The Examined Life and *The Nature of Rationality*

ie Examined Life, Nozick's third book – a book oout 'living and what is important in life' (1989: 11) has not made much impact on the philosophical orld. However, his fourth, *The Nature of Rationality* 993), marks a return to issues of decision theory and tionality, and so contributes to ongoing debates thin the analytic tradition.

Nozick was the first to present Newcomb's oblem to the philosophical world, and his discuson has remained a classic work in decision theory, nphasizing the distinction between evidential and usal decision theory. In *The Nature of Rationality* introduces a new idea: symbolic utility. An action decision may be symbolic – expressive of an notion or attitude, for example – and so may have lue not so much in its effects, but by its standing as symbol. To illustrate, Nozick points out that for me people minimum wage legislation may have lue as a way of symbolizing the idea of helping the oor, even if it turns out to be ineffective as a policy. cting rationally, on Nozick's view, is a matter of aximizing decision-value', which is a weighted sum causal, evidential and symbolic utility (see DECIoN AND GAME THEORY).

e also: KNOWLEDGE, CONCEPT OF; PERSONAL ENTITY; RATIONAL CHOICE THEORY; SCEPTICISM

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JONATHAN WOLFF

NUMBERS

Numbers are, in general, mathematical entities whose function is to express the size, order or magnitude of something or other. Historically, starting from the most basic kind of number, the positive integers (1, 2, 3, . . .), which appear in the earliest written records, the notion of number has been generalized and extended in several different directions – often in the face of considerable opposition.

Other than the positive integers, the most venerable are the rational numbers (fractions), which were known to the Egyptians and Mesopotamians. The discovery, by Pythagorean mathematicians, that there are lengths that cannot be expressed as fractions occasioned the introduction of irrational numbers, such as the square root of 2, though the Greeks managed only a geometric understanding of these. The number zero was recognized, first in Indian mathematics, by the seventh century; the use of negative numbers evolved after this time; and complex numbers, such as the square root of −1, appeared first at the end of the Middle Ages. Infinitesimal numbers were developed by the founders of the calculus, Newton and Leibniz, in the seventeenth century (and were later to disappear from mathematics – for a time); and infinite numbers (ordinals and cardinals) were introduced by the founder of modern set theory, Cantor, in the nineteenth century.

The introductions of three of these kinds of number, in particular, occasioned crises in the foundations of mathematics. The first (concerning irrational numbers) was finally resolved in the nineteenth century by the work of Cauchy and Weierstrass. The second (concerning infinitesimals) was also resolved then, by the work of Weierstrass and Dedekind. The third (concerning infinite numbers), which involves paradoxes such as Russell's, still awaits a convincing solution.

It is seemingly impossible to give a rigorous definition of what it is to be a number. The closest *one can get is a family-resemblance notion, with very ill-defined boundaries.*

1 **Natural numbers**
2 **Rational and irrational numbers**
3 **Negative and complex numbers**
4 **Infinitesimal numbers**
5 **Transfinite numbers**
6 **Weierstrass and Dedekind**
7 **Set-theoretic reduction**
8 **Developments from logic**
9 **Number in general**

1 Natural numbers

All human societies would appear to have some form of counting, however limited, both for size – cardinality (one, two, three, ...) – and for order – ordinality (first, second, third, ...). The recognition that an infinitude of positive integers could be used for these purposes was already in place in the earliest civilization from which we have written records, that of Egypt *circa* the fourth millennium BC. The investigation of basic arithmetic operations (addition, multiplication and so on) was also well under way then. This was continued in the next great Middle Eastern civilization, that of Mesopotamia (Babylon), *circa* the second millennium BC. The investigations were greatly facilitated by the Mesopotamian invention of place-notation, that is, the idea that a single numeral can represent different quantities depending on where it occurs in a string of numerals. Thus, for example, in decimal notation, the first '2' of '2,121' represents two thousands and the second represents two tens. (The Mesopotamians actually preferred a sexagesimal to a decimal base.)

The *natural* numbers comprise the positive integers together with zero. The idea of zero as a number was not to be found in either of these civilizations, although the Mesopotamians did sometimes leave a gap where we would now write a zero. (Thus '2 2' might represent 202 or 2,020, and so on.) Though the origin of the idea is uncertain, zero was certainly being used by the seventh century AD by Indian mathematicians, who had also developed place-notation and used a decimal base. They therefore possessed, in effect, our modern number system (though the symbols they used were different). This system was taken up in the Arabian civilization around the eighth century, and thence passed into Europe around the twelfth.

