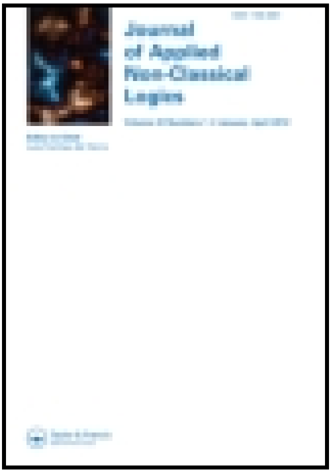

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The Trivial Object and the Non-Uiviality of a Semantically Closed Theory with Descriptions

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The Trivial Object and the Non-Uiviality of a Semantically Closed Theory with Descriptions

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ABSTRACT. *After indicating why this is needed, the paper proves a non-triviality result for a paraconsistent theory containing arithmetic, naive truth and denotation predicates, and descriptions. The result is obtained by dualising a construction of Kroon. Its most notable feature is that there is a trivial object – one that has every property.*

KEY WORDS: *non-triviality, denotation, descriptions.*

Introduction: the Trivial Object

In his tract *On Learned Ignorance*, the fifteenth century Neoplatonist Nicholas of Cusa described God in the following terms:[1]

> [I]n no way do they [distinctions] exist in the absolute maximum. The absolute maximum ... is all things, and whilst being all, is none of them...

It is not my purpose here to discuss why Cusanus held these views, or their ramifications for various issues.[2] All I want to do is note that Cusanus held God to be the trivial object, an object possessing all properties, even mutually inconsistent ones. (Assuming the identity of indiscernibles, there can obviously be only one such object.)

The idea that sensible theories might be inconsistent is not now a new one.[3] No one has suggested that the trivial theory (one in which everything holds) is a sensible one; but it has at least played a role in relevant/paraconsistent logic. For example, it defines a perfectly good world (situation, set up) in the canonical model structures for such logics.[4] Inconsistent objects, i.e., objects with

[1] [Her 54], I, 4.
[2] For some discussion, see [Pri 95], 1.8-9.
[3] See, e.g., [PRN 89].
[4] See, e.g., [RMPB 82].

inconsistent properties, have also figured in recent paraconsistent literature.[5] The trivial object has not surfaced—till now.

Actually, the trivial object is not quite as straightforward as one might think. If one is to make sensible use of it, as Cusanus would clearly like to do, it cannot have all properties, or triviality itself quickly ensues. For example, if φ is a closed formula, it cannot have the property $\lambda x(x = x \wedge \varphi)$ unless φ is itself true. It is natural to restrict its range of triviality to atomic properties—at least in the first instance; other inconsistent properties may follow from this.

Even then, the presence of the trivial object means that various standard logical principles have to be constrained. For example, if we write the trivial object as ∞, then we have, for any objects, a and b, $a = \infty$ and $\infty = b$. Since we do not want it to follow that all is one, transitivity of identity must fail. Similarly, for any relation, R, and object, a, we have $aR\infty$. We do not want to infer that every object R-relates to something (is married, is older than something, taller than something). Hence the rule of existential generalisation, which would allow us to infer that $\exists x aRx$, must also be curbed.

With a few such restrictions in place, however, the trivial object does have a sensible logic. In due course we will see what that is. I leave the matter for the time being, and turn to the main topic of this paper.

1 Triviality and Descriptions

One of the main ideas behind the development of paraconsistent logic has always been that it is possible to have sensible theories incorporating principles such as the naive abstraction schema of set theory and the T-schema of naive semantics, which naturally recommend themselves but give rise to inconsistency. Even if such theories are inconsistent, one does not want them to be trivial. Hence the question of proving suitably formulated theories to be non-trivial is an important one in studies of paraconsistent logic. (Non-triviality proofs play the same role in paraconsistent logic as consistency proofs play in classical logic.) The most important result of this kind so far is Brady's [Bra 89], which shows that naive set-theory (suitably formulated) is non-trivial. It is not difficult to show that a truth-predicate can be defined in this theory.[6] It therefore follows that a naive theory of truth—and even a joint theory of truth and sets—is non-trivial.

Even though such theories are non-trivial, it could of course happen that they become non-trivial when other logical machinery is added. One important such piece of machinery, not included in Brady's results, is that of descriptions. The addition of descriptions to set-theory is not, perhaps, terribly exciting: one already has set abstracts, which are term-forming devices that do most of what one would want descriptions for. The situation with semantics is quite different. Descriptions are implicated in an essential way in semantic paradoxes of self-

[5] E.g., the inconsistent numbers of [Pri 94].

[6] See [Pri 90], fn. 9.

reference, such as Berry's paradox and Köning's paradox.[7] Hence, the non-triviality of a semantically closed theory whose language contains descriptions is of central importance to a paraconsistent solution to such paradoxes.

