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The Paradoxes of Denotation

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1 Introduction

This talk is about those paradoxes of self-reference that deploy the notion of denotation, reference, and related notions—paradoxes such as Berry's and Richard's. These paradoxes are rarely singled out for special treatment. It is often assumed that if one can provide a method for solving other semantic paradoxes of self-reference, it will dispose equally of these. This tendency has been accentuated in recent years by logicians' obsession with the liar paradox. Solve that, and the rest will take care of themselves. This is, in fact, far from the case.

One important feature of the paradoxes of denotation is that they employ descriptions, or something logically equivalent, in an essential way. Of course, other paradoxes can be set up employing descriptions. We might formulate the liar paradox in the following way, for example. The fourth sentence of this paragraph is not true. But descriptions can always be avoided in these contexts. This is not the case with the paradoxes of denotation. The essential use of descriptions provides an extra dimension along which a solution may be sought; but it also means that considerations and constructions that are deployed in connection with, e.g., the liar paradox are by no means guaranteed to transfer—at least in a straightforward way.

Another important feature of the paradoxes of denotation is as follows. Most of the semantic paradoxes instantiate the naive biconditional that governs some semantic notion, such as truth or satisfaction, to generate something of the form $\alpha \leftrightarrow \neg\alpha$. From this, by principles

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such as the Law of Excluded Middle, $\alpha \wedge \neg\alpha$ follows. Thus, for example, truth seems to be governed by the T-schema. If we write 'x is true' as Tx, and use $\langle\cdot\rangle$ to indicate an appropriate naming device, we can write this as $T\langle\alpha\rangle \leftrightarrow \alpha$ (where α is any closed sentence). We instantiate the schema with a sentence, λ, of the form $\neg T\langle\lambda\rangle$ to get $T\langle\lambda\rangle \leftrightarrow \neg T\langle\lambda\rangle$, and so $T\langle\lambda\rangle \wedge \neg T\langle\lambda\rangle$.

Now, the notion of denotation seems to be governed by the D-schema. If we write 'y denotes x' as Dyx, then we can write this as $D\langle t\rangle x \leftrightarrow t = x$ (where t is any closed term). But though the paradoxes of denotation may make use of the schema, they proceed by giving independent argument for each arm of the contradictory conclusion, $\alpha \wedge \neg\alpha$. The arguments do not go through an equivalence of the form $\alpha \leftrightarrow \neg\alpha$.[1]

Differences of the kind just indicated have important consequences. In what follows, we will explore these. It should be said, straight away, that—with the exception of the material in the appendix to the paper—there is nothing new in what follows: everything can be found somewhere in the literature. However, I think it worth drawing the points together here, since their import is generally insufficiently appreciated.

2 Standard Examples and their General Features

Let us start with a brief reminder of the standard paradoxes of denotation. The best known paradoxes of these are Berry's, König's, and Richard's. To these I add a fourth: Berkeley's.[2] (There are, of course, others. We will note some in due course.)

Berry's Paradox. Any language with a finite vocabulary, such as English, has a finite number of names of any pre-assigned length. (I use 'name' here to refer to any—non-indexical—designator, not just proper names.) *A fortiori*, there is only a finite number of names that length that refer to natural numbers. Hence there is a finite number of numbers that can be referred to by name of less than, say, 100 words. There must therefore be a least. By construction, this number cannot be referred to by any name of less than 100 words. But 'the least number that cannot be referred to by any name of less than 100 words' refers to it. And this has only 16 words.

1. You argue that there is only a finite number of numbers that can be referred to by names with a given upper bound on their length, but '100 words' does not give such an upper bound. One could use '100 symbols' instead.

König's Paradox. König's paradox is similar to Berry's, but since we have a non-denumerable infinitude of ordinals to play with, in some ways it is easier. By a very standard cardinality argument, any language

[1] Most of the set-theoretic paradoxes work via an equivalence of the form $\alpha \leftrightarrow \neg\alpha$, too. A notable exception is Burali-Forti's paradox.

[2] See Priest (1995a), 4.4-4.9, 9.3.

with a finite vocabulary, such as English, has a countable number of names. Since the number of ordinals is not denumerable, there are ordinals that are not referred to by any name. Hence there is a least. By construction, this is not denoted by any name. But 'the least ordinal that is not referred to by any name' refers to it.

