dealt interestingly with the distinction between philosophy and science. Philosophy deals with what he called facts of consciousness, whose distinctive feature is that their *esse* is *percipi,* in the sense in which René Descartes had said that, so far as philosophy is concerned, there is no difference between seeing something and thinking one sees it.

The result of this careful phenomenological analysis (the word *phenomenology* had been introduced by Mansel's masters, Hamilton and Cousin) was that Mansel saw human experience as inherently complex and mysterious. In the background of Mansel's philosophy there was always an explicit contrast with a rival kind of reductive analysis that regarded man as being as unmysterious in his inner workings as a pocket watch. This contrast was the key to the controversies aroused by Mansel's Bampton lectures, "The Limits of Religious Thought," delivered in 1858. Mansel held that reason tells us that if evil exists, then God cannot be both perfectly good and all-powerful. However, God's omnipotence and perfect goodness must be accepted as a matter of faith. Although God is perfectly good, we cannot know the nature of his goodness. Man's finite goodness cannot explain God's infinite goodness; they are the same by analogy, not identity.

Mansel's lectures were attacked by F. D. Maurice and Goldwin Smith, and by John Stuart Mill, who devoted Chapter 7 of his *Examination of Sir William Hamilton's Philosophy* to Mansel's views. Mill wrote, "I will call no being good, who is not what I mean when I apply that epithet to my fellow creatures, and if such a being can sentence me to hell for not so calling him, to hell I will go." Mansel replied in *The Philosophy of the Conditioned,* and Mill in turn replied in numerous footnotes in later editions of the *Examination,* listing Mansel first among his critics. For Mansel man's goodness was not clear and God's goodness was inscrutable; both were equally a mystery.

Mansel's *Letters, Lectures, and Reviews,* published posthumously, contains, among other things, interesting articles on the philosophy of language and on mathematical logic.

See also Cousin, Victor; Descartes, René; Green, Thomas Hill; Hamilton, William; Locke, John; Logic, History of; Mill, John Stuart; Phenomenology; Language, Philosophy of; Reid, Thomas; Stewart, Dugald.

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MANY-VALUED LOGICS

An orthodox assumption in logic is that (declarative) sentences have exactly one of two values, true (1) and false (0). Many-valued logics are logics where sentences may have more than two values. Aristotle (*De Interpretatione*, chapter 9) was perhaps the first logician to countenance the thought that some sentences (future contingents) may be neither true nor false; Aristotle's ideas were discussed by many logicians in the Middle Ages. However, contemporary work on many-valued logics commenced with the work of the Polish logician Jan Łukasiewicz early in the twentieth century. One hundred years later there are many well-known many-valued logics, and the properties of such logics are well established. The logics have important philosophical applications (e.g., in articulating the views that some sentences are neither true nor false, or both true and false, or that truth comes by degrees). They also have important technical applications (e.g., in establishing various independence results).

In what follows, p, q, … will be used for propositional parameters (variables); A, B, … for arbitrary sentences; and Σ, Δ, … for sets of sentences. For references, see the last section of this entry.

ŁUKASIEWICZ LOGICS

To illustrate the notion of a formal many-valued logic, consider classical propositional logic with the following connectives: $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation), and $\rightarrow$ (conditional). This may be formulated as follows. The set of semantic values, *Val*, is {0, 1}. The set of designated values, *Des*, is {1}. An evaluation, *v*, assigns every propositional parameter (pp), a member of *Val*. All formulas are then assigned such values recursively by the clauses:

$$v(\neg A) = 1 - v(A)$$
$$v(A \wedge B) = Min(v(A), v(B))$$
$$v(A \vee B) = Max(v(A), v(B))$$
$$v(A \rightarrow B) = 1 \quad \text{if } v(A) \leq v(B)$$
$$= 1 - (v(A) - v(B)) \text{ otherwise}$$

(*Max*(*x, y*) is the maximum of *x* and *y*; *Min*(*x, y*) is the minimum of *x* and *y*. $v(A \rightarrow B)$ takes the maximum value minus any amount one has to drop to get from *A* to *B*.) The inference from Σ to *A* is valid ($\Sigma \vDash A$) just if there is no evaluation that makes all the premises designated but not the conclusion (i.e., there is no *v* such that for all $B \in \Sigma$, $v(B) \in Des$, but $v(A) \notin Des$).

