa \not\models b\Psi(Pa \vee Qa)$. Observe that:

$[Pa]^+ = \{@\}$
$[Pa \vee Qa]^+ = \{@, w\}$

@ $\Vdash^+ b\Psi Pa$, since $[Pa]^+ \in \Psi^+_{\langle @,\, \delta(b)\rangle}$. But @ $\nVdash^+ b\Psi(Pa \vee Qa)$, since $[Pa \vee Qa]^+ \notin \Psi^+_{\langle @,\, \delta(b)\rangle}$.

For (3):

$W = N = \{@, w\}$
$D = \{d, e\}$
$\delta(a) = d$
$P^+_@ = \{d\}$, $P^+_w = \{e\}$
$\Psi^+_{\langle @,d\rangle} = \{\{@\}, \{w\}\}$

Other information is irrelevant. The interpretation shows that $\mathfrak{A}x\, a\Psi Px \not\models a\Psi\mathfrak{A}x Px$. Observe that:

$[Pk_d]^+ = \{@\}$
$[Pk_e]^+ = \{w\}$
$[\mathfrak{A}x Px]^+ = \phi$

$@ \Vdash^+ a\Psi Pk_d$, since $[Pk_d]^+ \in \Psi^+_{\langle @,\delta(a)\rangle}$; similarly, $@ \Vdash^+ a\Psi Pk_e$. Hence, $@ \Vdash^+ \mathfrak{A}x\, a\Psi Px$. But $@ \nVdash^+ a\Psi\mathfrak{A}x Px$, since $[\mathfrak{A}x Px]^+ \notin \Psi^+_{\langle @,\delta(a)\rangle}$.

§4. Logical omniscience: The most virulent form. Unfortunately, there is one further form of logical omniscience that even these semantics do not break:

(5) If $\vDash A \leftrightarrow B$ then $a\Psi A \vDash a\Psi B$

For if $\vDash A \leftrightarrow B$, then in any interpretation $[A]^+ = [B]^+$. Hence, for any d, $[A]^+ \in \Psi^+_{\langle @,d\rangle}$ iff $[B]^+ \in \Psi^+_{\langle @,d\rangle}$. Yet this is just as implausible. For example, as in all standard relevant logics $\vDash Pa \leftrightarrow \neg\neg Pa$. But one can certainly, for example, believe that $\neg\neg Pa$ without believing Pa. Intuitionists believe in such a way.

In TNB, the inference (5) is rendered invalid by the presence of open worlds. In the present semantics, we do not have to invoke a whole new class of worlds; we can simply tweak the notion of proposition employed. We do this with the help of the TNB machinery of matrices.

Given any closed formula, A, of the language, we obtain its *matrix* as follows. We may suppose that the variables of the language are enumerated: $v_0, v_1, \ldots$. Let m be the least number greater than every n such that v_n occurs bound in A. Starting on the left of A, and moving right, we replace every occurrence of an individual constant with v_m, v_{m+1}, $v_{m+2}, \ldots$, in that order. Note, in particular, that if a constant occurs more than once, different variables will be used to replace it on each occasion. The following table illustrates:

Formula	**Matrix**
$Sab \vee Pc$	$Sv_0v_1 \vee Pv_2$
$\mathfrak{A}v_6 Sav_6b$	$\mathfrak{A}v_6 Sv_7v_6v_8$
$\mathfrak{S}v_3 Sv_3v_3$	$\mathfrak{S}v_3 Sv_3v_3$
$\neg Pa \rightarrow \mathfrak{A}v_0 Sv_0a$	$\neg Pv_1 \rightarrow \mathfrak{A}v_0 Sv_0v_2$

Clearly, different formulas can have the same matrix, and each formula is a substitution instance of its matrix. We will call a formula itself a matrix if it is the matrix of some closed formula or other. If A is a formula, I will write its matrix as A^M.