Richard's Paradox. Richard's paradox is constructed employing diagonalisation. As for König's paradox, the number of names in English is countable. Order these in a definable way, say lexicographically. Let us write the ith place in the decimal expansion of a real number, n, as n_i.[3] We now define the real number, r, by diagonalisation as follows. *r is the number such that: (i) if the ith name in the enumeration does not refer to a number, then r_i is 5; (ii) if the ith name in the enumeration does refer to a real number, s, then if $s_i \neq 5$, $r_i = 5$ else, $r_i = 6$.* r is distinct from every number referred to on the list, since it differs in some decimal place. Hence it is not on the list. Yet the italicised description does refer to it and is one of the names on the list.

2. Footnote 3: The single superscript dots have been replaced by ellipses, since the dots where hardly visible. Furthermore, 'infite' has been replaced by 'infinite'.

Berkeley's Paradox. There is an infinitude of objects in the world. You will never think about most of them, so there will be many objects that you will never think about. Consider any one such—εx(you will never think about x)—where this is an indefinite description. By definition, you will never think about this. But if you have been following what I have written, you just have.

It might be thought a mistake to classify Berkeley's paradox with the other three, since it does not make use of the notion of denotation explicitly. But in fact, there are good reasons to group it with the better-known paradoxes. To think of (conceive of) an object—at least in the sense relevant to this paradox—is to bring before the mind something that represents it. In particular, one may bring before the mind a name which denotes the object. Thus if the representation is a description, as it is in the case of Berkeley's paradox, the notion of denotation is deployed implicitly.

Secondly, all the standard paradoxes of self-reference—set-theoretic and semantic—share a common form. All the paradoxes instantiate the structure of an inclosure. This is constituted by a function, $\delta(X)$, and predicates, $\varphi(x)$ and $\psi(X)$, such that:

1. $\Omega = \{x : \varphi(x)\}$ exists, and $\psi(\Omega)$
2. If $X \subseteq \Omega$ and $\psi(X)$ then:

[3] The decimal expansion of a number may not be unique. Thus $0.3000\cdots$ is the same as $0.2999\cdots$. In case the expansion of a number is not unique, we always mean the one with the infinite string of zeros.

(a) $\delta(X) \notin X$ (Transcendence)
(b) $\delta(X) \in \Omega$ (Closure)

Contradiction arises for the case when X is Ω; for we then have $\delta(\Omega) \in \Omega \wedge \delta(\Omega) \notin \Omega$.[4]

Berkeley's paradox is clearly in this family. Given its use of descriptions, it is also in the same sub-family as the other denotation paradoxes, as the following table illustrates (where μ is the least-number operator, and, in Richard's paradox, $D(x, X)$ expresses the construction which diagonalises x out of a definable set, X).[5]

	$\boldsymbol{\delta(X)}$	$\boldsymbol{\varphi(x)}$	$\boldsymbol{\psi(X)}$
Berry	$\mu x(x \notin X)$	x is a nat. no. definable in < 100 words	X is definable in < 90 words
König	$\mu x(x \notin X)$	x is a definable ordinal	X is definable
Richard	$\iota x D(x, X)$	x is a definable real number	X is definable
Berkeley	$\varepsilon x(x \notin X)$	x is, was, or will be thought about	X is conceivable

3. Concerning table: All symbols in first row are now in bold face (rather than just some of the symbols). Vertical spacing of first row corrected. As before, 'words' should probably be replaced by 'symbols'.

Before we proceed to look at the paradoxes themselves, let me make one further comment arising from this classification. There is a general principle of adequacy on paradox-solutions, which can be called the *Principle of Uniform Solution* (*PUS*): same kind of paradox same kind of solution. Putative solutions that apply to only some of a family of paradoxes cannot be getting to the heart of the matter. I will not discuss the Principle here,[6] but simply note a corollary: we have a right

[4]For full details concerning these and the other paradoxes of self-reference, see Priest (1995a), chs. 9–11. Yablo's paradox also fits the schema. See Priest (1997a).