If everything is exactly the same, except that *Val* = {0, ½, 1}, one has the three-valued Łukasiewicz logic $Ł_3$. The semantic conditions for the connectives can be depicted in the form of tables, thus:

$\rightarrow$	1	1/2	0
1	1	1/2	0
1/2	1	1	1/2
0	1	1	1

	$\neg$
1	0
1/2	1/2
0	1

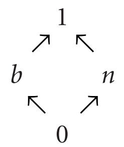
$\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$\wedge$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

More generally, if $n>1$ and everything is the same, except that $Val = \{i/(n-1) : 0 \leq i \leq n-1\}$, one has the Łukasiewicz *n*-valued logic $Ł_n$. Finally, if everything is the same, except that *Val* = [0, 1] (the set of all real numbers between 0 and 1, inclusive), one has the Łukasiewicz continuum-valued logic $Ł_{\aleph}$. (The relationship between these logics is that $Ł_n$ is a [proper] sublogic of $Ł_m$ if and only if [iff] *m* divides *n*; and $Ł_{\aleph}$ is a [proper] sublogic of all the $Ł_n$. The logic in which *Val* is the set of rationals between 0 and 1 turns out to be equivalent to $Ł_{\aleph}$.)

BOTH/NEITHER LOGICS

The values of a many-valued logic need not be numbers (and the designated values do not need to be a singleton). In another well-known family of logics, *Val* = {1, *b*, *n*, 0}. (1 can be thought of as *true and only true*; 0 as *false and only false*; *b* as *both true and false*; and *n* as *neither true nor false*.) *Des* = {1, *b*}. One can order these values as follows:



If *v* is an evaluation of the pps into *Val*, it is extended to all formulas by the following conditions:

$$v(A \vee B) = Lub\ \{v(A), v(B)\}$$
$$v(A \wedge B) = Glb\ \{v(A), v(B)\}$$

(*Lub X* is the least element of the lattice greater than or equal to every member of *X*. *Glb X* is the greatest element of the lattice less than or equal to every member of *X*.) The conditions for negation can be represented as follows:

	$\neg$
1	0
b	*b*
n	*n*
0	1

$A \rightarrow B$ can be defined as $\neg A \vee B$. Note that all these conditions agree with classical logic when the values are just 0 and 1.

These semantics give the logic often called First Degree Entailment (FDE). If one ignores the value *n*, one gets the three-valued logic LP. If one ignores the value *b*, one gets the strong Kleene three-valued logic, K_3. FDE and K_3 have no logical truths; LP (and $Ł_3$) does. LP and FDE are paraconsistent (i.e., the inference $A, \neg A \vdash B$ is not valid); K_3 is not. FDE is a sublogic of both logics, but neither is a sublogic of the other (and all three are sublogics of classical logic). The weak Kleene three-valued logic, B_3, is the same as K_3, except that any truth function with an n as an input gives n as an output.

For the first-order versions of all the logics in this section and the last, the quantifiers ∀ and ∃ can be thought of as the infinitary generalizations of ∧ and ∨, in the usual way. Thus, if *Dom*, is the domain of quantification, and every $d \in Dom$, has a name, c_d, (and if not just add them):

$$v(\forall x A\ (x)) = Glb\{v(A(c_d)) : d \in Dom\}$$

$$v(\exists x A\ (x)) = Lub\{v(A\ (c_d)) : d \in Dom\}$$

where the bounds are with respect to the appropriate orderings.