The modified semantics are exactly the same as the old, except that we change the relevant notion of proposition. Call an *iproposition* (intensional proposition), anything of the form $\langle A^M, X\rangle$, where X is a proposition. The thought here can be illustrated as follows. Suppose that Pa and Qa have the same truth values every world of an interpretation, and hence that they express the same proposition with respect to that interpretation. Then $[Pa]^+$ and $[Qa]^+$ are the same. Yet $\langle Pv_0, [Pa]^+\rangle$ and $\langle Qv_0, [Qa]^+\rangle$ are different ipropositions, since the formulas have different matrices. We now take $\Psi^+_{\langle w,d\rangle}$ and $\Psi^-_{\langle w,d\rangle)}$ to be sets, not of propositions, but of ipropositions; correspondingly, the truth/falsity conditions of a sentence containing Ψ are:

$w \Vdash^{+} a\Psi A$ iff $\langle A^{M}, [A]^{+}\rangle \in \Psi^{+}_{\langle w,\delta(a)\rangle}$
$w \Vdash^{-} a\Psi A$ iff $\langle A^{M}, [A]^{-}\rangle \in \Psi^{-}_{\langle w,\delta(a)\rangle}$

It is easy enough to modify our former counter-models to show that (1)–(4) still fail. But now (5) fails as well. Here is a counter-model.

$W = N = \{@\}$
$D = \{d\}$
$\delta(a) = \delta(b) = d$
$P^{+}_{@} = \{d\}$
$\Psi^{+}_{d,@} = \{\langle Pv_0, \{@\}\rangle\}$

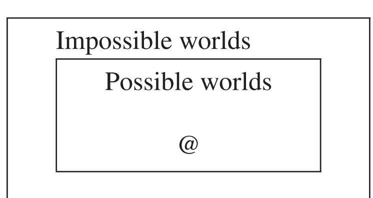
Other information is irrelevant. $\vDash Pa \leftrightarrow \neg\neg Pa$, but the interpretation shows that $b\Psi Pa \nvDash b\Psi\neg\neg Pa$. Observe that:

$[Pa]^{+} = [\neg\neg Pa]^{+} = \{@\}$

$@ \Vdash^{+} b\Psi Pa$, since $\langle Pv_0, \{@\}\rangle \in \Psi^{+}_{\langle @,\delta(b)\rangle}$; but $@ \nVdash^{+} b\Psi\neg\neg Pa$, since $\langle \neg\neg Pv_0, \{@\}\rangle \notin \Psi^{+}_{\langle @,\delta(b)\rangle}$.

The notion of an iproposition is a quite natural one. In some sense, for example, Pa and $\neg\neg Pa$ express different propositions (thoughts). Thus, the first could be grasped by someone who has no understanding of negation; the second could not. *Real* propositions cannot be individuated simply as sets of worlds, even when we have impossible worlds of the kind employed here at our disposal. Building certain syntactic concepts into the notion of a proposition, as ipropositions do, is a way of capturing this "fine-graining". It remains the case that different formulas can have the same matrices. Thus, suppose that in some interpretation $a = b$ holds at @, and (for the moment) that we have the substitutivity of identicals (SI). Then $[Pa]^{+} = [Pb]^{+}$, and $\langle Pv_0, [Pa]^{+}\rangle = \langle Pv_0, [Pb]^{+}\rangle$. So $c\Psi Pa$ is true at @ iff $c\Psi Pb$ is.[12] As we shall see in Section 7, however, once SI fails, $[Pa]^{+}$ and $[Pb]^{+}$ may be distinct, even when $a = b$ holds at @.

§5. Do the semantics work? So much for the semantics. The world structure of these is simpler than that of TNB, since we have eschewed open worlds, and so have just:



True, we have complicated the semantics of the intentional operators a little; but, overall, the structure does seem simpler.

The issue to be faced at this point is whether the semantics work. With respect to this, one might raise (at least) two questions. Do the quantifiers work properly? Do the impossible worlds function as required?

[12] The situation is effectively the same in the matrix semantics of TNB. Under the same conditions, Pa is true at any world iff Pb is—whether the world is closed or open. Hence, $c\Psi Pa$ is true at @ iff $c\Psi Pb$ is.