2 Petersen's Arguments

Nor is the possibility of triviality here a merely academic one. Here are three triviality-style arguments that can be formulated in such a context. (All of the arguments are due, in one way or another, to Uwe Petersen.) They all conclude that $0 = 1$. Depending on the context, this may not be complete triviality, but it is close enough to make no difference.[8] I note in advance that all the arguments use the transitivity of identity at some crucial place.

For definiteness, we suppose that we are working in arithmetic, where self-reference can be achieved by a suitable gödelisation. (Any other form of self-reference would do just as well.) We suppose there to be a description operator, ε, satisfying the following, which I will call the *description principle*:[9]

$$\exists x\varphi \vdash \varphi(x/\varepsilon x\varphi)$$

The stroke denotes substitution. (Here and in what follows, I assume that bound variables have been relabelled if necessary to avoid clashes.) The description principle is justified intuitively by the thought that '$\varepsilon x\varphi$' denotes one of the things that satisfy φ, if there is such a thing.

I A Liar

Suppose that we could find a truth-term, τ, which satisfied the condition:

$$\tau\langle\varphi\rangle = 0 \dashv\vdash \varphi$$

for all closed φ. (If e is some syntactic entity, I write $\langle e\rangle$ for the numeral of its gödel code.) Call this the *functional T-schema*. Then by standard self-referential constructions,[10] we can construct a fixed point formula, ψ, of the form $\mathrm{sg}(\tau\langle\psi\rangle) = 1$, where sg represents the *signum* function defined in the usual way:

$$\begin{aligned}\mathrm{sg}(x) = 0 \quad &\text{if } x = 0\\ \mathrm{sg}(x) = 1 \quad &\text{if } x > 0\end{aligned}$$

The functional T-schema for ψ, give us: $\tau\langle\psi\rangle = 0 \dashv\vdash \mathrm{sg}(\tau\langle\psi\rangle) = 1$. Now $\tau\langle\psi\rangle = 0 \vee \tau\langle\psi\rangle > 0$. In the first case, $\mathrm{sg}(\tau\langle\psi\rangle) = 0$ and $\mathrm{sg}(\tau\langle\psi\rangle) = 1$; in the second case, $\mathrm{sg}(\tau\langle\psi\rangle) = 1$, so $\tau\langle\psi\rangle = 0$, and $\mathrm{sg}(\tau\langle\psi\rangle) = 0$. In either case, then, $0 = 1$.

[7] See, e.g., [Pri 95], 9.4.

[8] For example, in a formal arithmetic with a very weak logic, given $0 = 1$, one can show that for any numbers, n, m, $m = n$. One can then prove all equations, all quantifier free formulas, and hence anything of the form $\exists x_1 ... \exists x_n\varphi$, where φ is quantifier-free.

[9] I formulate this and some other principles in terms of deducibility, rather than the more normal conditionality. This is because the presence of a conditional operator introduces irrelevant complexities.

[10] See, e.g., [BJ 74], p. 176.

Now, given a truth predicate, T, satisfying the T-schema, $T\langle\varphi\rangle \dashv\vdash \varphi$ for all closed φ, it looks easy enough to define τx as follows:

$$\varepsilon y((y = 0 \wedge Tx) \vee (y = 1 \wedge \neg Tx))$$

Given that $Tx \vee \neg Tx$, it is easy to show that $\exists y((y = 0 \wedge Tx) \vee (y = 1 \wedge \neg Tx))$. Hence, by the description principle, $(\tau x = 0 \wedge Tx) \vee (\tau x = 1 \wedge \neg Tx)$; and the functional T-schema would then seem to follow from the ordinary one.

In fact, it does not in a paraconsistent context; a moment's thought suffices to show that the deduction requires the disjunctive syllogism, which will break down for paradoxical sentences of the kind of $T\langle\psi\rangle$.

3 Denotation

The second and third arguments concern not truth but denotation. We suppose that we have a two place predicate, Δ, satisfying the naive denotation schema, which I break up into two parts:

$$\Delta(\langle s\rangle, t) \vdash s = t$$
$$s = t \vdash \Delta(\langle s\rangle, t)$$

for every closed term s. The first part can be thought to express the uniqueness of denotation, since if $\Delta(\langle s\rangle, t_1)$ and $\Delta(\langle s\rangle, t_2)$ it entails that $t_1 = t_2$. The second can be thought of as expressing existence; for since $s = s$, we have $\Delta(\langle s\rangle, s)$, and so $\exists x\Delta(\langle s\rangle, x)$. The Existence half may be thought problematic when descriptions are concerned, since these may suffer from denotation-failure. We will return to this point later.