2 Rational and irrational numbers

Positive integers quantify wholes. Once one starts to measure, it becomes clear that a way is required to quantify parts. *Rational* numbers, as ratios of natural numbers (with non-zero denominator), clearly serve this function. These were known to the Egyptians, at least in the form of unit fractions (of the form $1/n$) and, in a more sophisticated and general form, to the Mesopotamians. Decimal fractions (of the form 0.23) appear to have been developed by Arabic mathematicians, and by the time we reach Stevin in late sixteenth-century Europe, the modern computational system for rational numbers is essentially in place.

The creation of modern mathematics is usually reckoned to have occurred in Ancient Greece in the second half of the first millennium BC. What was distinctive about the Greek approach was that mathematics was freed from its practical roots and took on a purely theoretical form. In particular, the central concern of Greek mathematics was proof, especially proofs in geometry. As far as arithmetic goes, the first part of this epoch was dominated by Pythagoreanism, according to which everything can be explained in terms of the positive integers. (It follows that rational numbers are of little theoretical importance: given two lengths to compare, if one chooses one's unit of measurement small enough, the lengths will be integral.)

The discovery that the Pythagorean assumption was wrong came as a distinct shock. Most simply, the diagonal of a unit square (whose length is the square root of two) is not commensurable with its sides. The proof of this (one of the first *reductio* arguments in the history of mathematics) is now celebrated. It is not known who discovered it, though it was probably one of the later Pythagoreans, *c.*400 BC. At any rate, it inaugurated the first of three crises in the foundations of mathematics that have been associated with the introduction of a new kind of number.

The new numbers in this case are *irrational* numbers (that is, numbers which cannot be written as a ratio of integers – written as decimals they are infinite and non-repeating), such as the square root of two (and pi (π), although the irrationality of this was not proved until the eighteenth century by Lambert). They are required, together with rational numbers, for general quantification of length: in fact, the only conception of such magnitudes available to the Greeks was as geometric lengths. Given this representation, some geometric account of arithmetic operations had to be given. The hardest of these is division, which was solved by Eudoxus (*c.*360 BC) in his theory of proportions, given in book 5 of Euclid's *Elements*. But even for Eudoxus, ratios were not entities in their own right. Operations which treat irrational numbers in the form of surds (expressed in ways such as $\sqrt{2}$, $\sqrt[3]{71}$), as bona fide entities, can be found in Indian mathematics and, in particular, in the seventh-century mathematician Brahmagupta. Some use of surds was made by Arabic mathematicians, but their use did not become common until the sixteenth century (for example, in the work of Stevin and Cardan). A clear statement to the effect that irrational numbers are first-class numerical citizens can be found in Newton, in the seventeenth century; however, an adequate understanding of what this citizenship amounted to had to wait until the nineteenth century.

3 Negative and complex numbers

Subtraction had been familiar to mathematicians since the earliest times. The idea that a sensible quantity might itself be negative, however – and what the sense of it might be – took a long time to catch on. *Negative* numbers, the rules for operating with them, and their use as roots of equations appear in Brahmagupta, and were consequently known to Arabic mathematicians – though they did not use them as roots of equations. The first time they appear in this way in European mathematics is in the work of fifteenth-century mathematician Nicholas of Chuquet (who, however, did not allow zero to be a root!). Even Cardan, a century later, who employed negative roots extensively, called them 'fictional'. The idea that negative numbers might correspond to direction reversal is to be found in the early seventeenth-century mathematician, Girard; but, even after the invention of analytic geometry by Descartes a little later, many mathematicians simply ignored the negative parts of the plane. And even as late as the middle of the eighteenth century some textbook writers were still disputing the claim that the product of a pair of negative numbers is positive.