GENERAL DEFINITION

In general terms, in a semantics for a formal many-valued propositional logic, there is an arbitrary set of semantic values, *Val*. (If the cardinality of *Val* is n, the logic is called n-valued; if it is finite, the logic is called finitely many-valued; if it is infinite, the logic is called infinitely many-valued.) *Des*, the set of designated values, is an arbitrary subset of *Val*. Each n-ary connective in the language, #, is assigned an n-place (total) function, $f_\#$, with inputs and outputs in *Val*. An evaluation of the language, v, assigns each pp a member of *Val*. Semantic values are assigned to all sentences recursively by the equations $v(\#(A_1, \dots, A_n)) = f_\# (v(A_1), \dots, v(A_n))$. An inference is valid if there is no evaluation that makes all the premises designated and the conclusion undesignated. (Slightly more general definitions are also possible here.)

For quantifiers, a domain of quantification, *Dom*, and denotation function, δ, are added. For every constant c, $\delta(c) \in Dom$; if P is an n-place predicate, $\delta(P)$ is a (total) n-place function with inputs and outputs in *Dom*. $v(Pc_1, \dots c_n) = \delta(P)(\delta(c_1), \dots, \delta(c_n))$. Each quantifier, Q, is assigned a (total) function, f_Q, with inputs that are subsets of *Val* and outputs in *Val*. Assuming that each object in the domain has a name: $v(QxA(x)) = f_Q(\{v(A(c_d)) : d \in Dom\})$.

It is not difficult to check that any many-valued logic is a Tarski consequence relation. That is, it satisfies the following properties. (Here, Σ,Δ means Σ∪Δ; and set braces for singletons are omitted.)

If $A \in \Sigma$, $\Sigma \vDash A$

If $\Sigma \vDash A$ and $\Sigma \subseteq \Delta$, then $\Delta \vDash A$

If $\Sigma \vDash A$ and $\Delta, A \vDash B$, then $\Sigma, \Delta \vDash B$.

If $\Sigma \vDash A$, then any uniform substitution is valid.

(A uniform substitution is obtained by replacing each occurrence of any pp with the same formula.)

In many cases, the set of values (*Val*), together with the operations on it (the $f_\#$s), is a special case of an algebra of a certain kind. In classical logic, these are Boolean algebras; in the case of FDE, these are De Morgan algebras; and in the case of $Ł_\aleph$, these are MV algebras. Another notion of validity can be obtained by appealing to all the algebras of a kind. At this point, many-valued logic slides into algebraic logic.

PROOF PROCEDURES

All finitely many-valued logics are decidable (and *a fortiori* axiomatizable, though not necessarily finitely axiomatizable). A uniform algorithm is a generalization of truth tables (often there are more efficient ones). Consider all the possible assignments of values to the relevant pps. In each case, compute the values of the premises and the conclusion, and see if there is any assignment in which all the premises are designated and the conclusion is not.

A simple axiom system for $Ł_3$ is as follows:

$A \to (B \to A)$

$(A \to B) \to ((B \to C) \to (A \to C))$

$(\neg A \to \neg B) \to (B \to A)$

$((A \to \neg A) \to A) \to A$

The only rule of inference is *modus ponens* ($A, A \to B \vdash B$); $A \vee B$ is defined as $(A \to B) \to B$; and $A \wedge B$ is defined as $\neg(\neg A \vee \neg B)$. In each $Ł_n$ a family of *J*-functions can be defined, where $v(J_i A) = 1$ if $v(A) = i$, and $v(J_i A) = 0$ otherwise (i, here, being any value of the logic). These can be exploited to give a uniform procedure for producing an axiom system for each $Ł_n$. Similar techniques work for other finitely many-valued logics in which analogues of the J-functions can be defined. (Much technical effort has gone into investigating which functions can be defined in various many-valued systems.) An axiom system for $Ł_\aleph$ is obtained by replacing the last axiom cited earlier with:

$$((A \to B) \to B) \to ((B \to A) \to A)$$

If the designated values are changed to $[r, 1]$ (closed at the left end) or $(r, 1]$ (open at the left end), for some rational number, r, the systems are also axiomatizable. If r is an irrational number, they may not be.

