Geometries and Arithmetics

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1 Introduction: Geometry and Arithmetic

Arithmetic, geometry and logic are the three great *a priori* sciences of Kant's *Critique of Pure Reason*. According to Kant, the mind has certain cognitive structures which, when imposed on our "raw sensations", produce our experiences. The first two, space and time, are dealt with in the Transcendental Aesthetic. The third, the categories, is dealt with in the Transcendental Analytic. In the case of all three, a certain body of truths holds good in virtue of these *a priori* structures; and these constitute the three corresponding sciences; geometry in the case of space, arithmetic in the case of time, and logic in the case of the categories. As the difference in location in the *Critique* indicates, the sciences are not entirely on a par: geometry and arithmetic are synthetic; logic is analytic. None the less, each, as a science, is certain and, essentially, complete. This gives us Euclidean geometry, (standard) arithmetic, and Aristotelian logic.

There are few now who would agree with the Kantian picture of these three sciences—at least in its entirety. It has disintegrated, not just under the pressure of philosophical criticism, but under the pressure of developments in science itself. The science with the clearest modern status is, perhaps, geometry. There are many geometries; and which one is to be applied to actual space is an *a posteriori* matter. Following Frege, many twentieth-century philosophers have taken arithmetic and logic to be both *a priori* and analytic.[1] The purpose of this paper is to argue that arithmetic, at least, is

[1]For example, Wright (1983) shows that standard arithmetic may be derived in second order logic augmented by "Hume's Principle": if X and Y are in one-to-one correspondence then the number of Xs is the same as the number of Ys. This certainly looks as though it could be analytic.

in exactly the same camp as geometry. I will say a few words about logic at the very end of the paper.

The status of arithmetic has been a contentious issue this century, and I am hardly the first to argue for the position I have just stated. Quine's celebrated arguments in 'Two Dogmas of Empiricism' result in the same conclusion. I think that Quine's argument does establish the a posteriority of arithmetic.[2] However, it is not my aim to go over this ground here. Neither do I think that the central philosophical argument I shall employ is very new. (It is simply a sophisticated version of the old saw: if we kept counting two rabbits and two rabbits and getting five, arithmetic would have been refuted.) But I shall build the argument on the existence of relatively recent work on inconsistent arithmetics. I hope that will give it a novel twist.

I will start by reviewing the modern situation concerning geometry. I will then discuss what alternative arithmetics look like. Finally, I will argue that the applicability of an arithmetic is an *a posteriori* matter.

2 Non-Euclidean Geometry

Until the Nineteenth Century, 'geometry' just meant Euclidean geometry; but in the first part of that century some different geometries were developed. Initially, these were obtained by Lobachevski and Bolyi simply by negating one of the postulates of Euclidean geometry in order to try to find a *reductio* proof of it. But under Riemann, the subject developed into one of great generality and sophistication. In particular, he developed a highly elegant theory concerning the curvature of spaces in various geometries. Whether non-Euclidean geometries were to be called geometries *in strictu sensu* might have been a moot point; after all, they did not describe the structure of physical space. But they were at least theories about objects called 'points', 'lines', etc., whose behaviour bore important analogies to that of the corresponding objects in Euclidean geometries. Moreover, Riemann realised that it might well be an empirical question as to which geometry should be applied in physics.[3]

Within another 50 years, and even more shocking to Kantian sensibilities, Riemann had been vindicated. The General Theory of Relativity postulated

[2] Though, as an attack on analyticity, I think the argument fails. See Priest (1979).

[3] On the history of non-Euclidean geometry, see Gray (1994); and on Riemann in particular, see Bell (1953), vol. 2, ch. 26.

a connection between mass and the curvature of space (or space-time) which implied that space may have non-zero curvature, and so be non-Euclidean. Predictions of this theory were borne out by subsequent experimentation, and the Theory is now generally accepted.