Once the use of negative numbers became common mathematical practice, the thought that there might be a new kind of number to provide for their square roots was not far behind. If we call all of the numbers mentioned so far *real* numbers, and write the square root of -1 as i (following Euler), the product of i with any real number is called an *imaginary* number. More generally, the sum of a real number and an imaginary number $(a+ib)$ is called a *complex* number. One of the earliest occurrences of complex numbers in mathematics is in the work of Cardan, who observed, as we would now put it, that a cubic equation with a real root may also have complex roots, though Cardan himself regarded this observation as useless. A few years later, Bombelli articulated the use of complex numbers (in the same context) much further. Opposition to them did not then cease, however; for example, their use was frowned upon by Newton. But by the time of, and especially in the work of, Euler, in the eighteenth century, complex numbers had been shown to have numerous uses, both as the solutions of equations and elsewhere – for example, in important relationships between trigonometric functions. And by the time that Gauss, in the early nineteenth century, showed that every polynomial equation with complex coefficients has complex roots, complex numbers were well entrenched. With the development of electromagnetic theory later in the century, complex numbers were shown to be capable of representing even physical quantities (for example, impedance).

The problem of what sense to make of complex numbers was also solved by Gauss and later contemporaries, such as Argand. The complex number $x+iy$ could be thought of as the point (x,y) in two-dimensional Euclidean space (now usually called the Argand plane), with the arithmetic operations on complex numbers defined in a suitable way. This prompted the idea that points in a higher-dimensional space could also be thought of as numbers of a certain kind. Defining a suitable notion of multiplication for points in a three-dimensional space turned out to be impossible, but the problem was solved for four dimensions by the Irish mathematician William Hamilton later in the nineteenth century. Hamilton defined a class of numbers known as 'quaternions', of the form $x+iy+jz+kw$, where $ijk=i^2=j^2=k^2=-1$. The properties of quaternions were investigated for some years, but no real use was found for them, so the investigations lapsed. By this time, the observation that a somewhat different set of operations on points in Euclidean space might be fruitful had been made, and vector algebra was born. (If one thinks of complex numbers as two-dimensional vectors, then complex addition and vector addition are the same thing, but complex multiplication and the various vector multiplications are distinct.)

4 Infinitesimal numbers

The numbers discussed so far have all been finite. But numbers of some kinds are infinite – or their intuitive inverse; infinitesimal. It is sometimes (dubiously) claimed that infinitesimals can be found in Greek mathematics. What certainly can be found, from about the third century BC on, is a method called 'the method of exhaustion', (probably) invented by Eudoxus and developed by Archimedes. The main purpose of this method is the computation of areas and volumes of non-rectilinear geometric figures, and requires their approximation by rectilinear figures. For example, a circle is approximated by a regular polygon inscribed within it. As the length of the sides of the polygon decreases (and the number of sides increases) the approximation gets closer. By the late sixteenth century we find mathematicians such as Stevin and Wallis suggesting, in effect, that if we make the sides of the polygon infinitesimally small, the approximation will be exact – or, at least, will differ only infinitesimally from the true value. Hence, a new kind of number, *infinitesimals*, had to be recognized, to quantify infinitesimal lengths.

The use of infinitesimals in essentially this way was developed by Newton and Leibniz in the next century as the foundation of modern calculus (see ANALYSIS, PHILOSOPHICAL ISSUES IN §1). Though their inventions were independent, and used quite different notations, the essential ideas were the same. For example, to determine the gradient of the curve $y = x^2$ at the point $x = a$, we consider a point on the curve an infinitesimal distance d away along the x-axis. The slope of the line joining these two points is

$$\frac{(a+d)^2 - a^2}{d} = \frac{2ad + d^2}{d} = 2a + d$$

but since d is infinitesimally small, we can disregard it. Hence the gradient is $2a$.

The power of infinitesimal methods was so great, in both pure and applied mathematics, that they soon became entrenched. However, many people – most notably Berkeley – were severely critical. In particular, infinitesimals had at the same time to be both non-zero (since one divided by them) and zero (since one ignored them). This dilemma inaugurated the second foundational crisis in mathematics associated with the introduction of a new kind of number. But this one would have to wait only 200 years for a solution.

5 Transfinite numbers

It is clear that there are infinite quantities in mathematics, and hence that there must be a size that these quantities can be, namely, infinite. But for most of the history of mathematics, infinitude was thought of as mathematically indeterminate, and hence there was little to be said about the quantity 'infinity'. (One does, however, sometimes find mathematicians – even of the stature of Euler – taking it as a definite quantity, in particular, as the reciprocal of zero.) Even in the heyday of the infinitesimal calculus, infinite numbers played little part (though Leibniz certainly held that the reciprocals of distinct infinitesimals were distinct infinite numbers). The recognition of different orders of infinity, and consequently of the sense of infinite numbers, therefore came as another highly contentious idea.