Appropriate tableau and natural deduction systems for many-valued logics can often be found. For example, here is a tableau system for FDE. Lines of the tableau are of the form A: + or A: – . (Intuitively, + means "is designated" and – means "is not designated".) To test the inference $A_1, \dots, A_n \vdash B$, start with lines of the form A_1: + , …, A_n: + , B: – . The rules are as follows (± can be disambiguated uniformly either way):

$$\begin{array}{ccc} \alpha\wedge\beta:+ & \alpha\wedge\beta:- & \neg(\alpha\wedge\beta):\pm \\ \downarrow & \swarrow\ \searrow & \downarrow \\ \alpha:+ & \alpha:- \quad \beta:- & \neg\alpha\vee\neg\beta:\pm \\ \beta:+ & & \end{array}$$

$$\begin{array}{ccc} \alpha\vee\beta:- & \alpha\vee\beta:+ & \neg(\alpha\vee\beta):\pm \\ \downarrow & \swarrow\ \searrow & \downarrow \\ \alpha:- & \alpha:+ \quad \beta:+ & \neg\alpha\wedge\neg\beta:\pm \\ \beta:- & & \end{array}$$

$$\begin{array}{c} \neg\neg\,\alpha:\pm \\ \downarrow \\ \alpha:\pm \end{array}$$

A branch closes if it contains lines of the form A: + and A: – . Adding closure whenever there are lines of the form A: + and $\neg A$: +, gives K_3. Adding closure whenever there are lines of the form A: – and $\neg A$: –, gives LP. (Adding both gives classical logic.) The first-order versions of all the finitely many-valued logics already mentioned also have sound and complete proof procedures. However, first-order $Ł_{\aleph}$ is not axiomatizable. By contrast, the logics that are the same as $Ł_{\aleph}$, except that for some rational number, $r<1$, $Des = (r, 1]$ (open at the left end) or $[r, 1]$ (closed at the left end) are axiomatizable.

MANY-VALUED AND OTHER LOGICS

A number of important logics, notably intuitionist logic, standard modal, and relevant logics, are demonstrably not finitely many-valued. Specifically, suppose that a logic validates the inferences $\vdash A \rightarrow A$ and $A \vdash A \vee B$. Then for any a,b$\in$ *Val*, $f_{\rightarrow}(a, a) \in Des$, and if $a \in Des$, $f_{\vee}(a, b) \in Des$. Now suppose that the logic is n-valued, and that $p_0, \dots, p_n$ are distinct pps. Let A be the disjunction of all formulas of the form $p_i \rightarrow p_j$ (for $0 \leq i \neq j \leq n$). Consider any evaluation. For some i and j, p_i and p_j must have the same value; hence, $p_i \rightarrow p_j$, and so A, are designated. Hence, A is a logical truth. The logics just cited can be shown to have no logical truths of this form (where $\rightarrow$ is the intuitionist, strict, and relevant conditional, respectively).

However, nearly all logics have an infinitely many-valued semantics of a rather unilluminating kind. Consider the set of logical truths of any logic closed under uniform substitution. Let *Val* be the set of formulas of the language; $Des = \{A : \vdash A\}$; $f_{\#}(A_1, \dots, A_n) = \#(A_1, \dots, A_n)$. Then $\vdash A$ iff $\vDash A$.

[Proof: Suppose that A is a logical truth. Consider any interpretation, ν. It is easy to check that $\nu(A)$ is A with every pp, p, replaced by $\nu(p)$. Since the logic is closed under uniform substitution $\nu(A)$ is a logical truth; that is, it is designated. Conversely, suppose that A is not a logical truth. Consider the interpretation, ν, which maps every pp to itself. It is easy to check that $\nu(A) = A$, which is not designated.]

The construction can be extended to show that any Tarski consequence relation with finite sets of premises has a many-valued semantics iff it satisfies one condition. This is called uniformity, and is, loosely speaking, to the effect that pps not involved in an inference are irrelevant to it. Specifically, if $\Gamma, \Delta \vDash A$, then $\Gamma \vDash A$, provided that:

1.) Δ is nontrivial (that is, for some B, $\Delta \nvDash B$)

2.) No formula in Δ contains a pp that occurs in a formula in $\Gamma \cup \{A\}$

It should be noted that not all logics are uniform. In Ingebrigt Johansson's minimal logic, $\emptyset \cup \{p, \neg p\} \vDash \neg q$, but $\{p, \neg p\}$ is nontrivial, and $\emptyset \nvDash \neg q$.