II A Version of Berry

The second argument,[11] is a version of Berry's paradox. By standard techniques,[12] we may construct a fixed point term, s, such that:

$$s = \varepsilon x\neg\Delta(\langle s\rangle, x)$$

By the naive denotation schema (Existence), we have $\Delta(\langle s\rangle, \varepsilon x\neg\Delta(\langle s\rangle, x))$. But by the description principle, we have:

$$\exists x\neg\Delta(\langle s\rangle, x) \vdash \neg\Delta(\langle s\rangle, \varepsilon x\neg\Delta(\langle s\rangle, x))$$

So contraposing and detaching, we get $\forall x\Delta(\langle s\rangle, x)$. Thus, in particular, $\Delta(\langle s\rangle, 0)$ and $\Delta(\langle s\rangle, 1)$. Hence, by the denotation schema (Uniqueness), $s = 0 = 1$.

A natural thought is that the argument fails, since s fails to denote, and so the Existence half of the denotation schema cannot be applied in the proof. Natural as this thought is, it is not correct, since s can be shown to denote as follows. By Uniqueness, we have $\Delta(\langle s\rangle, 0) \wedge \Delta(\langle s\rangle, 1) \vdash 0 = 1$. Using the

[11] See [Pet 92].

[12] See, e.g., [Pri 9+].

fact that $0 \neq 1$, and contraposing, we get $\neg\Delta(\langle s\rangle, 0) \vee \neg\, \Delta(\langle s\rangle, 1)$, and hence $\exists x \neg\Delta(\langle s\rangle, x)$. But then, '$\varepsilon x \neg\Delta(\langle s\rangle, x)$', i.e., s, must denote something.

In fact, I take the argument to fail for other reasons: the description principle does not contrapose in a paraconsistent context.[13] The intuitive reason for this is easy enough to see. If $\exists x\varphi$ is true, then something satisfies φ; and if so, then $\varepsilon x\varphi$ denotes one such thing. But if the thing in question satisfies not only φ, but also $\neg\varphi$, then $\neg\varphi(x/\varepsilon x\varphi)$ will be true. Nonetheless, $\neg\exists x\varphi$ can be just plain false provided that not everything satisfies $\neg\varphi$. (Constructing a formal counter-model is left as an exercise.) Argument II applies contraposition to the description principle, and therefore fails. In particular, s both does and does not denote $\varepsilon x \neg\Delta(\langle s\rangle, x)$, as we have, in effect, seen. Hence $\Delta(\langle s\rangle, \varepsilon x \neg\Delta(\langle s\rangle, x))$ is both true and false, but s still does not denote everything.

III Hilbert and Bernays' Paradox

The third argument, which combines features of the first two whilst adding novelties of its own,[14] draws on a proof of Hilbert and Bernays. Let us define $\delta(x)$ as $\varepsilon y\Delta(x, y)$. The Existence half of the denotations schema gives us, for any closed term, t, $\Delta(\langle t\rangle, t)$; hence $\exists x\Delta(\langle t\rangle, x)$. So by the description principle $\Delta(\langle t\rangle, \varepsilon x\Delta(\langle t\rangle, x))$. But then $t = \varepsilon x\Delta(\langle t\rangle, x)$, by the Uniqueness half, i.e., $t = \delta(\langle t\rangle)$.

Argument II uses the fact that for any functor, f, we can find a fixed point term of the form $s = f(\langle s\rangle)$. Using the fact established in the last paragraph, we can construct an even stronger fixed point. If f is any functor, consider $f\delta(x)$. By the standard fixed-point property we can construct a fixed point $s = f\delta(\langle s\rangle)$. But then $s = fs$.

Now let f be the parity function, i.e.:

$$fx = 0 \text{ if } x \text{ is odd}$$
$$fx = 1 \text{ if } x \text{ is even}$$

If s is its fixed point, we have $fs = 0 \vee fs = 1$. In the first case, $s = 0$, and so $fs = 1$. Hence $0 = 1$. Similarly in the second case.

Suspicion again falls on the Existence half of the denotation schema. And in this case, it would seem to be entirely justified.) However, let θ be $(\exists z\Delta(x, z) \wedge \Delta(x, y)) \vee (\neg\exists z\Delta(x, z) \wedge y = 0)$, and define $\delta'(x)$ as $\varepsilon y\theta$. As with argument I, it is straightforward to show that $\exists y\theta$. Hence any closed δ' term denotes. If $s = f\delta'(\langle s\rangle)$, and we can show that $\delta'(\langle s\rangle) = s$, the argument can be run as before.

What needs to be shown is that:

$$\varepsilon y((\exists z\Delta(\langle s\rangle, z) \wedge \Delta(\langle s\rangle, y)) \vee (\neg\exists z\Delta(\langle s\rangle, z) \wedge y = 0)) = s$$

Call the left hand side r. Then the description schema gives us:

$$(\exists z\Delta(\langle s\rangle, z) \wedge \Delta(\langle s\rangle, r)) \vee (\neg\exists z\Delta(\langle s\rangle, z) \wedge r = 0)$$

[13] See [Pri 91], fn. 4. The proof that s denotes also applies contraposition to the denotation schema. This is suspect too. [Pri 87] 4.9 argues that contraposition fails for the T-schema. Similar considerations apply, presumably, to the denotation-schema.