[5]One might argue that indefinite descriptions are not really denoting terms. I think that this view is incorrect, but, if one does subscribe to it, Berkeley's paradox can be reformulated in various ways using only definite descriptions. Consider, for example, the first thing thought about by Ceasar on the Ides of March that will never be thought about by you—or if, by chance, everything thought about by Caesar on that day is something you have thought or will think about, just choose somebody else.

[6]It is discussed at length in Priest (1995a), 11.5. See also 17.6 of the second

to expect a uniform solution not only to the paradoxes of denotation, but to *all* of the semantic paradoxes.[7] Anything that does not apply to all of them must be latching on to accidents, not essences.

3 Berry's Paradox in More Detail

The precise structure of the paradoxical argument in the case of Berkeley's paradox is simplicity itself. If we write Cx as 'x is being thought about (conceived)', then it concerns a term $\varepsilon x \neg Cx$. Since, as a matter of fact, not everything is conceived of, $\exists x \neg Cx$. By an appropriate description principle, $\exists x \alpha(x) \rightarrow \alpha(\varepsilon x \alpha(x))$, it follows that $\neg C \varepsilon x \neg Cx$. But because we can (and do) bring this description before the mind, we have $C \varepsilon x \neg Cx$. The structures of the arguments in the other paradoxes enumerated are more complex, but it will be helpful for what follows to spell out at least Berry's paradox in a little more detail.

Let $N(y, z)$ mean that the number of symbols in y is less than z. Let $\beta(x)$ be the formula $\neg \exists y(Dyx \wedge N(y, 100))$ ('x is not denoted by a name with less than 100 symbols'). Considerations of size allow us to establish that:

$$\exists x \beta(x)$$

Let d be the description:

$$\mu x \beta(x)$$

Then by the appropriate description principle, $\exists x \alpha(x) \rightarrow \alpha(\mu x \alpha(x))$:

$$\neg \exists y(Dyd \wedge N(y, 100))$$

But by simple counting, it is easy to establish that $N(\langle d \rangle, 100)$. And since $d = d$, the D-schema gives us: $D\langle d \rangle d$. Putting these two things together gives us: $D\langle d \rangle d \wedge N(\langle d \rangle, 100)$. Thus:

$$\exists y(Dyd \wedge N(y, 100))$$

Hence we have a contradiction.

Note that we have employed the D-schema and a description principle, but otherwise the logic we have employed is pretty minimal—just the law of identity and conjunction introduction. Conceivably, it might be thought that the law of identity fails if it concerns a non-denoting term. But the very fact that $\exists x \beta(x)$ shows us that the term d denotes. Hence, $d = d$ does not fail on this count.[8]

edition.

[7]Indeed, to all the inclosure paradoxes, semantic and set-theoretic.

[8]A fuller formalisation, including the argument for the truth of $\exists x \beta(x)$, is given in the appendix to ch. 1 of Priest (1987). This uses slightly more logic, but no inferences that are invalid in, say, First Degree Entailment. In particular, the Law of Excluded Middle is not used.

4 General Solutions

With this preliminary material out of the way, let us move on to look at solutions to the denotation paradoxes. Do standard solutions to the semantic paradoxes apply to the paradoxes of denotation? This is not the place to review all standard approaches. We will look at just three kinds of solution: Kripkean, Tarskian, and dialetheic (in that order). One might object to all these solutions in their own terms. I shall not be concerned with such objections here. My concern is whether these solutions can be applied to the denotation paradoxes at all.

Kripkean Solutions. What characterises Kripkean solutions is the use of a non-classical logic with truth value gaps. This may be supplemented with a story about the conditions under which truth-value gaps arise. (In Kripke's case, this is his account of grounded and ungrounded sentences.) The central feature of such accounts is the failure of the Law of Excluded Middle: $\alpha \vee \neg\alpha$. As I have already observed, in, say, the Liar Paradox, the Law is applied to an equivalence of the form $\alpha \leftrightarrow \neg\alpha$ to give $\alpha \wedge \neg\alpha$.[9]

This solution cannot be applied uniformly to the paradoxes of denotation, for the simple reason that they do not all use the Law of Excluded Middle. This is patent in the case of Berkeley's paradox, but as the formalisation of the previous section demonstrates, it is equally so of Berry's.[10]