The finiteness constraint can be dropped if the notion of uniformity is strengthened in an appropriate fashion. (Some interesting differences between single-conclusion inference and multiple-conclusion inference emerge in this case.)

PHILOSOPHICAL APPLICATIONS

Many-valued logics have been claimed to have numerous philosophical applications. Like all interesting philosophical matters, these applications are debatable.

Łukasiewicz interpreted Aristotle's argument in *De Intepretatione* (chapter 9) as showing that, though true statements about the past and present are now necessarily true, contingent statements about the future (such as "There will be a sea battle tomorrow") currently have an indeterminate truth status. He suggested deploying $Ł_3$ in an analysis of this situation, reading the truth values {1, ½, 0} as necessarily true, indeterminate, and necessarily false, respectively. As one would expect $A \vee \neg A$ is not logically valid in $Ł_3$.

Łukasiewicz suggested adding an operator to the language, □, representing necessity, whose truth conditions may be represented as follows:

	□
1	1
1/2	0
0	0

Its dual, possibility, ◇, that is, ¬□¬, is as follows:

	◇
1	1
1/2	1
0	0

This makes the inference $A \vdash \Box A$ valid—which is reasonable enough on the Aristotelian picture. However, it also makes the inference $\Diamond A, \Diamond B \vdash \Diamond (A \wedge B)$ valid—which it is not, even for Aristotle. (Just let B be $\neg A$.) As has already been seen, normal modal logics are not finitely many-valued.

Future contingents are just one example of sentences that have been suggested as being neither true nor false (truth value gaps). Others include: sentences with reference failure ("The king of France is bald," "3 = 1/0"), category mistakes and other "nonsense" ("This stone is thinking of Vienna"), paradoxical sentences of self-reference ("This sentence is false"), sentences attributing a vague property in a borderline case ("This is a child"—said of someone around puberty), and sentences unverifiable by the appropriate mathematical or scientific procedure ("There are ten consecutive '7's in the decimal expansion of π", "This electron has a velocity of exactly 100 m/sec").

It is often claimed that K_3 (or, sometimes, B_3) is the appropriate logic for such cases: Gappy sentences take the value n. (In the last case, quantum logic and intuitionist logic have also been suggested to handle the matter.) In these logics $A \vee \neg A$ is not a logical truth, but neither is anything else. In particular, then, $A \wedge \neg A$ is not a logical falsity. Even if "The king of France is bald" is neither true nor false, "The king of France is bald and not bald" would seem to be logically false.

One way around this problem is to deploy the method of supervaluations. If v is any K_3 evaluation, let μ be a supervaluation of v ($v \preceq \mu$) iff:

$$\mu(p) \text{ is never } n \text{, and if } v(p) \neq n,\ v(p) = \mu(p)$$

An important feature of this logic, not shared by $Ł_3$, is that if $v(A)$ is 1 or 0, and $v \preceq \mu$, then $\mu(A)$ has the same value.

Now define the supertruth-value, v_s of a sentence under v as follows:

$$\begin{aligned} v_s(A) &= 1 \text{ if for all } \mu \text{ such that } v \preceq \mu,\ \mu(A) = 1 \\ &= 0 \text{ if for all } \mu \text{ such that } v \preceq \mu,\ \mu(A) = 0 \\ &= n \text{ otherwise} \end{aligned}$$

Define an inference as supervaluation valid if it preserves supertruth-value 1. The inferences that are supervaluation-valid now turn out to be exactly those that are classically valid.

[Proof: If an inference is not classically valid, let v be an evaluation that makes the premises true and the conclusion false. But v is a K_3 evaluation and $v \preceq v$. Hence the inference is not supervaluation-valid. Conversely, suppose that an inference is not supervaluation valid. Then there is a K_3 valuation, v, such that every supervaluation of v gives all the premises the value 1, but not the conclusion. Hence, there is some supervaluation that gives all the premises value 1, but the conclusion value 0. This is a classical evaluation. Hence, the argument is classically invalid.]