[14] See [Pri 9+].

Since s denotes (being an arithmetic functor applied to a term guaranteed to denote), we can rule out the right hand disjunct. Hence, $\Delta(\langle s \rangle, r)$; and since $\Delta(\langle s \rangle, s)$, we have $r = s$ by the denotation schema, as required.

Fortunately, the repair fails in a paraconsistent context. As with argument I, ruling out the second disjunct uses the disjunctive syllogism. And this cannot be relied upon for statements about paradoxical objects of s's kind.

4 Non-Triviality Proofs: a first approach

Although all the arguments fail, they underline the desirability of a non-triviality argument for a semantically closed theory with descriptions and self-reference. The rest of this paper will provide a non-triviality proof for such a theory (where the self-reference is obtained arithmetically). It does not give non-triviality for a theory containing *everything* that might be desired, but it comes close.

Let us start with Kripke's well known construction for a semantically closed theory with truth-value gaps, but no descriptions [Kri 75]. If we start with a ground model of arithmetic plus a totally undefined truth-predicate, and define the least fixed point on this, we end up with a model of classical arithmetic and a truth predicate satisfying the T-schema (couched in terms of deducibility). The construction works for various partial logics. We will fix on the case where the strong Kleene logic is used. There is a simple duality between this logic and the paraconsistent logic LP. Roughly, where one sees gaps, the other sees gluts. Using this duality, it is possible to dualise Kripke's construction to give the same result, but where the underlying logic is LP.[15]

It is natural to try to extend these fixed point results to a language that contains descriptions. If we add descriptions to the theory of the fixed point, this can be done in such a way as to give a conservative extension in the usual way: given any model of the theory, we (recursively) just let a description $\varepsilon x\varphi$ denote one of the things that satisfies φ. (How it behaves when there is no such thing is not important here.) This gives a model of the theory, and also of the description principle. Unfortunately, it is not guaranteed to give us the T-schema for sentences that contain descriptions. The problem is not that sentences containing descriptions do not have gödel codes; we may suppose them to have been coded from the start. Rather, the problem is that there is no guarantee that the right sentences are in the extension of the truth-predicate.

It is worth noting that we can obtain a model of the T-schema expressed as a material biconditional for the LP fixed point, simply by putting every closed formula containing a description in both the extension and the anti-extension of the truth predicate.[16] The biconditional for sentences not containing descriptions is provided by the original construction. If φ contains a description, $T\langle\varphi\rangle \equiv \varphi$ is true, by the properties of $\equiv$ in LP. A similar trick works for a

[15] See [Dow 84].

[16] Strictly speaking, what we put in is the code of the formula. But to simplify things, I shall often identify syntactic entities and their codes; this is inessential and harmless.

denotation predicate. Without descriptions, the denotation predicate behaves quite unproblematically. The only terms are arithmetic ones; the extension of the denotation predicate for these is fixed at the ground model and never changes. If we add descriptions in at the fixed point, we can just put everything of the form $\langle t, d\rangle$ into both the extension and anti-extension of the denotation predicate, where t is any closed term containing a description. This gives the naive denotation schema in the form of a material biconditional.

If the semantic schemas in the form of material biconditionals were all that were wanted, the constructions of the previous paragraphs would suffice. Presumably, they are stronger than this, however.[17] To obtain the bi-deducible semantic principles it would appear to be necessary, then, to add descriptions in at the ground model and prove anew the existence of a fixed point. When we do this, a problem arises. As the extension of an open formula φ changes, the denotation of $\varepsilon x\varphi$ may change in an uncontrolled way, as, therefore may the truth value of $P\varepsilon x\varphi$, where P is some predicate. And this is sufficient to destroy monotonicity from stage to stage, on which the existence of the fixed point depends.

5 Kroon's Proof

A way around this problem was found by Kroon [Kro 91]. Working in the context of truth-value-gaps, Kroon introduces the totally undefined object, ∞, which is in neither the extension nor the anti-extension of any predicate. Arranging the other details in a suitable way, this is sufficient to give monotonicity, and so the fixed point, back.[18] In this section and the next, I will give a version of Kroon's construction.[19] I will then consider its dualisation.

The language is that or first-order arithmetic augmented by a two place predicate, Δ and a description operator, ε. (Later on, we will see that it might as well have a truth predicate too.) It will also expedite matters to suppose the language has a single extra constant symbol, #. We may assume that the language contains a function symbol for every primitive recursive function.

[17] Though this has recently been questioned for interesting reasons by Goodship [Goo 96].