This recognition is almost entirely due to Georg Cantor in the late nineteenth century. He realized that, given some iterable operation (for example, forming the set of topological limit points of some set of points), it made sense to consider the result of having applied the operation an infinite number of times, and then to apply the operation again, and again... indefinitely (see CANTOR, G. §1). To quantify this ordering, Cantor introduced a new class of infinite numbers, *transfinite ordinals*, the least of which he denoted by a lower-case omega, ω (see SET THEORY §2). He also defined arithmetic operations on these numbers which generalize the operations on finite numbers, but which have rather unusual properties. For example, addition is not commutative: $\omega + 1 \neq 1 + \omega$.

Having generalized the notion of order into the infinite, Cantor next generalized the notion of size. To do this he needed a criterion of size that would work (even) in infinite domains. He adopted the criterion (also suggested by FREGE (§8)) that two collections have the same size just if their members can be put into one-to-one correspondence (that is, each member of either collection corresponds to exactly one member of the other). He observed that adding members to an infinite set may not increase its size. (This fact had been noted by some previous mathematicians – for example, Galileo – who thought it so absurd that they rejected infinity as a quantity.) In another of the most famous *reductio* arguments in the history of mathematics, Cantor also established that for any infinite collection, there is one of greater size. This is now called 'Cantor's theorem' (see CANTOR'S THEOREM). To quantify the different sizes Cantor introduced a new kind of infinite number, *transfinite cardinals* (see SET THEORY §3), denoted by the Hebrew letter aleph ($\aleph$) with a subscript ($\aleph_0$ being the smallest, the size of the natural numbers); and defined arithmetic operations on these which, again, generalize the finite case. These operations also have striking properties: for example, double any aleph and you get what you started with.

It should be noted that natural numbers can be thought of indifferently as ordinals or cardinals. This is because, canonically, we count the size of a finite collection by ordering, and its cardinality is n just if the last object in the ordering is the nth. However, transfinite ordinals and cardinals are quite distinct since, as just observed, if one adds further objects to an ordered set, the size may remain the same. Transfinite ordinal and cardinal arithmetics are also quite distinct. Cardinal addition, for example, is commutative, where ordinal addition is not. It is worth noting that there are very simple questions concerning basic transfinite arithmetic operations that are still unanswered (see CONTINUUM HYPOTHESIS).

Cantor's introduction of transfinite numbers into mathematics did not entail the use of infinitesimals. This is because the unusual properties of (cardinal and ordinal) multiplication allow no obvious sense to be given to the notion of division, and so not to reciprocation. However, the use of infinite numbers brought its own problems. In particular, the totality of all ordinal numbers is an ordered collection and hence must have an ordinal, which must be greater than, and so distinct from, all ordinals. Similarly, the totality of all objects must have the largest possible cardinal, but, by Cantor's theorem, there is a larger. These paradoxes are now called the Burali-Forti paradox and Cantor's paradox, respectively; Russell's paradox, concerning the set of all sets that are not members of themselves, is a stripped-down version of Cantor's (see PARADOXES OF SET AND PROPERTY §4). They heralded the third crisis in the foundations of mathematics associated with the introduction of a new kind of number; a crisis which, unlike the first two, is as yet not satisfactorily resolved. These paradoxes fuelled the rejection of Cantor's ideas by some, including Kroneker; but despite this, they were absorbed into orthodox mathematics within about fifty years.

6 Weierstrass and Dedekind

The final resolution of the first two crises we met (concerning irrationals and infinitesimals) took place in the nineteenth century, a period when mathematicians became particularly concerned with the rigorous foundations of their subject. In this context, the work of Weierstrass and DEDEKIND is particularly significant.

Early in the nineteenth century, the notion of a 'limit' appeared in Cauchy's formulation of the calculus. His method differed from that set out in §4 above as follows: instead of taking d to be some infinitesimal quantity, we let it be a finite quantity, and then consider the limit of what happens when d approaches zero (the limit being a quantity that may be approached as close as we please, though never, perhaps, attained). Despite the fact that Cauchy possessed the notion of a limit, he mixed both infinitesimal and limit terminology, and it was left to Weierstrass, later in the century, to replace all appeals to infinitesimals by appeals to limits. At this point infinitesimal numbers disappeared from mathematics (though they would return, as we shall see).