How to understand the status of physical geometry as it emerged from this affair is still philosophically contentious. The simplest interpretation is a realist one.[4] Geometry in physics is a theory about how certain things in physical space, i.e., points, lines, etc., behave; and a non-Euclidean geometry gets it right. The alternative to realism is non-realism, of which there are many kinds. One is reductionism: talk of geometric points and lines is to be translated without loss into talk of relationships between physical objects. This is the view most famously associated with Leibniz,[5] but has found few modern adherents. Another kind of non-realism is instrumentalism.[6] Geometry has no descriptive content, literal or reductive. It is merely auxiliary machinery for the rest of physics. As such, we may choose whatever such machinery makes life easiest elsewhere; and a non-Euclidean geometry does just that. One notable version of instrumentalism is that according to which, once we have chosen a geometry, its claims become true by convention. Such conventionalism is often associated with the name of Poincaré.[7]

Whichever of these spins one puts on the ball, the changes in geometry have forced us to draw a crucial distinction. We must distinguish between geometries as pure mathematical structures and geometries as applied theories. As pure mathematical structures, there are many geometries. Each is perfectly well-defined proof-theoretically or model-theoretically. What holds in it may be *a priori*. By contrast, which pure geometry to apply to the cosmos as a physical geometry is neither *a priori* nor certain, but is to be determined by the usual criteria of physical science.

3 Non-Standard Arithmetics

Having spelled out the situation for geometry, let me now address the question of whether the situation is the same for arithmetic. There are two issues to be addressed here: whether there are alternative arithmetics, as there are

[4]This is endorsed by Nerlich (1976).

[5]And also Aquinas, *Summa Theologia* I, *quaest.* 46.

[6]This is endorsed in Hinckfuss (1996).

[7]See Poincaré (1952).

alternatively geometries; and whether the question of which one to apply is *a posteriori*. I will take the questions in that order.

Normal arithmetic is the set of sentences of the usual first-order language that are true in the standard model, the natural numbers, $0, 1, 2, ...$, as subject to the usual arithmetic operations. We may take an alternative arithmetic simply to be one that is inconsistent with this. In other words, we form an alternative arithmetic by throwing in something false in the standard model. Naturally, if consistency is to be preserved, other things must be thrown out. There are two possibilities here.

The first is that we retain all the axioms of Peano Arithmetic, but add the negation of something independent of Peano Arithmetic but true in the standard model. We then have a theory that has a classical non-standard model.[8] As a rival arithmetic, such theories are a little disappointing, however. For, as is well known, any model of such a theory must have an initial section that is isomorphic to the standard model. In a sense, then, such theories are not rivals to standard arithmetic, but extensions thereof.

The second, and more radical, way of obtaining a nonstandard arithmetic is to add something inconsistent with the Peano axioms, and jettison some of these. (This is the analogue of how non-Euclidean geometries were initially produced.) In principle, this could produce many different systems, but I know of only one to be found in the literature. This jettisons the axiom which says that numbers always have a successor, and adds its negation, producing a finite arithmetic.[9] Although there are such systems, then, there is no well-worked out theory of their general structure.

A more radical way still of producing a non-standard arithmetic, for which there is now a general theory, is to drop the consistency requirement. We may then add the negation of something true in the standard model *and jettison nothing.* This situation is novel enough to warrant an extended introduction.

[8]In this model there may even be solutions to diophantine equations that have no solution in the standard model—by the solution to Hilbert's tenth problem—though these solutions will have no name in the standard language of arithmetic.

[9]See van Bendegem (1987). Goodstein (1965) gives an arithmetic where a number can have more than one successor, though it would be more accurate to describe this as an arithmetic in which there is more than one successor function.

4 Solving Equations

A driving force behind the development of mathematics can be seen as the extension of the number system in such a way as to provide solutions to equations that had no solution. Thus, for example, the equation $x + 3 = 2$ has no solution in the natural numbers. Negative numbers began to be used for this purpose around the Fifteenth Century. Or consider the equation $x^2 = -1$, which has no solution in the domain of real numbers. This occasioned the introduction of complex numbers a little later. In each case, the old number system was embedded in a new number system in which hitherto insoluble equations found roots.