[18] One of Kroon's aims is to give a satisfactory theory of descriptions that solves semantic paradoxes which employ them. I do not think that it succeeds in this, for exactly the same reason that Kripke's original account of truth fails to solve its targetted paradoxes: the expressive power of the language is limited, since if φ lacks a truth value, there is no way of expressing this truly in the language; in particular, $\neg T\langle\varphi\rangle$ is neither true nor false. Similarly, if some term, say t, denotes ∞ there is no way of truly expressing this in the language. In particular, $\Delta(\langle t\rangle, \infty)$ lacks a truth value. (If it is suggested that ∞ is just a technical device for handling denotationless terms, and that t actually lacks a denotation, then it is this fact that cannot be truly expressed in the language: $\neg\exists x\Delta(\langle t\rangle, x)$ lacks a truth value.) In both cases, attempts to put this expressive power into the language just succeed in reproducing version of the paradoxes.

[19] Kroon writes ∞ using a window icon. More importantly, the version of the proof given here simplifies Kroon's in a couple of ways. First, Kroon uses another semantic object, which he writes as $*$, to be the denotation of descriptions that determinately apply to nothing. I let such terms denote 0. Secondly, Kroon's description operator is a a definite description operator; mine is a least-number operator.

Terms and formulas are defined by the usual joint recursion. I will write the numeral of the number n as $\mathbf{n}$.

A *Kroon model* is a pair, $\langle D, I\rangle$. $D = \mathcal{N} \cup \{\infty\}$, where $\mathcal{N}$ is the natural numbers. I assigns '$\mathbf{0}$' the number zero, and '#' the object ∞. To each arithmetic function symbol it assigns the appropriate arithmetic function, where the function has output ∞ if any of its inputs is ∞. I assigns every predicate an extension and anti-extension, which must be disjoint. Their union need not exhaust the set of all pairs of D. (The only predicates we have are binary.) In fact, any pair that contains ∞ must be in neither the extension nor the anti-extension of a predicate. The extension of $=$ is $\{\langle x, x\rangle\,; x \in \mathcal{N}\}$; the anti-extension is $\{\langle x, y\rangle\,; x, y \in \mathcal{N} \text{ and } x \neq y\}$.

Given a Kroon model, every closed term, t, is assigned a denotation, $|t|$, in D; and every closed sentence, φ, is assigned a truth value, $|\varphi|$, in $\{T, N, F\}$. This is done by a joint recursion.

If c is a constant, $|c| = I(c)$; if t is $ft_1...t_n$, $|t| = I(f)(f\,|t_1|\,...\,|t_n|)$. When t is the term $\varepsilon x\varphi$: if for some n, $|\varphi(x/\mathbf{n})| = T$ and for all $m < n$ $|\varphi(x/\mathbf{m})| = F$, $|t| = n$; if $|\varphi(x/\mathbf{n})| = F$ for all n, $|t| = 0$; otherwise, $|t| = \infty$.

If φ is of the form Pst then $|\varphi| = T$ if $\langle |s|, |t|\rangle$ is in the extension of P; $|\varphi| = F$ if $\langle |s|, |t|\rangle$ is in the anti-extension of P; otherwise $|\varphi| = N$. The truth-conditions for connectives are computed according to the standard strong Kleene conditions; and the truth conditions for the quantifiers are as follows: $|\exists x\varphi| = T$ if for some number, n, $|\varphi(x/\mathbf{n})| = T$; $|\exists x\varphi| = F$ if for every number, n, $|\varphi(x/\mathbf{n})| = F$; $|\exists x\varphi| = N$ otherwise. Note that the domain of quantification is $\mathcal{N}$, not D.

It is easy to check that any Kroon interpretation verifies all the truths of the standard model of arithmetic. It is also easy to check that it satisfies a description principle of the following form:

$$\exists x(\varphi \wedge \forall z(z < x \supset \neg\varphi(x/z)) \vdash \varphi(x/\varepsilon x\varphi)$$

For suppose the premise has the value T. Then for some number n, $|\varphi(x/\mathbf{n})| = |\forall z(z < \mathbf{n} \supset \neg\varphi(x/z))| = T$. If $m < n$ then $|\mathbf{m} < \mathbf{n}| = T$. Hence, $|\varphi(x/\mathbf{m})| = F$. Thus, $|\varepsilon x\varphi| = n$ and $|\varphi(x/\varepsilon x\varphi)| = T$.[20]

For future reference, the following Lemma and Corollary are useful.

Lemma

(i) For any n, $|t(\mathbf{n}/\#)| = |t|$ or ∞. (In particular, if $|t| = \infty$, $|t(\mathbf{n}/\#)| = \infty$.)

(ii) For any n, $|\varphi(\mathbf{n}/\#)| = |\varphi|$ or N. (In particular, if $|\varphi| = N$ then $|\varphi(\mathbf{n}/\#)| = N$.)