Concerning the first question, it is important that we can quantify into intentional contexts sensibly. Contra Quinean orthodoxy, one can make perfectly good sense of examples such as: Hob believes that a witch blighted his corn, and Nob believes her to have cursed his sow:

$$\mathfrak{S}x(h\Psi(x \text{ is a witch} \wedge x \text{ blighted } h\text{'s corn}) \wedge n\Psi(x \text{ cursed } n\text{'s sow})).$$

The following inferences also make perfectly good sense:

John believes Father Christmas to exist
Father Christmas does not exist

Something which does not exist is believed by John to do so

George believes Osama bin Laden to be alive
Dick believes Osama bin Laden to be alive

Someone is such that both George and Dick believes him to be alive

The behavior of quantifiers depends on two structural properties of interpretations.

Locality Lemma. Let $\langle @, W, N, R, D, \delta_1 \rangle$ and $\langle @, W, N, R, D, \delta_2 \rangle$ be two interpretations. Write the respective holding relations as $\Vdash_1$ and $\Vdash_2$. For any A in the language of the interpretations, if $\delta_1(c) = \delta_2(c)$, $\delta_1(P) = \delta_2(P)$, and $\delta_1(\Psi) = \delta_2(\Psi)$, for every constant, c, predicate, P, and intentional operator, Ψ, in A, then for every $w \in W$:

$w \Vdash_1^+ A$ iff $w \Vdash_2^+ A$
$w \Vdash_1^- A$ iff $w \Vdash_2^- A$

Denotation Lemma. Let $\langle @, W, N, R, D, \delta \rangle$ be an interpretation. Let A be any formula of the language of the interpretation with at most one free variable, x, and let a and b be any constants such that $\delta(a) = \delta(b)$. Then for any $w \in W$:

$w \Vdash^+ A_x(a)$ iff $w \Vdash^+ A_x(b)$
$w \Vdash^- A_x(a)$ iff $w \Vdash^- A_x(b)$

The lemmas can be proved by a straightforward induction. Proofs can be found in a technical appendix to this paper. Given these lemmas, quantifiers satisfy all the standard laws of quantification theory.[13]

In this context, it is worth noting the following. A natural thought is to take the iproposition expressed by B to be $\langle B, [B]^+ \rangle$. (So the first member of the pair is B itself, not its matrix.) If one were to do this, however, the proof of the Denotation Lemma would break down in the case for Ψ, as consulting it will show.

Turning to the second question, we must ask whether the impossible worlds in the construction function as required. In the semantics for $\rightarrow$ in use here, if B holds at every world of every interpretation, then $A \rightarrow B$ is a logical truth, even though A is absolutely unrelated to B. Similarly, if A fails at every world of every interpretation, then $A \rightarrow B$ is a logical truth even though, again, A is absolutely unrelated to B. If the conditional is to be a relevant one, then, we require:

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13 See, for example, Priest (2008, 14.7).

Maximum Variation Lemma. For every A:

for some worlds, w, in some interpretations, $w \Vdash^+ A$
for some worlds, w, in some interpretations, $w \nVdash^+ A$

The proof of this lemma is also to be found in the technical appendix.[14]

Note that in the semantics of TNB the Maximum Variation Lemma is trivial, because of the way that open worlds are defined. But it is also more important. Given these semantics, if A is true at all worlds of all interpretations, then anything of the form $a\Psi A$ is everywhere true; and if A is true at no world of any interpretation $\neg a\Psi A$ is nowhere true. Real intentional operators have no such degenerate properties. (There is nothing such that I must believe, fear, etc., *that*.) In the present semantics, if Maximum Variation were to fail, then there would be As for which $[A]^+ = W$ or $[A]^+ = \phi$; but since each of these sets may or may not be a member of $\Psi^+_{\langle w,\delta(a)\rangle}$, nothing general follows about intentional operators.

§6. Identity. The foregoing assumes that identity is not in the language. If it is added to the language, new complications arise. In relevant logics, $\delta_w(=) = \langle =^+_w, =^-_w \rangle$, subject to the constraint that:

If $w \in N$ then $=^+_w = \{\langle d, d\rangle : d \in D\}$

This says, essentially, that at possible worlds identity behaves in a standard fashion (the anti-extension of identity does no real work with respect to the properties of identity). But at impossible worlds identity can behave nonstandardly: in particular, its extension and anti-extension can vary arbitrarily, just like any other predicate. These semantics verify the standard laws of identity (Identity and SI), as well as preserving the Maximum Variation Lemma (which makes identity a good citizen of a relevant logic).[15]

Unfortunately, in the context of intentional operators, SI is not what is wanted. I can understand that Venus is Venus without understanding that Venus is the Morning Star. Hence, in TNB, chapter 2, a "contingent identity" semantics is given for identity. Neighborhood semantics, as such, does nothing, to provide counter-models to SI.[16] Hence the preceding semantics need to be modified for contingent identity in the same way.