On the other side of the street, it has been suggested that some sentences are both true and false (truth-value gluts). These include: paradoxical sentences of self-reference ("This sentence is false"), statements describing instantaneous transition states ("He is in the room"—said at the instant he is symmetrically poised between being in and out), statements of rights and obligations ("She is legally required to do such and such"—when the requirements are based on inconsistent legislation), and sentences attributing a vague property in a borderline case ("This is a child"—said of someone around puberty).

It is sometimes suggested that LP—or FDE if one wants to also take in the possibility of truth value gaps—is the appropriate logic for such cases. The glutty sentences take the value b. (Other paraconsistent logics have also been suggested for the job.) In these logics $A \wedge \neg A$ may take a designated value. In LP the negation of this is also a logical truth.

A way to regain classical logic with LP is by the use of subvaluations. Subvaluations and subvaluation validity are defined in the way dual to supervaluation (*b* replacing *n*, and *some* replacing *all*). In the case of subvaluations, one has the equivalence between classical validity and subvaluation validity only in the one-premise case. (But the duality between the two cases is exact. In a classical multiple-conclusion logic $A \vee B \vdash A, B$ is valid. It is not supervaluation-valid. The equivalence between classical and supervaluational validity holds only because in a single-conclusion inference, one is, in effect, disjoining all the conclusions. In the subvaluation case, this corresponds to conjoining all the premises, which reduces matters to the single premise case.) The technique of super/subvaluations can be generalized to FDE, where there are both gaps and gluts.

A weakness of both LP and FDE is that they do not have a detachable conditional, since $A, A \rightarrow B \nvDash B$. They can be augmented with such a conditional, though. Thus, the many-valued logic RM_3 augments LP with a detachable conditional, $\Rightarrow$, whose truth conditions can be represented as follows:

$\Rightarrow$	1	b	0
1	1	0	0
b	1	b	0
0	1	1	1

In the context of information processing, truth value gaps are often interpreted as incomplete information, and truth-value gluts as inconsistent information. While in the context of gaps and gluts, a word should be said about set theory. It is well known that the naive comprehension schema

$$x \in \{y : A(y)\} \leftrightarrow A(x)$$

leads to contradiction (and so triviality)—in the shape of paradoxes such as Russell's—when the underlying logic is classical. It has often been suggested that the principle might be consistent (or at least inconsistent but nontrivial) when the underlying logic is many-valued. Problems for such suggestions arise because the principle generates triviality if the logic contains contraction $((A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B))$ and *modus ponens*. Let $A(y)$ be $y \in y \rightarrow B$. Call the set that this defines *c*. Comprehension quickly gives: $c \in c \leftrightarrow (c \in c \rightarrow B)$. Contraction and *modus ponens* then give *B*. (This is Curry's paradox.) RM_3, K_3, B_3, and $Ł_n$ (for finite *n*) all contain *modus ponens* and, if not contraction, something closely related to it that will do the same job. However, the schema based on $Ł_{\aleph}$ is consistent. If the extensionality principle $(\forall x(x \in y \leftrightarrow x \in z) \rightarrow y = z)$ is added, though, then even $Ł_{\aleph}$ gives triviality. (Virtually the same comments can be made about the naive *T*-schema ("*A*" is true $\leftrightarrow$ *A*) when self-reference is present. Though here extensionality is, of course, not an issue.)

For a final example of the philosophical application of many-valued logics: It is often claimed that the appropriate semantics for a language with vague predicates is one with degrees of truth. Such logics now usually go under the rubric of fuzzy logics. $Ł_{\aleph}$ is a paradigm one such. (It is not the only one: $Ł_{\aleph}$ is one of a family of logics in which $Val = [0, 1]$. Each is based on a so-called t-norm—essentially a function stating the truth conditions for an appropriate conjunction connective.) The only logical inference that the simplest form of the Sorites paradox uses is *modus ponens*. This is valid in $Ł_{\aleph}$; but if one changes *Des* to, say, [0.8, 1], it is not. (Let $v(p) = 0.9 \in Des$, $v(q) = 0.7 \notin Des$. Then $v(p \rightarrow q) = 0.8 \in Des$.) Note that probability theory is not a many-valued logic. The probability of a compound sentence is not determined by the probabilities of it components. (Let *a* and *b* be independent fair coins. Let A_H be "Coin *a* will come down heads"; A_T be "Coin *a* will come down tails"; and B_H be "Coin *b* will come down heads." $\text{Prob}(A_H) = \text{Prob}(A_T) = \text{Prob}(B_H) = 0.5$. But $\text{Prob}(A_H \wedge A_T) = 0$ and $\text{Prob}(A_H \wedge B_H) = 0.25$.)