4. Footnote 10: 'emply' has been replaced by 'employ'.

This conclusion might be thought puzzling. Doesn't Kripke give us a construction which produces a consistent model of the T-schema; and can't this be applied in the same way to give us a consistent model of the D-schema? So how can inconsistency arise? The construction certainly can be applied in a natural way. But, first, it does not validate the T-schema or the D-schema, merely the rule-form of each. Secondly, the trouble is that the denotation paradoxes employ not just the D-schema, but also descriptions, and the most natural ways of treating descriptions and denotation ensure that monotonicity, and so the iterative construction that depends on this, fails.[11]

[9]If one supervaluates over the gaps, one may retain the Law of Excluded Middle itself. But then other pertinent principles of inference give way.

[10]Richard's paradox does require the Law of Excluded Middle, since this makes use of definition by cases. The business part of König's paradox does not employ it, but it is possible, as far as I know, that every proof of the uncountability of the ordinals (from which it follows that there are undenoted ordinals) deploys the Law.

[11]There are other ways, which do deliver monotonicity, as is demonstrated by Kremer (1990) and Kroon (1991). But one has to insist that non-denoting inputs give truth-valueless outputs. Thus if t is any non-denoting term and m is any denoting term, $t = m$ is valueless, as is $D\langle t\rangle m$. One cannot therefore say truly that $\neg\exists x D\langle t\rangle x$, as one should be able to do.

Tarskian Solutions. Let us move on to solutions of the Tarskian variety. According to these, we must enforce a hierarchy of metalanguages, L_i $(i > 0)$. If T_i is the truth predicate of L_i, it is guaranteed to satisfy the T-schema only for sentences of lower languages. If one can construct a sentence, α, of the form $\neg T_i\langle\alpha\rangle$, this is a sentence of L_i, and of no lower language. Hence the argument to contradiction is blocked. We can apply the same idea to the paradoxes of denotation. This time, each L_i—which now contains descriptions—contains a denotation predicate, D_i, which is guaranteed to work only for names of lower languages. Descriptions containing D_i itself are not such names. Thus, in the formalisation of the Berry paradox for level i, if d_i is the description $\mu x\neg\exists y(D_i yx \wedge N(y, 100))$, though we may have $d_i = d_i$, we cannot apply the D-schema of level i to infer $D_i\langle d_i\rangle d_i$.

The solution cannot be employed uniformly, however. Berkeley's paradox stands out. If we try to apply the Tarskian strategy to this paradox, the hierarchy of truth/denotation predicates gives rise to a hierarchy of "thinking of" predicates, C_n—whether these are now thought of as belonging to different languages is, in fact, irrelevant. We can argue as before that, for any i, $\exists x\neg C_i x$; hence, by an appropriate description principle, $\neg C_i\varepsilon x\neg C_i x$. But now the impossibility of using this description in the appropriate schema metamorphoses into the claim that one can think of an object specified by a level i description only by a thought of a higher level; and thus if $C_j\varepsilon x\neg C_i x$ then $j > i$. But nothing would seem clearer than that one can have a thought whose content is itself. I can, for example, think of this very thought. (I am thinking of this thought; therefore I exist.) Even granted that thoughts are stratified into levels, this must be a thought at some level. Suppose that it is i. Then we have $C_i n$ where $n = \varepsilon x C_i x$.

Dialetheic Solutions. Finally, and briefly, let us consider dialetheic solutions. In such solutions, the proofs of contradiction are allowed to stand; but the contradictions are quarantined by the use of an appropriate paraconsistent logic. Clearly, there is nothing in the denotation paradoxes we have examined to undercut this solution. One may legitimately ask, however, if the effects of paradox are really quarantined. The obvious arguments to triviality are broken, but are there other and more subtle arguments? I will return to this question later.

5 Solutions Concerning Descriptions

Unlike the more standard paradoxes of self-reference, the paradoxes of denotation use descriptions essentially. Perhaps the solution to the paradoxes lies in the fact that these descriptions misbehave in some way.