Specifically, an interpretation is the same as before, except that we add a new component, Q. Members of Q are to be thought of as the identities, or avatars, of objects at particular worlds.[17] The domain of quantification, D, is now taken to comprise maps, d, from W to Q. $d(w)$ is the identity of d at w. Moreover, if P is an n-place predicate, including identity, and $\delta_w(P) = \langle P^+_w, P^-_w \rangle$, each of P^+_w and P^-_w must be taken to be subsets of Q^n, not D^n. The truth/falsity conditions of atomic sentences are now as follows. I write $d(w)$ as $|d|_w$:

$w \Vdash^+ Pt_1 \dots t_n$ iff $\langle |\delta(t_1)|_w, \dots, |\delta(t_n)|_w \rangle \in P^+_w$
$w \Vdash^- Pt_1 \dots t_n$ iff $\langle |\delta(t_1)|_w, \dots, |\delta(t_n)|_w \rangle \in P^-_w$

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[14] The lemma is, of course, satisfied in the semantics of standard relevant logics. The point at issue is whether the addition of intentional operators preserves this property.

[15] See Priest (2008, 24.6).

[16] If $@ \Vdash^+ a = b$ then $\delta(a) = \delta(b)$. The Denotation Lemma then gives us that $@ \Vdash^+ A_x(a)$ iff $@ \Vdash^+ A_x(b)$.

[17] TNB, 2.9.

All else remains the same. Note that each of $=^+_w$ and $=^-_w$ is a subset of Q^2, subject to the constraint that if $w \in N$ then $=^+_w = \{\langle q, q\rangle : q \in Q\}$.

These semantics verify the Law of Identity, $\vDash \mathfrak{A}x\, x = x$, but SI fails in the scope of intentional operators. Thus, consider the following counter-model to the inference $a = b$, $c\Psi Pa \vDash c\Psi Pb$:

$W = \{@, w\}$
$N = \{@\}$
$D = \{f, g, h\}$
$Q = \{0, 1, 2\}$
$\delta(a) = f$, $\delta(b) = g$, $\delta(c) = h$, where

$|f|_@ = |g|_@ = |h|_@ = 0$
$f_w = 1$
$g_w = 2$

$P^+_@ = \{0\}$
$P^+_w = \{2\}$
$\Psi^+_{\langle @,0\rangle} = \{\langle Pv, \{@\}\rangle\}$

Other information is irrelevant. Since $|\delta(a)|_@ = |\delta(b)|_@$, $@ \Vdash^+ a = b$. Note that:

$[Pa]^+ = \{@\}$
$[Pb]^+ = \{@, w\}$

Hence, $@ \Vdash^+ c\Psi Pa$, since $\left\langle Pv, [Pa]^+\right\rangle \in \Psi^+_{\langle @,0\rangle}$, but $@ \nVdash^+ c\Psi Pb$, since $\left\langle Pv, [Pb]^+\right\rangle \notin \Psi^+_{\langle @,0\rangle}$.

It is not difficult to rework the counterexamples to the various forms of logical omniscience in a form appropriate for the contingent identity semantics, showing that these also avoid these problems. (Details are left as an exercise.) Further, the semantics meet the same adequacy requirements as before. In particular, the Locality and Denotation Lemmas continue to hold, the proofs being much the same as before; similarly for the Maximum Variation Lemma. Proofs may be found in the technical appendix.