TECHNICAL APPLICATIONS

Many-valued logics have various technical applications. Perhaps the most important of these, in a philosophical context, is their use in proving independence results. Thus, suppose that one has some axiom system, *T*, and wishes to know whether some formula, *A*, is deducible in it. One way to show that it is not is to construct a many-valued logic such that all the axioms of *T* always take a designated value, and all the rules of *T* preserved designated values. It follows that all theorems always take designated values. If one can find an interpretation of the logic in which *A* does not take a designated value, it follows that it cannot be proved.

For example, the following is a set of axioms for the $\rightarrow/\neg$ fragment of the relevant logic often called RW (R minus contraction). The only rule of inference is *modus ponens*:

$A \rightarrow A$

$(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$

$A \rightarrow ((A \rightarrow B) \rightarrow B)$

$\neg\neg A \to A$

$(A \to \neg B) \to (B \to \neg A)$

Now consider the three-valued Łukasiewicz logic, $Ł_3$. One can check (e.g., by truth tables) that all the axioms always take the designated value and that *modus ponens* preserves that property. Now let C be the formula: $(p \to (p \to q)) \to (p \to q)$. Take an evaluation, v, in which $v(p) = ½$ and $v(q) = 0$. Computation verifies that $v(C) = ½$. Hence C is not provable. Since $v(\neg C) = ½$ as well, $\neg C$ cannot be proved either. Hence, C is independent of RW.

A much more technically demanding example of the use of many-valued logics to prove independence is in set theory. If one takes the values of the logic to be those of any Boolean algebra, taking the top value as the only designated value, and interprets the connectives and quantifiers in appropriate ways, the logic delivered is classical logic. Choosing the Boolean algebra in an appropriately set-theoretic way, one can also show that the axioms (and so theorems) of Zermelo Fraenkel set theory, ZF, take the designated value. Choosing the algebra in more cunning fashions, one can show that various important set-theoretic principles, such as the continuum hypothesis, do not receive designated values. Hence, ZF does not entail the continuum hypothesis.

HISTORY, PERSONS, AND REFERENCES

This entry concludes by putting the investigations discussed earlier in their historical context. Relevant references that may be consulted for further details are also given at the end of each paragraph. For a gentle introduction to many-valued logics, see Graham Priest (2001, chapters 7, 8, 11); for a more detailed introduction, see Alasdair Urquhart (2001); and for further detailed technical discussions, see Richard Hähnle (2001). J. Michael Dunn and George Epstein (1977) provide a bibliography of work on many-valued logics up to 1974.

The first modern many-valued logic was $Ł_3$. This, and its generalization to n-valued logics, $Ł_n$, were published by Łukasiewicz around 1920. At about the same time, the U.S. mathematician Emil Post was also constructing finitely many-valued logics. (The most significant feature of Post's systems is its treatment of negation. If the values of the n-valued logic are 0, 1, … , n–1, then $v(\neg A) = | 1 + v(A)|$ (Mod n). Philosophical applications of this many-valued logic are difficult to find.) The logic $Ł_\aleph$ was published by Łukasiewicz and Alfred Tarski in 1930. Much of the early investigation of many-valued logics and their axiomatizations were carried out by Polish logicians including Mordechaj Wajsberg and Jerzy Słupecki. Finding a demonstrably complete axiom system for $Ł_\aleph$ turned out to be a hard problem. Reputedly, it was solved by Wajsberg, but the first proofs to be published were by Alan Rose and Berkeley Rosser and by Chen Chung Chang in the late 1950s. The unaxiomatizability of first-order $Ł_\aleph$ was proved by Bruno Scarpellini in 1962. (Łukasiewicz 1970, Rosser and Turquette 1952, Wójcicki 1988, Malinowski 1993.)