Now consider Boolean equations. A Boolean expression is a term constructed from some of an infinite number of variables, p, q, r... by means of the functors $\wedge$, $\vee$ and $^-$ (complementation). A Boolean equation is simply an equation between two Boolean expressions. The simplest interpretation for this language is the two-element Boolean algebra, $\mathcal{B}_2$, whose Hasse diagram is:

$$\begin{array}{c} \top \\ \updownarrow \\ \bot \end{array}$$

($\wedge$ is interpreted as meet, $\vee$ as join and $^-$ as order-inversion). Within this interpretation many Boolean equations have solutions. E.g., the equation $p \vee \overline{p} = q$ is solved by $q = \top$ and $p = \top$ (or $\bot$). But many equations have no solutions, e.g., $p = \overline{p}$. It is natural, then, to extend the algebra to one in which all equations have solutions. The simplest such one is the algebra $\mathcal{D}_3$, whose Hasse diagram is as follows:

$$\begin{array}{c} \top \\ \updownarrow \\ \mu \\ \updownarrow \\ \bot \end{array}$$

(where operations are interpreted in the same way; in particular, μ is a fixed-point for $^-$). In this structure the equation $p = \overline{p}$ is solved by $p = \mu$. More generally, it is not difficult to check that if every variable is assigned μ, any Boolean expression evaluates to μ. Hence every Boolean equation has a solution. (The one just given is rather trivial; but in general, there will be others.)

The construction here is a special case of a more general one. The three element algebra just given is a De Morgan algebra with a fixed point for negation.[10] By the same argument as before, every Boolean equation has a solution in an algebra of this kind. The general result is that every Boolean algebra can be embedded in such an algebra.[11] By Stone's theorem, every Boolean algebra can be embedded in a power-set algebra. Hence, it suffices to prove the result for power-set algebras. Let A be any set, and let $\langle \wp(A), \cap, \cup, ^{-} \rangle$ be its power set algebra. Let $S = \{-1, 0, 1\}$ and let $F = S^A$.[12] If f,$g \in F$, define the functions $f \wedge g$, $f \vee g$ and $\overline{f}$ as follows:

$$\begin{aligned} f \wedge g(x) &= \min\{f(x), g(x)\} \\ f \vee g(x) &= \max\{f(x), g(x)\} \\ \overline{f}(x) &= -f(x) \end{aligned}$$

It is straightforward to check that $\langle F, \wedge, \vee, ^{-} \rangle$ is a De Morgan algebra. ($f \leq g$ iff for all $x \in A$, $f(x) \leq g(x)$.) Also, if f_i is the constant function with value i, then $\overline{f_0} = f_0$. Finally, it is easy to check that if $B \subseteq A$, and $\hat{B}$ is the following function:

$$\begin{aligned} \hat{B}(x) &= 1 \quad \text{if } x \in B \\ &= -1 \quad \text{if } x \in B - A \end{aligned}$$

then the map $B \mapsto \hat{B}$ is an embedding, as required. (Note also that if, for some a, $A = \{a\}$, so that $\wp(A)$ is $\mathcal{B}_2$, then F is $\mathcal{D}_3$, with $\top = f_1$, $\mu = f_0$ and $\perp = f_{-1}$.)

The result is analogous to one of the fundamental theorems of algebra, that every field can be extended to one over which all equations have solutions (i.e., all non-constant polynomials have roots), an algebraic closure. (The algebraic closure of the reals is, of course, the field of complex numbers.) The above proof shows that any Boolean algebra can be closed in a similar way.[13]

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[10]A De Morgan algebra is a structure $\langle D, \wedge, \vee, ^{-} \rangle$, where $\langle D, \wedge, \vee \rangle$ is a distributive lattice, and for all a, $b \in D$, $a = \overline{\overline{a}}$ and $a \leq b \Rightarrow \overline{b} \leq \overline{a}$.

[11]This proof is due to Greg Restall.

[12]In general, S could be any subset of the reals containing 0 and closed under $-$.