§7. Restricted substitutivity. The contingent identity semantics of the last section invalidate SI within the scope of intentional operators. But they invalidate it also in the scope of any other operator that has a world-shift in its truth/falsity conditions: even if a and b have the same avatars at @, they may have different avatars at other worlds. In particular, SI will not be valid when substituting into the scope of an $\rightarrow$. The failure of SI in such contexts is moot.[18] Arguably it fails. It is clear that Venus is inhabited entails that Venus is inhabited. It is less clear that Venus is inhabited entails that the Morning Star is inhabited—at least without the extra information in the antecedent that Venus is the Morning Star. Or again, suppose that we have a statue, s, made out of a lump of gold, g. That s ceases to exist entails that s ceases to exist. But that s ceases to exist would not seem to entail that g ceases to exist: the lump of gold could survive the crushing of the statue. On the other hand, substitutivity is used all the time in conditionals in mathematical reasoning, so we do need it if such conditionals are employed in mathematics.

[18] See Priest (2008, 24.7.10).

Perhaps the simplest semantic way to regain substitutivity within conditionals, if this be needed, is as follows. Assuming the necessity of true identity statements, identities are constant across possible worlds. But they will be constant across some impossible worlds too. Let S be the class of worlds at which they are constant.[19] Add specific mention of this in the specification of an interpretation. That is, there is a set, S, such that $N \subseteq S \subseteq W$, and for every $d \in D$ and $w \in S$, $|d|_w = |d|_@$. We may now restrict the worlds relevant to the truth/falsity conditions of $\rightarrow$ to S. That is:

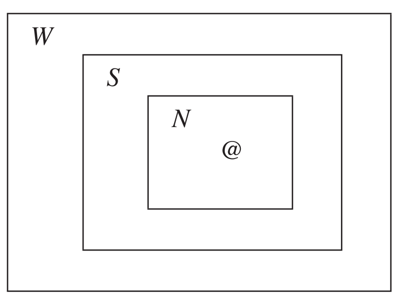
$w \Vdash^+ A \rightarrow B$ iff for all $x, y \in S$, such that $Rwxy$, when $x \Vdash^+ A$, $y \Vdash^+ B$
$w \Vdash^- A \rightarrow B$ iff for some $x, y \in S$, such that $Rwxy$, $x \Vdash^+ A$ and $y \Vdash^- B$

As far as the pure relevant logic goes, these semantics deliver exactly the same logic. The worlds in $W - S$ simply do no work. Moreover, the semantics still invalidate SI in intentional contexts. However, it now verifies SI provided that substitution is not being made into such a context:

SI Lemma. If x is not in the scope of a Ψ in A then $a = b, A_x(a) \vDash A_x(b)$.

The proof is in the technical appendix. The various principles of logical omniscience still fail. (In the old counter-models, just let $S = W$.) The proofs of the Locality, Denotation, and Maximum Variation Lemmas are still essentially the same.

The addition of S does revive the tripartite world structure:



This is not by adding a whole new class of worlds (the open worlds), however. It is just by distinguishing a special subset of the old impossible worlds. So this is still some kind of theoretical gain.

§8. Conclusion. We have seen how to provide neighborhood world-semantics for intentional operators. The semantics are no less technically adequate than those of TNB (chapters 1 and 2), and are just as intuitive conceptually. They would also seem to have the edge on simplicity.

§9. Acknowledgments. Thanks go to Giacomo Sillari, who refereed an earlier draft of the paper for the *Review*, for a very careful and helpful set of comments.

[19] In the semantics of TNB, one has substitutivity of identicals in the scope of $\rightarrow$s. The role of S is played, in effect, by the closed worlds (pp. 23, 45). In the present semantics, without S, all worlds are closed.

§10. Technical appendix. In this appendix, I will give the proofs of the various technical results cited in the paper.