Weierstrass also gave the first modern account of negative numbers, defining them as signed reals, that is, pairs whose first members are reals and whose second members are 'sign bits' ('+' or '−'), subject to suitable operations.

A contemporary of Weierstrass, Tannery, gave the first modern account of rational numbers. An 'equivalence relation' is a relation R that is reflexive (xRx), symmetric (if xRy then yRx) and transitive (if xRy and yRz then xRz). Given an equivalence relation on a domain of objects, an 'equivalence class' is the set of all those things in the domain related to some fixed object. Tannery defined the rationals as equivalence classes of pairs of natural numbers (the second of which is non-zero), under the equivalence relation '$\sim$' (tilde), defined as follows.

$$\langle m,n\rangle \sim \langle r,s\rangle \text{ iff } m.s = r.n$$

The problem of irrational numbers was finally solved independently by Weierstrass, Cantor and Dedekind, who gave different but equivalent constructions of real numbers. Weierstrass' is in terms of infinite decimal expansions; Cantor's is in terms of convergent infinite sequences of rationals. Dedekind's construction is the simplest: consider any splitting of the rational numbers (which can be taken to include the integers, since the integer n may be identified with the rational number $n/1$) into two non-empty disjoint collections L ('left') and R ('right'), such that anything less than a member of L is in L, and anything greater than a member of R is in R. This pair is now called a 'Dedekind cut' (or 'section'). Real numbers (including irrational numbers) can be thought of as such sections (or just one of their parts).

Dedekind also gave the first axiom system for the natural numbers, in terms of an initial number and the successor operation (+1). In modern form, the axioms are as follows:

(1) 0 is a number.
(2) The successor of any number is a number.
(3) 0 is the successor of no number.
(4) Any two numbers with the same successor are the same.
(5) If 0 has some property, and the successor of any number with that property also has it, then all numbers have it.

These axioms, together with the recursive definitions for addition and multiplication (also given by Dedekind), are now usually named after Peano, who gave a formalized version a few years later.

7 Set-theoretic reduction

The drive for rigour in the foundations of number theory reached its height in the reduction of all numbers to sets, due to the logicists Frege, Russell and Whitehead at the turn of the twentieth century (see LOGICISM). The key to this was a set-theoretic definition of cardinal and ordinal numbers. The cardinal numbers were defined as equivalence classes of sets under the equivalence relation 'can be put into one-to-one correspondence with'. (For Russell and Whitehead, this was restricted to sets of fixed type, in an attempt to avoid the paradoxes of infinite number – see THEORY OF TYPES.) A 'well-ordering' on a set is an ordering such that every non-empty subset has a least member in the ordering (see SET THEORY §2). Two well-ordered sets are 'order-isomorphic' if they can be put into a one-to-one correspondence that preserves the ordering. The ordinal numbers were defined as the equivalence classes of well-ordered sets (of a given type) under the equivalence relation of order-isomorphism. Cardinal and ordinal arithmetic operations were defined in an appropriate fashion.

Given the ordinals, other numbers could then be defined in a relatively straightforward way. Natural numbers are the finite ordinals (which can be shown to satisfy the Peano axioms – see §6 above); rational numbers can be defined by the Tannery construction; real numbers as Dedekind sections; negative numbers as Weierstrassian pairs; complex numbers can be defined as pairs of signed reals, thought of as points on the Argand plane. In each case, arithmetic operations can be defined in natural ways.

Since the Frege/Russell reduction, several other non-equivalent, but equally good, definitions of the cardinal and ordinal numbers have been discovered (and in virtue of this, the claim that these numbers just *are* certain sets is difficult to maintain). The most elegant of these is due to von Neumann. According to this, each ordinal is simply the collection of all smaller ordinals. (Thus, zero is simply the empty set, the least infinite ordinal is the set of natural numbers, and so on.) Cardinal numbers are identified with 'initial' ordinals, that is, least ordinals of each size.

8 Developments from logic

Work in mathematical logic in the twentieth century has provided several notable developments bearing on numbers. Three are particularly important. The first was the proof by Gödel in 1931 that the Peano axioms, and all other consistent axiom systems for arithmetic, are incomplete, in the sense that there are truths of arithmetic that cannot be proved from the axioms – at least if the underlying logic is first-order (see GÖDEL'S THEOREMS). The axioms are complete if the underlying logic is second-order and the induction principle (5) is formulated as a second-order axiom and not just a first-order schema; but second-order logic is not itself axiomatizable (see SECOND- AND HIGHER-ORDER LOGICS §1). This raises profound questions about the nature of both numbers and our knowledge thereof, that fall outside the bounds of this entry.