For example, perhaps the descriptions in question fail to refer. Thus, for example, the description employed in Berry's paradox, d, is 'the least number not referred to by a name with less than 100 symbols'. If this description fails to refer, then, one might argue, because of reference-failure, $d = d$ is untrue. Alternatively, existential generalisation may fail for terms such as d. Either of these facts would break the paradox argument. Let us, then, see whether the blame can be laid at the feet of descriptions.

Ambiguity. It may be suggested that a name may have more than one referent. 'Aristotle', for example, refers to more than one person. Normally, we resolve such ambiguity by fixing the context. But maybe even when we have fixed the context, a name may have more than one denotation. For example, why can I not simply christen every object in the world, and *a fortiori* every number, with the name 'Bruce'. In this case, every natural number, real number, ordinal, or object in general, is referable/conceivable, and the paradoxes lapse. (Richard's paradox lapses because the diagonal definition assumes that denotation is unique.)

The solution does not stand up to inspection, however. As is so often the case, it merely gives us the wherewithal to reformulate the paradoxes. Consider Berry's paradox as an example. Call a name *univocal* if it has a single denotation. The number of univocal names with less than 100 symbols must be no greater than the number of names with less than 100 symbols. Hence there must be numbers that are not referred to by them. Now consider the least number that cannot be referred to by a univocal name with less than 100 symbols. The name just used refers to it; and being a definite description, it is univocal. Hence it is referred to by a univocal name. Similar reformulations apply to the other paradoxes.

In reply, it might be suggested that perhaps definite descriptions are not univocal. A different theory of how they function is required. Whilst any evaluation of such a view would have to await the theory in question, the suggestion would seem to fail. Even if ordinary definite descriptions may have more than one denotation, it would seem to be within our power to create univocal names of the required kind. Thus, let us define an operator, ι^*, that works as follows: if there is a unique thing satisfying $a(x)$ let $\iota^* x a(x)$ refer to it and only it. Otherwise... (it refers to nothing, or a unique non-existent object, or whatever). Now, taking Berry as an example again, consider $\iota^* x(x$ is the least number not referred to by a univocal name with less than 100 symbols). This univocally refers to it.

Contexts. In the previous suggestion context made a brief appearance. Maybe it can be deployed in a different way. It is clear that the denotations of some terms depend on context. (Consider, e.g., 'the present president of the USA'). Perhaps, in these paradoxes, the crucial number is not definable in one context, but definable in another.[12]

The suggestion is something of an act of desperation. The descriptions employed in the denotation paradoxes do not contain indexicals of any obvious kind, and hence do not appear to change their denotation from context to context. It must be insisted that 'definable' is itself context-sensitive. In fact, more than this, it must be insisted not only that it is context-sensitive, but that the context changes in the course of the argument—something for which there is, *in general*, no clear rationale.[13]

5. Footnote 13: '100 words' should probably be '100 symbols'.

But in any case, there would appear to be formulations of the paradox that circumvent the issue of context. Call a number *Definable* (with a capital 'D') in i words if it can be defined in some context or other by a name of i words. We can simply run, e.g., Berry's paradox employing the notion of Definability. We consider the least number not Definable by less than 100 words. It is defined by that description in this particular context, and so in some context. Hence, it is Definable in less than 100 words. But now, it will probably be asked, how do we know that there are numbers that are not Definable in less than 100 words? Maybe every number can be so Defined in some context. And in that case, $\exists x \neg \exists y(Dyx \wedge N(y, 100))$ is false, and $\mu x \neg \exists y(Dyx \wedge N(y, 100))$ is an empty term. Now, it may be the case that every natural number can be referred to by some description with less than 100 words in some context or other, but it seems implausible to suppose that one might avoid the version of König's paradox which employs the notion of Definability in the same way. The number of ordinals is so large that it would seem to be impossible that every one can be referred to in some context. A context is fixed by a bunch of parameters: time, place, speaker, audience, etc. If this is the case, the number of contexts has

6. '100 words' should probably be '100 symbols'. Three occurrences.

[12] A solution along these lines is offered by Simmons (1994).