Proof of Locality Lemma.
I give the proof without identity. In the case of identity, the only thing that changes is the proof of the atomic case. I indicate the variation in a footnote. The proof is by a joint recursion on the structure of A. I will do the cases for $+$. The cases for $-$ are similar. If A is atomic:[20]

$$\begin{array}{lll} w \Vdash_1^+ Pt_1 \dots t_n & \text{iff} & \langle \delta_1(t_1), \ \dots, \delta_1(t_n)\rangle \in P_w^+ \\ & \text{iff} & \langle \delta_2(t_1), \ \dots, \delta_2(t_n)\rangle \in P_w^+ \\ & \text{iff} & w \Vdash_2^+ Pt_1 \dots t_n \end{array}$$

The proofs for the extensional connectives are all similar. Here is the one for $\neg$:

$$\begin{array}{llll} w \Vdash_1^+ \neg B & \text{iff} & w \Vdash_1^- B & \\ & \text{iff} & w \Vdash_2^- B & \text{IH (induction hypothesis)} \\ & \text{iff} & w \Vdash_2^+ \neg B & \end{array}$$

For the conditional:

$$\begin{array}{llll} w \Vdash_1^+ B \to C & \text{iff} & \text{for all } u, \ v, \text{ such that } Rwuv, & \\ & & \qquad \text{if } u \Vdash_1^+ B \text{ then } v \Vdash_1^+ C & \\ & \text{iff} & \text{for all } u, \ v, \text{ such that } Rwuv, & \\ & & \qquad \text{if } u \Vdash_2^+ B \text{ then } v \Vdash_2^+ C & \text{IH} \\ & \text{iff} & w \Vdash_2^+ B \to C & \end{array}$$

For intentional operators:

$$\begin{array}{llll} w \Vdash_1^+ a\Psi B & \text{iff} & \left\langle B^M, [B]^+ \right\rangle \in \Psi^+_{\langle w, \delta_1(a)\rangle} & \\ & \text{iff} & \left\langle B^M, [B]^+ \right\rangle \in \Psi^+_{\langle w, \delta_2(a)\rangle} & \text{IH} \\ & \text{iff} & w \Vdash_2^+ a\Psi B & \end{array}$$

Finally, here is the case for $\mathfrak{A}$. The case for $\mathfrak{S}$ is similar.

$$\begin{array}{llll} w \Vdash_1^+ \mathfrak{A}xB & \text{iff} & \text{for all } d \in D, w \Vdash_1^+ B_x(k_d) & \\ & \text{iff} & \text{for all } d \in D, w \Vdash_2^+ B_x(k_d) & \text{IH, since } \delta_1(k_d) = \delta_2(k_d) \\ & \text{iff} & w \Vdash_2^+ \mathfrak{A}xB & \end{array}$$

□

Proof of Denotation Lemma.
I give the proof without identity. In the case of identity, the only thing that changes is the proof of the atomic case. I indicate the variation in a footnote. The proof is by a joint recursion on the structure of A. I will do the cases for $+$. The cases for $-$ are similar. Suppose that A is atomic, and, for the sake of illustration, contains only one occurrence of x:[21]

$$\begin{array}{lll} w \Vdash^+ Pt_1 \dots a \dots t_n & \text{iff} & \langle \delta(t_1), \ \dots \delta(a), \dots \delta(t_n)\rangle \in P_w^+ \\ & \text{iff} & \langle \delta(t_1), \ \dots \delta(b), \dots \delta(t_n)\rangle \in P_w^+ \\ & \text{iff} & w \Vdash^+ Pt_1 \dots b \dots t_n \end{array}$$

---

[20] In the case of identity, replace everything of the form $\delta_i(c)$ with $|\delta_i(c)|_w$.

[21] In the case of identity, replace everything of the form $\delta_i(c)$ with $|\delta_i(c)|_w$.

The proofs for the extensional connectives are all similar. Here is the one for $\neg$:

$$\begin{array}{lll} w \Vdash^+ \neg B_x(a) & \text{iff} & w \Vdash^- B_x(a) \\ & \text{iff} & w \Vdash^- B_x(b) \quad \text{IH} \\ & \text{iff} & w \Vdash^+ B_x(b) \end{array}$$

For the conditional:

$$\begin{array}{lll} w \Vdash^+ (B \to C)_x(a) & \text{iff} & w \Vdash^+ B_x(a) \to C_x(a) \\ & \text{iff} & \text{for all } u,\ v, \text{ such that } Rwuv, \\ & & \qquad \text{if } u \Vdash^+ B_x(a) \text{ then } v \Vdash^+ C_x(a) \\ & \text{iff} & \text{for all } u,\ v, \text{ such that } Rwuv, \\ & & \qquad \text{if } u \Vdash^+ B_x(b) \text{ then } v \Vdash^+ C_x(b) \quad \text{IH} \\ & \text{iff} & w \Vdash^+ B_x(b) \to C_x(b) \\ & \text{iff} & w \Vdash^+ (B \to C)_x(b) \end{array}$$