Proof

The proof is by a joint recursion. Take (i) first. It is clearly true if t is just a constant. Functional application preserves the property, since functions are gap-in/gap-out. This just leaves descriptions. Suppose that t is $\varepsilon y\varphi$, and

[20] This (and some later arguments) assumes that the truth value of a sentence depends only on the denotation of the terms it contains. Strictly speaking, this requires a proof. But the proof is of a kind that is standard in proofs of soundness, and I omit it.

consider things of the form $|\varphi(y/\mathbf{m})|$. By recursion hypothesis (ii), when we substitute # for $\mathbf{n}$ in each of these, we obtain terms whose values are either the same or ∞. Moreover, if one of these had the value ∞ already, the substitution preserves this. It is not difficult to check, on a case by case basis, that the result follows.

The proof of (ii) is by a recursion on the way that formulas are constructed. Suppose that φ is of the form Pst. The result follows by recursion hypothesis (i). If φ is $\neg\psi$ or $\psi \wedge \theta$, the result follows simply from the recursion hypothesis and the Kleene truth conditions. (Other connectives can be thought of as defined from these.) Finally, φ may be of the form $\exists y\psi$. ($\forall$ may be taken as defined in the usual way.) The result then follows, in the same way, from the truth conditions for $\exists$.□

Corollary

If $|\exists x\varphi| = T$ then $|\varphi(x/\varepsilon x\varphi)| = T$ or N.
If $|\exists x\varphi| = N$ then $|\varphi(x/\varepsilon x\varphi)| = N$.
If $|\exists x\varphi| = F$ then $|\varphi(x/\varepsilon x\varphi)| = F$ or N.

Proof

For T: Suppose that $|\exists x\varphi| = T$. Then there is (a least) n such that $|\varphi(x/\mathbf{n})| = T$. Now, either for all $m < n$ $|\varphi(x/\mathbf{m})| = F$, in which case $|\varepsilon x\varphi| = n$ and $|\varphi(x/\varepsilon x\varphi)| = T$; or for some $m < n$ $|\varphi(x/\mathbf{m})| = N$, in which case $|\varepsilon x\varphi| = \infty$. By the lemma, $N = |\varphi(x/\#)| = |\varphi(x/\varepsilon x\varphi)|$.

For N: if $|\exists x\varphi| = N$ then for some n, $|\varphi(x/\mathbf{n})| = N$. By the lemma, $N = |\varphi(x/\#)| = |\varphi(x/\varepsilon x\varphi)|$.

For F: if $|\exists x\varphi| = F$ then for all n, $|\varphi(x/\mathbf{n})| = F$. By the lemma, $|\varphi(x/\varepsilon x\varphi)| = |\varphi(x/\#)| = F$ or N.□

6 A Fixed Point

It remains to construct a Kroon interpretation that verifies the denotation schema. This is done by the usual method. Crucial to it is the following monotonicity lemma.

Definition

If $\mathcal{A}$ and $\mathcal{B}$ are Kroon interpretations, $\mathcal{A} \preceq \mathcal{B}$ iff the extension and anti-extensions of $\mathcal{B}$ are at least as great as those of $\mathcal{A}$.

Monotonicity Lemma

If $\mathcal{A} \preceq \mathcal{B}$ then:

(i) if for some number n, $|t| = n$ in $\mathcal{A}$, the same is true in $\mathcal{B}$.

(ii) if $|\varphi| = T$ $[F]$ in $\mathcal{A}$, the same is true in $\mathcal{B}$.

Proof

The proof is by a joint recursion. Take (i) first. If this is true for ε-terms, it is clearly true for all terms. So suppose that t is $\varepsilon x\varphi$. If $|t| = n$ in $\mathcal{A}$ then either $n = 0$ and for all m $|\varphi(x/\mathbf{m})| = F$; or $|\varphi(x/\mathbf{n})| = T$ and for all $m < n$ $|\varphi(x/\mathbf{m})| = F$. In either case, by recursion hypothesis (ii) $|\varepsilon x\varphi| = n$ in $\mathcal{B}$.

The proof for (ii) is itself by recursion. For atomic formulas: suppose that φ is Pst and that $|Pst| = T$ $[F]$ in $\mathcal{A}$. Then $\langle |s|, |t| \rangle$ is in the [anti-]extension of P

in $\mathcal{A}$, and both components must therefore be numbers; by recursion hypothesis (i), the same is true in $\mathcal{B}$. The cases for the connectives and quantifiers are standard in strong Kleene logic, and so omitted.□

We now construct a fixed point in a standard way. We define a transfinite sequence of Kroon interpretations, $\langle M_\alpha;\ \alpha$ an ordinal$\rangle$. M_0 is the Kroon interpretation in which the extension and anti-extension of the predicate Δ are both empty. The [anti-]extension of Δ in $M_{\alpha+1}$ is $\{\langle t, d\rangle; d \in D$ and $|t = \mathbf{d}| = T$ $[F]$ in $M_\alpha\}$. (Here, if d is ∞, $\mathbf{d}$ is $\#$.) For limit λ, the [anti-]extension of M_λ is just the union of the [anti-]extensions of M_β for $\beta < \lambda$. Given the behaviour of ∞ with identity, it is clear that each interpretation is an Kroon interpretation.