[13] One way to bring this home is simply to make it explicit that the context is fixed. The paradoxes then reappear. Thus, take Berry's paradox for example. The place is here; the time is now; I am the speaker; you are the audience; the topic of discourse is Berry's paradox, and I am giving a version of it. Call this context c. The version of the paradox I give is as follows. There is only a finite number of names with less than 100 words. *A fortiori*, the number of numbers that I can refer to in this context, c, is finite. Consider the least number that I cannot refer to (in this context). By construction, I cannot refer to it (in c). But I have just referred to it by 'the least number that I cannot refer to in this context'. Similar reformulations apply to the other paradoxes.

some determinate cardinality, as, therefore, does the number of things that can be referred to in a context. But the number of ordinals is greater than this. It would seem that this conclusion can be avoided only by, again, playing fast and loose with the notion of context. Considerations of the kind we have just been considering apply similarly to Berkeley's paradox.

There may well, of course, be more to be said about each of the possibilities I have just discussed. The subject being what it is, I am sure there is. Equally, there may be different stories as to why something else to do with descriptions invalidates the arguments. Such possibilities have to be treated on their merits. But let me conclude with one final observation. If one solves the paradoxes of denotation by appealing to some doctrine specifically about naming—perhaps something about how descriptions work—then, since the other paradoxes of self-reference do not employ this notion, the solution will not be applicable to them. Such solutions will therefore fly in the face of the Principle of Uniform Solution.

6 A Paradox of Hilbert and Bernays

The discussion of the paradoxes of denotation has taken us quite a long way. But we have not finished yet. This is because there is a paradox of denotation that I have not mentioned so far; and in some ways this paradox is more virulent than the ones I have discussed. It was formulated originally (as far as I am aware) by Hilbert and Bernays.[14] In a nutshell, it goes as follows. Consider the name '1 + the denotation of this name'. Suppose it denotes n. Then it also denotes $1+n$. Since denotation is unique, $n = n+1$. And so, if you like, $0 = 1$.[15]

To make matters more precise, let me give a semi-formalisation of the paradox. By standard techniques of self-reference, one can construct a term, t, such that t has the form $1 + \mu x D\langle t\rangle x$. Now, $t = t$, and hence:

$$t = 1 + \mu x D\langle t\rangle x$$

But by the D-schema, $D\langle t\rangle t$. Hence:

$$\exists x D\langle t\rangle x$$

By the properties of the least number operator, $D\langle t\rangle \mu x D\langle t\rangle x$. So by the D-schema once more, $t = \mu x D\langle t\rangle x$. Substituting back in (1) gives: $t = t + 1$.

[14] For references and discussion, see Priest (1997b).

[15] A variation of Hilbert and Bernay's paradox is given by Simmons (1994). Essentially, he considers a term, t, of the form: the sum of all the numbers denoted by terms in the set $\{\langle 1\rangle, \langle t\rangle\}$. If $\langle t\rangle$ denotes n, then $n = n + 1$. If $\langle t\rangle$ fails to denote, $\langle t\rangle$ simply denotes 1. Contradiction in either case.

The result is classically intolerable. It might be thought that one could tolerate the result if one uses a paraconsistent logic. So what if $\exists x(x = x+1)$? After all, we still have $\neg\exists x(x = x+1)$. Indeed, there are even formal inconsistent arithmetics where both hold.[16] The trouble is that the above argument is just an *example* of Hilbert and Bernays' paradox. Instead of $1 + x$, we could use any arithmetic function, $f(x)$, to infer, in exactly the same way, that existence of a term, t, such that $t = f(t)$. Now let f be the function such that:

7. Should 'that existence' maybe be 'the existence'?

$$\text{if } x > 0 \text{ then } f(x) = 0, \text{ else } f(x) = 1$$

If $t = 0$, then $0 = f(0) = 1$; if $t > 0$ then $t = f(t) = 0$ and we are back with $0 = 1$ again. So the result cannot be accepted even by a dialetheist about numbers (unless they are of a very extreme kind!).

What to say about this paradox? There is, in fact, a quite obvious, and as far as it goes, adequate, response. (Another dialetheic response is discussed in the appendix to this paper.) This is simply to deny that the term t denotes. After all, if it did denote, it would have to denote both a number and its successor. And if t doesn't denote, we can fault the argument in a couple of places. We can deny that $t = t$. Alternatively, we can deny the instance of existential generalisation.