The case for the intentional operators is as follows. t is either a constant or x. If t is a constant, let t' and t'' be t; if t is x, let t' be a and t'' be b.

$$\begin{array}{lll} w \Vdash^+ (t\Psi B)_x(a) & \text{iff} & w \Vdash^+ t'\Psi(B_x(a)) \\ & \text{iff} & \left\langle (B_x(a))^M, [B_x(a)]^+ \right\rangle \in \Psi^+_{\langle w, \delta(t') \rangle} \\ & \text{iff} & \left\langle (B_x(b))^M, [B_x(b)]^+ \right\rangle \in \Psi^+_{\langle w, \delta(t'') \rangle} \quad (*) \\ & \text{iff} & w \Vdash^+ t''\Psi(B_x(b)) \\ & \text{iff} & w \Vdash^+ (t\Psi B)_x(b) \end{array}$$

For the line marked ($*$): By induction hypothesis, $w \Vdash^+ B_x(a)$ iff $w \Vdash^+ B_x(b)$. Hence, $[B_x(a)]^+ = [B_x(b)]^+$. Further, $(B_x(a))^M = B^M = (B_x(b))^M$. And whatever t is, $\delta(t') = \delta(t'')$.

Finally, here is the case for $\mathfrak{A}$. The case for $\mathfrak{S}$ is similar. Let A be $\mathfrak{A}yB$. If y is x, then there are no free occurrences of x, and the result is trivial. So suppose that x and y are distinct. Write $B_{y,x}(k_d, c)$ for $(B_x(k_d))_y(c)$.

$$\begin{array}{lll} w \Vdash^+ (\mathfrak{A}yB)_x(a) & \text{iff} & \text{for all } d \in D,\ w \Vdash^+ B_{y,x}(k_d, a) \\ & \text{iff} & \text{for all } d \in D,\ w \Vdash^+ B_{y,x}(k_d, b) \quad \text{IH} \\ & \text{iff} & w \Vdash^+ (\mathfrak{A}yB)_x(b) \end{array}$$

□

Proof of Maximum Variation Lemma.

We prove something stronger, namely that there is an interpretation with worlds, w_1 and w_2, such that for all A:

1. $w_1 \Vdash^+ A$ and $w_1 \Vdash^- A$
2. $w_2 \nVdash^+ A$ and $w_2 \nVdash^- A$

Here is the proof without identity. The proof with identity is exactly the same, except that one item of the counter-model is different, as indicated in a footnote. Consider an interpretation where:

$W = \{@, w_1, w_2\}$
$N = \{@\}$
$R@xx$, for all $x \in W$
$Rw_1w_1w_1$ and $Rw_2w_1w_2$

For every $w \in W$, and n-place predicate, P:[22]

$$P^+_{w_1} = P^-_{w_1} = D^n$$
$$P^+_{w_2} = P^-_{w_2} = \phi$$

For every $w \in W$, $d \in D$, and operator, Ψ:

$$\Psi^+_{\langle w_1, d\rangle} = \Psi^-_{\langle w_1, d\rangle} = \{\langle A, X\rangle : \text{where } A \text{ is a matrix, and } X \subseteq D^n\}.$$
$$\Psi^+_{\langle w_2, d\rangle} = \Psi^-_{\langle w_2, d\rangle} = \phi$$

We prove 1 first. The proof is by a joint recursion on A. Here are the cases for $+$. The cases for $-$ are similar.

Since the extension of every predicate is universal at w_1, every atomic sentence is true there. The cases for the extensional connectives are straightforward. For the quantifiers: $w_1 \Vdash^+ \mathfrak{A}xB$ iff for all $d \in D$, $w_1 \Vdash^+ B_x(k_d)$, which is true by induction hypothesis. The case for $\mathfrak{S}$ is similar. For intentional operators, $w_1 \Vdash^+ a\Psi B$ iff $\langle B^M, [B]^+\rangle \in \Psi^+_{\langle w_1, \delta(a)\rangle}$, which is true by definition. This leaves the case for $\rightarrow$.