[13]Strictly speaking, the result is not quite the analogue, since the algebra obtained is not a Boolean algebra. The exact algebraic analogue is that every De Morgan algebra (and *a fortiori* every Boolean algebra) can be extended to a De Morgan algebra in which all equations have solutions. And in fact, the techniques of the proof can be extended to show this. Every De Morgan algebra is isomorphic to a field of polarities (Dunn (1986),

5 Models of the Inconsistent

Algebras have many applications. The relevant one in the present context, is to provide a structure of truth-values for a language. The fact that every Boolean expression can be evaluated to μ in $\mathcal{D}_3$ means that if we take these as the semantic values of a propositional language, and interpret connectives in the obvious way, then, provided we take the designated values, Δ, to be those $\geq \mu$, every sentence is satisfiable; as, more generally, is every set of sentences. These semantics characterise the paraconsistent logic LP,[14] and the value μ may be thought of as *both true and false*. Such a value might seem a rather odd one. One might be tempted to call it an imaginary truth value, for the same reason that $\sqrt{-1}$ was called imaginary. But there is nothing really imaginary about imaginary numbers. Mathematically speaking they are just as *bona fide* as real numbers. Indeed, they even have applications in physics. Leave quantum mechanics out of this; even in classical physics, magnitudes such as impedance are given by complex numbers. Similarly, mathematically speaking, there is nothing imaginary about μ. And like imaginary numbers, μ may even have important applications. For example, it has been argued that one application for μ is to take it to be the truth value of paradoxical self-referential sentences.

Propositional LP can be extended to a first order logic in an obvious fashion. Let L be a first order language. An LP interpretation for L is a pair, $\langle D, d\rangle$, where D is a non-empty domain, d maps each constant into D, each n-place function symbol into an n-place function on D, and each n-place predicate to an n-place function from D into $\mathcal{D}_3$. For identity, we require that $d(=)(x, y) \in \Delta$ $(= \{\top, \mu\})$ iff $x = y$. The semantic values of formulas are assigned in the obvious way (the quantifiers being treated as the analogues of conjunction and disjunction). An inference is valid iff there is no interpretation for which all the premises are in Δ, but the conclusion is not.

It is clear that classical interpretations are special cases of LP interpretations. It follows that LP is a sub-logic of classical logic. It is a proper sub-logic, since in it $\alpha \wedge \neg\alpha \not\models \beta$. However, it is to be noted that the logi-

p. 189). The members of such a field are pairs of the form $\langle X, Y\rangle$ where each of X and Y is a subset of some underlying set, A. The closure is now $S^A \times S^A$, operations being defined in the natural way. The rest of the proof is then much as before.

[14] See, e.g., Priest (1987), ch. 5.

cal truths of LP are exactly those of classical logic.[15] As is to be expected, every set of first-order sentences has a model. There is always the trivial model, with a one-element domain, such that for every predicate, P, $d(P)$ is the constant function with value μ. But in general, there will be many non-trivial models too. One useful way of constructing these is as follows.

Let $\mathcal{I} = \langle D, d \rangle$ be any interpretation. Let $\sim$ be any equivalence relation on D, that is also a congruence relation on the denotations of the function symbols in the language (i.e., if g is such a denotation, and $d_i \sim e_i$ for all $1 \leq i \leq n$, then $g(d_1, ..., d_n) \sim g(e_1, ..., e_n)$). If $d \in D$ let $[d]$ be the equivalence class of d under $\sim$. Define the *collapsed interpretation*, $\mathcal{I}_\sim = \langle D_\sim, d_\sim \rangle$, as follows. $D_\sim = \{[d]; d \in D\}$; if c is a constant, $d_\sim(c) = [d(c)]$; if f is an n-place function symbol, $d_\sim(f)([d_1], ..., [d_n]) = [d(f)(d_1, ..., d_n)]$ (this is well defined, since $\sim$ is a congruence relation); and if P is an n-place predicate:

$$\begin{array}{lll} d_\sim(P)([d_1], ...[d_n]) & = \top & \text{if } \forall e_1 \sim d_1 ... \forall e_n \sim d_n, d(P)(e_1, ...e_n) = \top \\ & = \bot & \text{if } \forall e_1 \sim d_1 ... \forall e_n \sim d_n, d(P)(e_1, ...e_n) = \bot \\ & = \mu & \text{otherwise} \end{array}$$

(It is easy to check that $d_\sim(=)$ is as required for an LP interpretation.) In effect, the collapsed model identifies members of an equivalence class to produce a composite object. The predicates true or false of this are exactly those true of false of all the objects that compose it. In particular, then, if two members of the class have inconsistent properties the equivalence class is an inconsistent object.