The second development concerns the paradoxes surrounding transfinite numbers. The orthodox view that has emerged this century is that embedded in Zermelo–Fraenkel set theory (ZF). According to this, there just is no totality of all ordinals, all sets or other 'large' collections, and so the question of their size does not arise. Although this account provides enough set theory for most mathematics (though not all: category theory appears to require large sets of just this kind), it can hardly be said to be conceptually adequate. For example, standard logic defines the sense of a quantifier in terms of the domain (totality) over which it ranges. It is therefore unclear what the sense of the quantifiers of ZF is, if, as it claims, there is no such totality. (See Priest 1987.)

The third development is due to Robinson and is called 'nonstandard analysis' (see ANALYSIS, NON-STANDARD). As was proved originally by Löwenheim and Skolem, (first-order) theories of number have nonstandard models (see LÖWENHEIM–SKOLEM THEOREMS AND NONSTANDARD MODELS). In particular, any theory of the reals will have such models. Robinson showed that in all of these models, there are non-zero numbers that are smaller than any real number: infinitesimals. Using these, he demonstrated that the reasoning of the infinitesimal calculus (which is much more intuitive than limit reasoning) can be interpreted in a perfectly consistent manner. Hence, infinitesimals have been rehabilitated as perfectly good numbers.

9 Number in general

The preceding review of the development of the notion of number naturally prompts the question of what a number is. One might interpret this as the question of whether numbers are Platonic objects, mental constructions, or nothing more than mystified numerals. This is a central issue in the philosophy of mathematics.

Alternatively, in virtue of the plethora of kinds of numbers we have seen, one might interpret the question as asking what makes entities of certain kinds, but not others, numbers. Beyond the rather vague characterization with which this article began, it seems difficult to give a general characterization of number. The most fundamental numbers (both historically and conceptually), the natural numbers, measured size (or order), were subject to distinctive operations (such as addition) and could be the roots of equations. Each of these central features has played a role in generating new kinds of numbers (different concerns being dominant on different occasions). The result is a collection of entities which are related by family resemblance (as observed by Wittgenstein in §67 of *Philosophical Investigations*), though the boundaries of the family seem somewhat arbitrary. It is difficult to see why, for example, complex numbers (or quaternions) should be called numbers, but not vectors or numerical matrices; both of these share the central features of natural numbers.

This conclusion is reinforced by recent work by Conway (1976). He gives a (transfinite) recursive construction that generalizes both the Dedekind construction of the reals and the von Neumann construction of ordinals. Essentially, a *number* is any pair, $\langle L, R \rangle$, such that all the members of L and R are numbers, and every member of R is greater than or equal to ($\geqslant$) every member of L (see ANALYSIS, PHILOSOPHICAL ISSUES IN §2). '$\geqslant$' and the arithmetic operations are also defined in a natural recursive manner. The construction generates virtually all the numbers we have met in this article, including infinitesimals, but excluding, notably, the complex numbers (and if cardinals are to be identified with initial ordinals, a non-uniform definition of arithmetic operations is necessary). Moreover, the construction generates many novel numbers, for example, notably, numbers obtained by applying the full range of real-number operations to infinite numbers, for example, $\omega - 1$, $\sqrt{\omega}$, which make no sense on the usual understanding. Moreover, a simple generalization of the construction (dropping the ordering condition on L and R), produces even more number-like objects (which Conway calls 'games', because, in a certain sense, they code the strategic possibilities in a two-person game).

Just conceivably, a unifying account of number might eventually be found, but in the meantime the emergence of new kinds of numbers seems likely. For example, there are nonstandard inconsistent models of arithmetic which contain inconsistent numbers (natural numbers with inconsistent properties). These have some notable applications. For example, some of them can be shown to provide solutions for arbitrary sets of simultaneous linear equations. (See Mortensen 1995.) And just as the existence of nonstandard models of analysis made infinitesimals legitimate, so might these legitimize the notion of an inconsistent number.