Lemma

(i) For all $\alpha < \beta$ then $M_\alpha \preceq M_\beta$.

(ii) If $\langle t, d\rangle$ is in the [anti-]extension of Δ in M_β, $|t = \mathbf{d}| = T$ $[F]$ in M_β.

Proof

The proof is by induction on β. For $\beta = 0$ the clauses hold vacuously. We take the cases for successor and limit ordinals separately.

Suppose the hypothesis is true for β, and let $\alpha < \beta + 1$.

(i) If $\langle t, d\rangle$ is in the [anti-]extension of Δ in M_α then by induction hypothesis $\langle t, d\rangle$ is in the [anti-]extension of Δ in M_β. Thus, $|t = \mathbf{d}| = T$ $[F]$ in M_β by (ii). So $\langle t, d\rangle$ is in the [anti-]extension of Δ in $M_{\beta+1}$, as required.

(ii) Suppose that $\langle t, d\rangle$ is in the [anti-]extension of Δ in $M_{\beta+1}$. Then $|t = \mathbf{d}| = T$ $[F]$ in M_β. But by (i) $M_\beta \preceq M_{\beta+1}$. So by the Monotonicity Lemma, $|t = \mathbf{d}| = T$ $[F]$ in $M_{\beta+1}$, as required.

Suppose that λ is a limit ordinal and that the result holds for all $\alpha < \lambda$.

(i) If $\langle t, d\rangle$ is in the [anti-]extension of Δ in M_α then, by definition, $\langle t, d\rangle$ is in the [anti-]extension of Δ in M_λ.

(ii) Suppose that $\langle t, d\rangle$ is in the [anti-]extension of Δ in M_λ. Then for some $\alpha < \lambda$, $\langle t, d\rangle$ is in the [anti-]extension of Δ in M_α. Thus, $|t = \mathbf{d}| = T$ $[F]$ in M_α. By (i), $M_\alpha \preceq M_\lambda$; hence by the Monotonicity Lemma $|t = \mathbf{d}| = T$ $[F]$ in M_λ, as required.□

The Lemma shows that as α increases, the [anti-]extensions of the M_αs are non-decreasing. By the usual cardinality considerations, there must be a δ such that $M_\delta = M_{\delta+1}$. This is the required model. For $|\Delta(\langle t\rangle, s)| = T$ at $M_{\delta+1}$ iff $\langle t, d\rangle$ is in the extension of Δ at $M_{\delta+1}$ (where d is the denotation of s at $M_{\delta+1}$) iff $|t = \mathbf{d}| = T$ at $M_\delta = M_{\delta+1}$ iff $|t = s| = T$ at $M_{\delta+1}$, as required. Similarly, $|\Delta(\langle t\rangle, s)| = F$ at $M_{\delta+1}$ iff $|t = s| = F$ at $M_{\delta+1}$. It follows that $|\Delta(\langle t\rangle, s)| = N$ at $M_{\delta+1}$ iff $|t = s| = N$ at $M_{\delta+1}$. Thus, at $M_{\delta+1}$ $|\Delta(\langle t\rangle, s)| = |t = s|$. This more than suffices to verify the naive denotation schema.

If a truth predicate were added to the language, it could be shown to have a fixed point in a similar way (and the denotation and truth predicates to have a common fixed point). Alternatively, a truth predicate can simply be defined. If φ is (the code of) any closed formula, let f be the arithmetic functor that maps it to (the code of) the formula $\varepsilon x(\varphi \wedge x = \mathbf{1})$; and write Tx for $\Delta(fx, \mathbf{1})$. If, at the fixed point, φ is true, then $\varphi \wedge \mathbf{0} = \mathbf{1}$ is false and $\varphi \wedge \mathbf{1} = \mathbf{1}$ is true. Hence $|\varepsilon x(\varphi \wedge x = \mathbf{1})| = 1$, so $\varepsilon x(\varphi \wedge x = \mathbf{1}) = \mathbf{1}$ is true; as, therefore, is $\Delta(f\langle\varphi\rangle, \mathbf{1})$.