It is now not difficult to prove the *Collapsing Lemma*: if the value of α is in Δ in $\mathcal{I}$, then it is in Δ in $\mathcal{I}_\sim$. First we show that for any term, t, $d_\sim(t) = d([t])$. Applying this fact secures the atomic case. The result is then proved by induction.[16] The Collapsing Lemma assures us that if an interpretation is a model of some set of sentences, then any interpretation obtained by collapsing it will also be a model. This allows us to construct non-trivial models of inconsistent theories. Let us see how by returning to arithmetic.

[15] For a proof, see Priest (1987), ch. 5.

[16] See Priest (2000a).

6 Inconsistent Arithmetic

Let L be the usual language of first order arithmetic, with function symbols for successor, addition and multiplication, and one binary predicate, $=$. Let N be the natural numbers; let $\mathcal{N}$ be the standard model of L; and let A be the set of sentences true in $\mathcal{N}$. Any model of A, I will call a *model of arithmetic.*

Example one. Let n, $p \in N$ and $p > 0$. Define a relation, $\sim$, on N, thus:

$$x \sim y \text{ iff } (x,\, y < n \text{ and } x = y) \text{ or } (x,\, y \geq n \text{ and } x = y \pmod{p})$$

It is easy to check that $\sim$ is a congruence relation on N. Let $\mathcal{N}_n^p$ be the model obtained by collapsing with respect to it. The Collapsing Lemma assures us that it is a model of arithmetic. It is finite; it has an initial tail of length n that behaves consistently. The other numbers form a cycle of period p. The successor graph can be depicted as follows:

$$\begin{array}{ccccccccc} 0 & \to & 1 & \to & \dots & \to & n & \to & n+1 \\ & & & & & & \uparrow & & \downarrow \\ & & & & & & n+p-1 & \leftarrow & \cdots \end{array}$$

Example two. Let $\mathcal{M}$ be any non-standard classical interpretation of A. Define the relation $\sim$ as follows:

$$x \sim y \text{ iff } (x,\, y \in N \text{ and } x = y) \text{ or } (x,\, y \text{ are non-standard})$$

Again, it is easy to check that $\sim$ is an equivalence relation that is also a congruence on the arithmetic operators. Let $\mathcal{N}_\Omega$ be the model obtained by collapsing with respect to it. $\mathcal{N}_\Omega$ contains the standard interpretation, plus an inconsistent "point at infinity". The successor graph can be depicted as follows:

$$\begin{array}{cccccc} 0 & \to & 1 & \to & \dots & \Omega \\ & & & & & \circlearrowright \end{array}$$

Just as the logic LP allows the solution of Boolean equations that do not have a solution in a Boolean algebra, the inconsistent arithmetics allow solutions to arithmetic equations that do not have them in the standard model. (Any equation that has a solution in $\mathcal{N}$, has a solution in any model of arithmetic.) For example, $x = x + p$ has solutions in any $\mathcal{N}_n^p$, and if s and t are any non-constant polynomials in x, then $s = t$ has a solution in $\mathcal{N}_n^1$ (namely, n) and $\mathcal{N}_\Omega$ (namely, Ω).[17] This fact will be important later.

[17] On solutions of equations in inconsistent models, see, further, Mortensen (1995).

7 The General Structure of Models

The foregoing provides all that we need to pursue the philosophical issues raised, but let me digress for a section to describe the general structure of LP models of arithmetic (which include, of course, all the classical models).[18]

Let $\mathcal{M} = \langle M, d\rangle$ be any model of A, and let us define the ordering on M in the usual way: $x \leq y$ iff $\exists z\ x + z = y$. If $i \in M$, define $N(i)$ (the *nucleus* of i) to be $\{x \in M; i \leq x \leq i\}$. (In a classical model, $N(i) = \{i\}$, but this need not be the case in an inconsistent model. For example, in $\mathcal{N}_n^p$ $\{x; n \leq x\}$ is a nucleus.) Any two members of a nucleus define the same nucleus. Now, if N_1 and N_2 are nuclei, define $N_1 \preceq N_2$ iff for some $i \in N_1$ and $j \in N_2$, $i \leq j$. $\preceq$ is a linear ordering.