$w_1 \Vdash^+ B \rightarrow C$ iff for all u, v, such that Rw_1uv, if $u \Vdash^+ B$ then $v \Vdash^+ C$. This is true by induction hypothesis, since $Rw_1w_1w_1$, and w_1 accesses nothing else.

Next we prove 2. The proof is by a joint recursion on A. Here are the cases for $+$. The cases for $-$ are similar.

Since the extension of every predicate is empty at w_2, no atomic sentence is true there. The cases for the extensional connectives are straightforward. For the quantifiers: $w_2 \Vdash^+ \mathfrak{A}xB$ iff for all $d \in D$, $w_2 \Vdash^+ B_x(k_d)$, which is not true, by induction hypothesis. The case for $\mathfrak{S}$ is similar. For intentional operators, $w_2 \Vdash^+ a\Psi B$ iff $\langle B^M, [B]^+\rangle \in \Psi^+_{\langle w_2, \delta(a)\rangle}$, which is not true by definition. This leaves the case for $\rightarrow$.

$w_2 \Vdash^+ B \rightarrow C$ iff for all u, v, such that Rw_2uv, if $u \Vdash^+ B$ then $v \Vdash^+ C$. This is not true. $Rw_2w_1w_2$; by 1, $w_1 \Vdash^+ B$; and by induction hypothesis, $w_2 \nVdash^+ C$. Hence, $w_2 \nVdash^+ B \rightarrow C$. □

Proof of the SI Lemma.
Suppose in an interpretation that $@ \Vdash^+ a = b$. Then $|\delta(a)|_@ = |\delta(b)|_@$; and so for all $w \in S$, $|\delta(a)|_w = |\delta(b)|_w$. We prove that for all $w \in S$ (and so, in particular, @):

$$w \Vdash^+ A_x(a) \text{ iff } w \Vdash^+ A_x(b)$$
$$w \Vdash^- A_x(a) \text{ iff } w \Vdash^- A_x(b)$$

The result follows. The result is proved by a joint recursion on A. A is made up of atomic formulas, which may contain x free, and formulas that do not contain x free, by means of the extensional connectives, quantifiers, and $\rightarrow$. I will do the cases for $+$. Those for $-$ are similar. If A does not contain x free, the result is trivial. Suppose that A is atomic, and, for the sake of illustration, contains only one occurrence of x:

$$\begin{array}{lll} w \Vdash^+ Pt_1 \ldots a \ldots t_n & \text{iff} & \langle |\delta(t_1)|_w, \ldots |\delta(a)|_w, \ldots |\delta(t_n)|_w\rangle \in P^+_w \\ & \text{iff} & \langle |\delta(t_1)|_w, \ldots |\delta(b)|_w, \ldots |\delta(t_n)|_w\rangle \in P^+_w \\ & \text{iff} & w \Vdash^+ Pt_1 \ldots b \ldots t_n \end{array}$$

[22] If we are dealing with identity, replace D^n with Q^n in this case and the next.

The proofs for the extensional connectives and quantifiers are as in the Denotation Lemma. For the conditional:

$$
\begin{aligned}
w \Vdash^{+} (B \rightarrow C)_x(a) \quad &\text{iff} \quad w \Vdash^{+} B_x(a) \rightarrow C_x(a)\\
&\text{iff} \quad \text{for all } u,\ v \in S \text{ such that } Rwuv,\\
&\qquad\qquad \text{if } u \Vdash^{+} B_x(a) \text{ then } v \Vdash^{+} C_x(a)\\
&\text{iff} \quad \text{for all } u,\ v \in S \text{ such that } Rwuv,\\
&\qquad\qquad \text{if } u \Vdash^{+} B_x(b) \text{ then } v \Vdash^{+} C_x(b) \quad \text{IH}\\
&\text{iff} \quad w \Vdash^{+} B_x(b) \rightarrow C_x(b)\\
&\text{iff} \quad w \Vdash^{+} (B \rightarrow C)_x(b) \qquad \Box
\end{aligned}
$$

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