See also: ANALYSIS, PHILOSOPHICAL ISSUES IN; ANTIREALISM IN THE PHILOSOPHY OF MATHEMATICS; ARITHMETIC, PHILOSOPHICAL ISSUES IN; LOGICAL AND MATHEMATICAL TERMS, GLOSSARY OF; REALISM IN THE PHILOSOPHY OF MATHEMATICS; SET THEORY, DIFFERENT SYSTEMS OF

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GRAHAM PRIEST

NUMENIUS (*fl. c.* mid 2nd century AD)

Numenius was a Platonist philosopher. He came from Apamea (Syria) and wrote in Greek. His work – now lost – is usually considered Neo-Pythagorean in tendency, and exercised a major influence on the emergence of Neoplatonism in the third century. A radical dualist, he postulated the twin principles of god – a transcendent and changeless intellect, equated with the Good of Plato's Republic *– and matter, identified as the Pythagorean Indefinite Dyad: god is good, matter evil. In addition to this supreme god, he added at a secondary level a creator-god, one of whose aspects is the world-soul, itself further distinguished into a good and an evil world-soul. He had a strong interest in Oriental wisdom, especially Judaic, and famously called Plato 'Moses speaking Attic'.*

1 Life, work and influence

Nothing is known of Numenius' life, but he can be dated with reasonable accuracy by the fact that he is attested as the teacher of one Harpocration, who was also influenced by the Athenian Platonist Atticus, who in turn flourished in the AD 170s. He is often mentioned in conjunction with a 'companion', Cronius, who was presumably associated with his school, and who may possibly be the addressee of Lucian's treatise on Peregrinus.

Of his works none has survived, but some extracts of his dialogue *On the Good* are preserved by Eusebius in his *Preparation for the Gospel*, as also are some considerable passages from a lively polemical work, *On the Apostasy ['Diastasis'] of the Academics from Plato*, which helps to clarify Numenius' own position, while providing some useful data on the New Academy. Alongside this, we know of the works *On the Indestructibility of the Soul* and *On the Secret Doctrines of Plato*, treatises *On Numbers* and *On Place*, and a work called *Epops*, or 'The Hoopoe', which probably embodies a pun on *epopteia* (mystical vision). We also have an extended account of his doctrine on matter preserved by the late Roman commentator on Plato's *Timaeus*, CALCIDIUS, who may well be more extensively indebted to him than he acknowledges.

His philosophical importance is considerable. He was a major influence, through the mediation of Ammonius Saccas (not to be confused with Ammonius, son of Hermias), on the father of Neoplatonism Plotinus and his followers Amelius and Porphyry, as well as the Christian theologian Origen (see NEOPLATONISM §1; PORPHYRY §4; ORIGEN). His Pythagoreanism consists of presenting Plato as a disciple of Pythagoras (see, for example, fragments 7, 24.57), although without derogating from Plato's greatness (as was done by more extreme Pythagoreans, such as Moderatus of Gades).

Numenius was much interested in the wisdom of the East and in comparative religion. He attracted the interest of Church Fathers by his references to Jahveh, Moses and even Jesus (fr. 1). Indeed, he described Plato as 'Moses speaking Attic' (fr. 8), which seems to imply an acceptance of something like Philo's wholesale allegorization of the Pentateuch (see PHILO OF ALEXANDRIA §1). There has been speculation that he was himself of Jewish stock – his hospitality to the Jewish tradition is certainly notable – but this is not a necessary inference. Numenius may simply be reflecting the syncretistic religious and philosophical milieu in which he lived.

2 Metaphysics

Numenius' views on ethics and logic are not known (although his ethical stance may be assumed to be austere), so we may confine ourselves to his metaphysics and psychology. He is at odds with previous Pythagoreans in maintaining a radical dualism between the first principles of god (the Monad, the Good) and matter (the Dyad), instead of subordinating the material Dyad to the all-generating Monad, as is done by his Pythagoreanizing predecessors from Eudorus through Moderatus to Nicomachus of Gerasa. Numenius' dualism allies him rather with Plutarch and Atticus, and leads him, like them, to postulate an evil world-soul, derived from a reading of Plato (*Laws* X) to balance the beneficent world-soul (see PLUTARCH OF CHAERONEA §§3–4).

Numenius proffered a system of three levels of spiritual reality: a primal god (the Good, or the Father), who is almost supra-intellectual; a secondary, creator-god (the demiurge of Plato's *Timaeus*); and a world-soul. In this he anticipates to some extent Plotinus, although he was more strongly dualist than Plotinus in his attitude to the physical world and