If $i \in M$, i *has period* $p \in M$ iff $i + p = i$. (In a classical model every number has period 0 and only 0. But again, this need not be the case as $\mathcal{N}_n^p$ demonstrates.) All members of a nucleus have the same periods. We may thus speak of the periods of the nucleus itself. If a nucleus has a period distinct from 0, I will call it *proper*. If $N(i) \preceq N(j)$ then any period of $N(i)$ is a period of $N(j)$. Hence any nucleus after a proper nucleus is a proper nucleus.
If i is in a proper nucleus, its successor, i' is in the nucleus. i may have more than one predecessor (as $\mathcal{N}_n^p$ demonstrates). However, i has a unique predecessor, $'i$, in that nucleus. Thus, if i is in a proper nucleus, its *chromosome*:

$$...,'' i,' i, i, i', i''...$$

is contained in the nucleus. A chromosome is either a finite cycle or is isomorphic to the integers. Moreover, all the chromosomes in any one nucleus are either finite cycles of some minimum non-zero period (in which case the minimum non-zero periods of all subsequent nuclei are divisors of it—not necessarily proper), or are isomorphic to the integers. (Both sorts of nuclei are possible.)

Thus, the general structure of a model is a liner sequence of nuclei. There are three segments (any of which may be empty). The first segment contains only improper nuclei. The second segment contains proper nuclei with linear chromosomes. The final segment contains proper nuclei with cyclical chromosomes of finite period. A period of any nucleus is a period of any subsequent nucleus; and in particular, if a nucleus in the third segment has minimum

[18] I shall not give proofs. These can be found in Priest (200a).

non-zero period, p, the minimum non-zero period of any subsequent nucleus is a divisor of p. Thus we might depict the general structure of a model as follows (where m is a multiple of n):

$$0, 1, \ldots \quad \boxed{\begin{array}{c} \ldots a \to a' \ldots \\ \ldots b \to b' \ldots \\ \vdots \end{array}} \quad \ldots \quad \boxed{\begin{array}{ccc} d_0 \ldots d_i & e_0 \ldots e_i & \\ \uparrow \quad \downarrow & \uparrow \quad \downarrow & \ldots \\ d_{m-1} \ldots d'_i & e_{m-1} \ldots e'_i & \end{array}} \quad \ldots \quad \boxed{\begin{array}{ccc} f_0 \ldots f_i & g_0 \ldots g_i & \\ \uparrow \quad \downarrow & \uparrow \quad \downarrow & \ldots \\ f_{n-1} \ldots f'_i & g_{n-1} \ldots g'_i & \end{array}} \quad \ldots$$

An interesting question is that of what orderings the proper nuclei in a model may have. It is known that they can have the order-type of any ordinal. But they can also have the order type of the rationals, or of any ordering that can be embedded in the rationals in a certain continuous fashion. A general solution to this problem (and many others) is still open.

8 Empirical Applications

Let us now return to the philosophical issues. Each model of arithmetic (or its theory)—save the standard one—is an alternative pure arithmetic, in the same sense that there are alternative pure geometries: abstract mathematical structures dealing with objects (numbers or points) that behave in a way recognisably similar to the corresponding objects of the standard theories. Moreover, as we have seen, there are arithmetics which do not even have initial sections isomorphic to the natural numbers (whilst still verifying all of standard arithmetic). Thus, the first point of similarity with geometry is established.[19]

To establish the second point, it must be shown that which of these arithmetics one is to apply, is an *a posteriori* matter. And this may well be doubted. Could there be applied non-standard arithmetic in the same way that there is applied non-Euclidean geometry?