If, at the fixed point, φ is false, then for all n, $\varphi \wedge \mathbf{n} = \mathbf{1}$ is false. Hence $|\varepsilon x(\varphi \wedge x = \mathbf{1})| = 0$, so $\varepsilon x(\varphi \wedge x = \mathbf{1}) = \mathbf{1}$ is false; as, therefore, is $\Delta(f\langle\varphi\rangle, \mathbf{1})$. Finally, if, at the fixed point, φ is neither, then for all n, $\varphi \wedge \mathbf{n} = \mathbf{1}$ is false, unless $n = 1$, when it is neither. Hence $|\varepsilon x(\varphi \wedge x = \mathbf{1})| = \infty$, so $\varepsilon x(\varphi \wedge x = \mathbf{1}) = \mathbf{1}$ is neither; as, therefore, is $\Delta(f\langle\varphi\rangle, \mathbf{1})$.[21]

7 Dualising

The paraconsistent logic LP[22] is well known to be dual to the strong Kleene logic. Specifically, given a Kleene interpretation, if one thinks of the extension of a predicate as containing those things of which it is true only, the anti-extension as containing those things of which it is false only, the middle ground as containing those things of which it is both true and false, and thinks of the value N as *being both true and false*, we obtain an LP interpretation. A dual construction takes us in the opposite direction. The only formal difference between the two logics is that in Kleene's logic, the middle value is not designated, whilst in LP it is.

Using this duality it is possible reinterpret any construction concerning Kleene logic as one concerning LP. In particular, any fixed point construction of Kripke's kind can be reinterpreted as one for LP.[23] If we dualise a Kroon interpretation, we obtain what I will call a *Cusanus interpretation*. In a Cusanus interpretation, ∞ is exactly the trivial object: any atomic sentence of the form $Pt\#$ or $P\#t$ is true (and false).

Anything that takes the value T in a Kroon model is true-only in the dual Cusanus model. In particular, every sentence of the language of arithmetic that is true in the standard model is true only in any Cusanus interpretation. Moreover, any Cusanus interpretation satisfies the description principle: $\exists x\varphi \vdash \varphi(x/\varepsilon x\varphi)$. This follows from the Corollary at the end of section 5. Finally, the Cusanus dual of the Kroon fixed point, $\mathbf{M}_\delta$, satisfies the naive denotation principle, since $|\Delta(\langle t\rangle, s)| = |t = s|$. If we define a truth predicate as indicated at the end of the last section, it also verifies the T-schema. Hence, the set of sentences true at the dual of $\mathbf{M}_\delta$ is a non-trivial LP theory containing all of arithmetic, the description principle and the naive denotation and T-schemas. This is our non-triviality result.

8 Limitations of the Result

The non-triviality result goes part-way to a complete result, but it has its limitations. It does not model everything that one might want. As I noted in

[21] Interestingly enough, the construction will not work for a satisfaction predicate. This is because the sentence $\mathbf{0} = \mathbf{0} \vee \# = \#$ (for example), is true. Hence if S is a predicate satisfying the satisfaction schema for formulas of one free variable, the pair $\langle\langle 0 = 0 \vee v = v\rangle, \infty\rangle$ would have to be in its positive extension. Hence the interpretation could not be a Kroon interpretation.

[22] See, e.g., [Pri 87], ch. 5.

[23] The dualisation for Kripke's original construction is spelled out in [Dow 84].

the first section, both quantifier principles and identity principles have to be restricted if the trivial object is present. Take these in turn.

Existential generalisation (and its dual, universal instantiation) fail in the model (as they do, for different reasons, in Kroon interpretations). For example, $\mathbf{0} = \# \wedge \# = \mathbf{1}$, is true, but $\exists x(\mathbf{0} = x \wedge x = \mathbf{1})$ is not. There are, however, restricted versions that are verified. Existential generalisation works for purely arithmetic terms. More generally, call a term *grounded* if its denotation at M_δ is not ∞. All arithmetic terms are grounded; but so is any ε-term that does not contain Δ, as are a number that do (e.g., $\varepsilon x(x = \mathbf{1} \vee \Delta(\langle \mathbf{0} \rangle, \mathbf{0}))$. Existential generalisation works for all grounded terms.

The law of identity, $t = t$, holds in Cusanus models, but substitutivity of identicals fails. (In Kroon models, it is the other way around.) In particular, the transitivity of identicals fails since $\mathbf{0} = \# \wedge \# = \mathbf{1}$, but $\mathbf{0} = \mathbf{1}$ is not true. This is one reason why all the triviality arguments of sections 3 and 4 are unsound in the model. As before, substitutivity does hold when the terms in question are grounded. It should be noted that we also have substitutivity in the form or a material (non-detachable) equivalence: $s = t \vdash \varphi \equiv \varphi(s/t)$. This is a simple corollary of the Lemma of section 5.

These restrictions are essential to the proof. As the third triviality argument shows, if we had standard identity and quantifier principles, triviality would result, since the denotation schema holds in unrestricted form. As we saw there, it is natural to replace the existential half of it with something more guarded, such as: $\exists x \Delta(\langle t \rangle, x) \vdash \Delta(\langle t \rangle, t)$. A triviality or non-triviality result for this restricted version, but with usual quantifier and identity principles (or at least, ones more normal in a free logic[24]), is still an open problem.

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