Unfortunately, we can formulate a version of the paradox that takes the possibility of non-denotation into account. We simply use a notion of description that guarantees denotation. Specifically, let us define the description operator μ^* as follows:

$$\mu^* x\alpha(x) = \mu x\big((\exists y\alpha(y) \wedge x = \mu x\alpha(x)) \vee (\neg\exists y\alpha(y) \wedge x = 0)\big)$$

Using the Law of Excluded Middle, it is easily shown that:

$$\exists x\big((\exists y\alpha(y) \wedge x = \mu x\alpha(x)) \vee (\neg\exists y\alpha(y) \wedge x = 0)\big) \tag{1}$$

Hence, all μ^*-terms denote. Moreover, by the appropriate description principle:

$$(\exists y\alpha(y) \wedge \mu^* x\alpha(x) = \mu x\alpha(x)) \vee (\neg\exists y\alpha(y) \wedge \mu^* x\alpha(x) = 0)$$

Hence, if $\exists y\alpha(y)$ it follows that:

$$\mu^* x\alpha = \mu x\alpha(x). \tag{2}$$

We now run the argument using μ^*-terms. We construct a term, t^*, of the form $1 + \mu^* x D\langle t^*\rangle x$. Since all μ^*-terms denote, we have $t^* = t^*$, and hence:

$$t^* = 1 + \mu^* x D\langle t^*\rangle x$$

[16] See, e.g., Priest (1997c). In such arithmetics, although $\forall x\forall y(x + 1 = y + 1 \supset x = y)$ holds, we cannot use $\exists x(x = x + 1)$ to infer that $0 = 1$ since the disjunctive syllogism fails.

But by the D-schema, $D\langle t^*\rangle t^*$. Hence:

$$\exists x D\langle t^*\rangle x$$

By the properties of the least number operator, $D\langle t^*\rangle \mu x D\langle t^*\rangle x$. So by the D-schema once more, $t^* = \mu x D\langle t^*\rangle x$. By an application of (2), $t^* = \mu^* x D\langle t^*\rangle x$, and so $t^* = t^* + 1$, as before.[17]

8. Footnote 17: Doesn't the Tarskian approach always consist in building a *well-founded* hierarchy of meta-languages $L_1, L_2, L_3, \ldots$ where each L_i is a meta-language of L_{i-1}? Here you are talking about a *non-wellfounded* hierarchy of meta-languages $\ldots, L_{-3}, L_{-2}, L_{-1}, L_0$, which I guess was never Tarski's intention.

9. A non-matched right parenthesis has been removed from the formula α and a missing '*' and 'x' have been added.

Appealing to truth-value gaps does solve this version of the paradox. If there are such gaps, we may take it that the Law of Excluded Middle fails. Hence, we can no longer show (1), and that all μ^*-terms denote. In particular, to show that $\mu^* x D\langle t^*\rangle x$ denotes, we would have to show that $\exists x D\langle t^*\rangle x \vee \neg\exists x D\langle t^*\rangle x$. And there is no way of establishing this. Hence, appealing to truth value gaps, and specifically, assuming that the sentence $\exists x D\langle t^*\rangle x$ is neither true nor false, will solve the problem. Unfortunately, as we have already seen, it will not solve the ordinary Berry paradox.

Fortunately, then, dialetheism will solve this version of the paradox as well—though not by simply accepting the contradiction as true. (As we have seen, this is to no avail.) Given that we now have the Law of Excluded middle, we can establish that every μ^*-term denotes. But to establish (2) requires the disjunctive syllogism $\alpha \vee \beta, \neg\alpha \vdash \beta$. (We have to rule out one disjunct to infer the other.) Now the syllogism is invalid in paraconsistent logics; it fails to be truth preserving if α is both true and false. The α in question here is $\neg\exists x D\langle t^*\rangle x \wedge \mu^* x D\langle t^*\rangle x = 0$; and this may be both true and false if $\exists x D\langle t^*\rangle x$ is. Hence, we may solve the paradox by supposing this sentence to be a dialetheia.