I take the answer to be 'yes'. As I said in the introduction, there are general Quinean arguments why this is the case, though I do not intend to discuss these here. Instead, I will attempt to establish the point by telling

[19]It might, at this point, be suggested that *real* arithmetic is second order, and that this is categorical. Hence there is only one arithmetic. The reply is that we can still obtain non-standard arithmetics by varying the Peano postulates. Moreover, the Collapsing Lemma applies to *second order LP* as well, and this can be used to give non-standard second order paraconsistent arithmetics.

a story to show how we might come to replace a standard application of ordinary arithmetic with a different one.[20] I do not want to claim that the situation described is a possibility in any real sense: it is merely a thought experiment. But if it succeeds it will show the required conceptual possibility. And this suffices to show that applied arithmetic is *a posteriori*, in the same way that applied geometry is.

There are a few such stories already in the literature, notably, those of Gasking (1940). Gasking was criticised by Castañeda (1959), who argued that if we use a different arithmetic and preserve our standard practices of counting, we end up with either a simple change of terminology or inconsistency. I will not discuss whether Castañeda's arguments work, here. I intend to finesse them by describing a situation that might motivate an inconsistent arithmetic, specifically, one of the form $\mathcal{N}^1_n$.

Let us suppose that we come to predict a collision between an enormous star and a huge planet. Using a standard technique, we compute their masses as x_1 and y_1, respectively. Since masses of this kind are, to within experimental error, the sum of the masses of the baryons (protons and neutrons) in them, it will be convenient to take a unit of measurement according to which a baryon has mass 1. In effect, therefore, these figures measure the numbers of baryons in the masses. After the collision, we measure the mass of the resulting body, and obtain the figure z, where z is much less than $x_1 + y_1$. Naturally, our results are subject to experimental error. But the difference is so large that it cannot possibly be explained by this. We check our instruments, suspecting a fault, but cannot find one; we check our computations for an error, but cannot find one. We have a puzzle. Some days later, we have the chance to record another collision. We record the masses before the collision. This time they are x_2 and y_2. Again, after the collision, the mass appears to be z (the same as before), less than $x_2 + y_2$. The first result was no aberration. We have an anomaly.

We investigate various ways of solving the anomaly. We might revise the theories on which our measuring devices depend, but there is no obvious way of doing this. We could say that some baryons disappeared in the collision; alternatively, we could suppose that under certain conditions the mass of a

[20]There are certainly other possible examples. For example, even if arithmetic is not the form of the intuition of time, as Kant thought, one might tell a story where circumstances suggested the possibility of calibrating time with a non-standard arithmetic. In virtue of the cycles in inconsistent arithmetics, stories of time-travel would appear to be particularly fruitful here.

baryon decreases. But either of these options seems to amount to a rejection of the law of conservation of mass(-energy), which would seem to be a rather unattractive course of action.

Then someone, call them Einquine, fixes on the fact that the resultant masses of the two collisions were the same in both cases, z. This is odd. If mass has gone missing, why should this produce the same result in both cases? An idea occurs to Einquine. Maybe our arithmetic for counting baryons is wrong.[21] Maybe the appropriate arithmetic is $\mathcal{N}_z^1$. For in this arithmetic $x_1 + y_1 = x_2 + y_2 = z$, and our observations are explained without having to assume that the mass of baryons has changed, or that any are lost in the collisions! Einquine hypothesizes that z is a fundamental constant of the universe, just like the speed of light, or Planck's constant.

While she is thus hypothesising, reports of the collisions start to come in from other parts of the galaxy. (The human race had colonised other planets some centuries before.) These reports all give the masses of the two new objects as the same, but all are different from each other. Some even measure them as greater that the sum of their parts. Einquine is about to give up her hypothesis, when she realises that this result is quite compatible with it. Even if the observer measures the mass as z', provided only that $z' > z$ then $z = z'$ in $\mathcal{N}_n^1$, and their results are the same!

But this does leave a problem. Why do observers consistently record result that differ from each other? Analysing the data, Einquine sees that values of z (hers included), are related to the distance of the observer from the collision, d, by the (classical) equation $z = z_0 + kd$ (where z_0 and k are constants). In virtue of this, she revises her estimate of the fundamental constant to z_0, and hypothesizes that the effect of an inconsistent mass of baryons on a measuring device is a function of its distance from the mass. Further observational reports bear this hypothesis out; and Einquine starts to consider the mechanism involved in the distance-effect.