Canonical statements of the other many-valued logics mentioned in this entry were given by the following: B_3, Dmitryi Anatol'evich Bochvar, 1939; K_3, Stephen Kleene, 1952; FDE and RM_3, Alan Ross Anderson and Nuel Belnap, 1975; LP, Graham Priest, 1979. (Rescher 1969, Priest 2001.)

The proof that intuitionist logic is not many-valued was first given by Kurt Gödel in 1933. The idea was applied to modal logic by James Dugunji in 1940. The earliest versions of the idea that every logic has a many-valued semantics are usually attributed to Adolf Lindenbaum in the 1920s. Generalizations are due to Jerzy Łos and Roman Suszko in 1958. (Hughes and Cresswell 1968, Shoesmith and Smiley 1978, Wójcicki 1988.)

The applicability of many-valued logics to the view that some sentences are neither true nor false was pursued by many people in the second half of the twentieth century. These include Richard Routley, Leonard Goddard, Saul Kripke, Kit Fine, and Scott Soames. Supervaluations were invented by van Fraassen in 1969. Toward the end of the twentieth century, their application to vagueness became a very standard idea. The application of many-valued logics to the view that some sentences are both true and false, though less popular, has been pursued by various paraconsistent logicians. These include Newton da Costa, Priest, Routley, and Dominic Hyde. The generalization of supervaluation to logics with gluts as well as gaps was developed by Achille Varzi in the 1990s. (Rescher 1969, Scott 1974, Haack 1978, Dunn and Epstein 1977, Humberstone 1998, Varzi 2000, Priest 2001.)

The possibility of basing the naive comprehension schema for sets on $Ł_\aleph$ was investigated by Thoralf Skolem and Chang in the 1950s. The consistency of the schema (and the inconsistency of extensionality) was proved by Richard White in 1979. (White 1979.)

Fuzzy logics and their applicability to vagueness have been investigated fairly intensely since about the 1970s, by many people, including Kenton Machina and Patrick Grim, and, on the technical side, Lotfi Zadeh, Petr Hájek, and Daniele Mundici. (Keefe 2000, Hájek 1998, Cignoli, D'Ottaviano, and Mundici 2000.)

The use of many-valued logics in independence investigations goes back to the early years of the subject, though this has flourished with the proliferation of non-classical logics in the second half of the twentieth century. One of the earliest techniques for proving independence results in set theory is that of forcing, developed by Paul Cohen in the early 1960s. That similar things could be done with Boolean-valued models was realized by Robert Solovay, Dana Scott, and others a few years later. (Anderson and Belnap 1975, Bell 1985.)

See also Fuzzy Logic; Intuitionism and Intuitionistic Logic; Logic, History of; Logic, Non-Classical; Modal Logic; Paraconsistent Logics; Relevance (Relevant) Logics; Set Theory.

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Graham Priest (2005)

MANY WORLDS/MANY MINDS INTERPRETATION OF QUANTUM MECHANICS

The many worlds/many minds formulations of quantum mechanics are reconstructions of Hugh Everett III's (1957a, 1957b, 1973) relative-state formulation of quantum mechanics. Each is presented as a proposal for solving the quantum measurement problem. Much of the philosophical interest in these theories derives from the metaphysical commitments they suggest. They illustrate the roles played by traditional metaphysical distinctions both in formulating and in evaluating physical theories. They also illustrate the range of metaphysical options one must consider if one wants a metaphysics that is consistent with the structure of the physical world suggested by the best physical theories.

The quantum measurement problem is a consequence of the orthodox quantum-mechanical representation of physical properties. In order to account for interference effects, the orthodox view requires that one allows for a physical system to be in a *superposition* of having mutually incompatible classical physical properties. An electron *e* might, for example, be in a superposition of being in New York City and being in Los Angeles. If the unit-length vector $(NYC)_e$ represents the electron