Of course, this does not show that the paradox cannot be reconstructed in some other way. But what can be shown is that assuming that descriptions may fail to denote, one can construct an inconsistent model of the D-schema, principles concerning descriptions and self-reference, in which, e.g., $0 = 1$ is not true.[18] These principles do not, therefore, engender triviality. (The model has to be inconsistent since it validates the argument of Berry's paradox.)

[17] It is also possible to give a Yabloesque version of this paradox (as was pointed out by Simmons in a talk at the conference *Heaps and Liars*, University of Connecticut, October 2002). A simple version is as follows. Consider a sequence of terms, t_i, $i \le 0$, such that $t_i = \langle 1+\delta(t_{i-1})\rangle$, where $\delta(t)$ is the denotation of t if it has one, and 0 otherwise. All the terms in the sequence therefore denote. Suppose that $\delta(t_0) = k$. Either $k = 0$ or $k > 0$. In the latter case, $k = \delta(t_0) = 1 + \delta(t_{-1})$. So $\delta(t_{-1}) = k - 1$. Either $k-1 = 0$ or $k-1 > 0$. In the latter case, carry on in the same way. Eventually, we must find an i such that $\delta(t_i) = 0$. But then $0 = \delta(t_i) = 1 + \delta(t_{i-1})$. Contradiction. (Note that the sequence of terms can be accommodated in an infinite regress of Tarski metalanguages, L_i, $i \le 0$. Hence, a Tarskian approach cannot solve this paradox.)

[18] For details, see Priest (1999).

7 Conclusion

In this paper, I have reviewed the paradoxes of denotation and some of the lessons to be learned from them. Most importantly, these paradoxes have distinctive features that make the applicability of a number of standard consistent solutions to the other semantic paradoxes highly problematic when applied to them. Nor do these features appear to give other satisfactory avenues for consistent solutions. A dialetheic solution is the only simple and uniform solution to all the paradoxes in question.

Appendix: Another Solution to the Paradox of Hilbert and Bernays.

10. Numbering of appendix removed.

In this appendix, I will discuss another possible solution to the Hilbert and Bernays Paradox. This is to the effect that the crucial term has—not fewer than one denotation, but—more than one denotation. Intuitively, this is just as plausible. After all, if the naughty term, t, denotes some number, it would appear to denote its successor too. But this second approach will be more acceptable to a Meinongian, who is committed to the view that all terms denote (though the denotations might be non-existent).

For a start, what shape does logic have if terms may have multiple denotations? The most crucial question here is how to define the truth conditions of atomic sentences with such terms. If we suppose that a predicate, P, has an extension, P^+, and an anti-extension, P^-, and that any term has a set of denotations $den(t)$, then the natural truth conditions are as follows:

$$Pt_1 \cdots t_n \text{ is true iff } \exists x_1 \in den(t_1) \cdots \exists x_n \in den(t_n) \langle x_1, \ldots, x_n \rangle \in P^+$$

$$Pt_1 \cdots t_n \text{ is false iff } \exists x_1 \in den(t_1) \cdots \exists x_n \in den(t_n) \langle x_1, \ldots, x_n \rangle \in P^-$$

Note that even if P is a classical predicate (so that its extension and anti-extension are complements), a sentence may still be both true and false. This approach still, therefore, requires dialetheism.[19]

Does this approach solve the paradox? In fact, it validates all the inferences employed in the argument except the substitutivity of identity. To see why, just consider a simple case of this, the transitivity of identity. The extension of identity is, as usual, $\{\langle x, x\rangle : x \in D\}$, where D is the domain of quantification. Now consider the inference $t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3$. Suppose that $den(t_1) = \{a, b\}$, $den(t_2) = \{b, c\}$, $den(t_3) = \{c\}$ (where a, b, and c are distinct objects). Then as is easy to check, $t_1 = t_2$ and $t_2 = t_3$ are both true, but $t_1 = t_3$ is not. Hence, these semantics break the argument. In fact, it is possible to construct

[19] For a full discussion of multiple denotation, see Priest (1995b).

an inconsistent but non-trivial model of the D-schema, the appropriate description principles and self-reference in a logic of multiple denotation.[20] This approach therefore solves the problem too.

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11. Replacement: 'Dordrect' → 'Dordrecht'.

[20] See Priest (200a).