We could continue the story indefinitely, but it has gone far enough. For familiar reasons, there are likely to be theories other than Einquine's that could be offered for the data. Some of them might preserve orthodox arithmetic by jettisoning conservation laws, or by keeping these but varying some physical auxiliary hypotheses. Others might modify arithmetic in some other, but consistent, way (which would be just as good for present purposes). An

[21] We already know that different sorts of fundamental particles satisfy different sorts of statistics.

obvious suggestion here is that we might use, instead, a finite consistent arithmetic, with maximum number z_0, which is its own successor. (For all we have seen of the example so far, this would do just as well.) However, there might well be reasons that lead us to prefer the inconsistent arithmetic. Notably, this arithmetic gives us the full resources of standard arithmetic, whilst the finite arithmetic does not. For example, in the inconsistent arithmetic it is true that for any prime number there is a greater prime, which is false in the finite arithmetic. This extra strength might cause us to prefer it. Alternatively, the difference might even occasion different empirical predictions, which verify the inconsistent arithmetic. Indeed, whatever arithmetical theories we consider applying, these theories may become more or less plausible in the light of further experimentation, etc.[22]

Just as with geometry, the question of the status of the applied arithmetic is another matter. One might tell a realist story about this. Collections of physical objects have a certain physical property, namely size; and sizes, together with the operations on them, have the same objective structure as the numbers and corresponding operations in the non-standard arithmetic. Hence, facts about the mathematical structure transfer directly to the physical structure, and this is why it works. Alternatively, one could tell instrumentalist stories of various obvious kinds. For example, Gasking (1940), runs a conventionalist line on arithmetic that is similar to Poincaré's on geometry. But it is not necessary to go into this here: the mainpoint is made. It is quite possible that we might vary our arithmetic for empirical reasons, even to an inconsistent one. There can, therefore, be alternative applied arithmetics, just as there are alternative applied geometries.[23]

[22]The revision of arithmetic I have just described is a local one, in that it is only the counting of *baryons* that is changed. It would be interesting to speculate on what might happen which could motivate a global change, i.e., a move to a situation where everything is counted in the new way.

[23]Some of the philosophical issues surrounding inconsistent arithmetics, were aired in Priest (1994). There, it is argued that our ordinary arithmetic might be an inconsistent one. A critique of this is to be found in Denyer (1995), with replies in Priest (1996). Denyer's criticisms are not relevant here. For present purposes, I accept that our ordinary arithmetic is the usual one. My concern is with inconsistent arithmetics as revisionary.

9 Non-Standard Logics in Science

It may failry be pointed out that the revision of arithmetic emvisaged in the previous section is more radical than was the adoption of a non-Euclidean geometry. For the arithmetic revision requires the adoption of a non-standard logic, whilst the geometric revision did not.

It might be argued that this shows such revision to be impossible after all. For logic, it might be thought, cannot be revised. Such an argument, though would be incorrect. Logic can be revised: it has been revised. The adoption of Frege/Russell logic early in the 20th century did just that. To give but one example, traditional logic counts the following inference as valid—it is an instance of the syllogism *Darapti*—whilst modern quantification theory does not:

All men are mortal.
All men are liars.
―――――――――――
Some mortals are liars.

There is therefore nothing sacrosanct about logical theories.[24]

The observation, rather, cuts in the other direction. Science uses mathematics like a multi-national company uses the resources of an under-developed country. It steps in and helps itself to anything that it wants to use. Thus, scientists have always been mathematical opportunists. Whether it was infinitesimals, imaginary numbers, bizarre functions (like the Dirac δ-function), it did not matter whether these had an accepted logical foundations. If the mathematics gave the right answers, science employed it, and everything else went along for the ride. Thus, if an example of the kind described in the last section were ever to arise, it would *force* a revision of logic. (If it had not already been revised, since there are good and quite independent reasons for revising our logical theory anyway.) None of the Kantian trilogy—geometry, arithmetic and logic—is immune from revision under the pressure of scientific advance.[25]

[24]The question of the revision of logical theories is discussed at much greater length in Priest (200b), which paper partly overlaps with this one.

[25]An earlier version of this paper was given at the European meeting of the Association of Symbolic Logic, held at Donostia / San Sebastián, July 1996.

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