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INCONSISTENT MODELS OF ARITHMETIC
PART I: FINITE MODELS

ABSTRACT. The paper concerns interpretations of the paraconsistent logic LP which model theories properly containing all the sentences of first order arithmetic. The paper demonstrates the existence of such models and provides a complete taxonomy of the finite ones.

1. INTRODUCTION

Let A be the standard model of arithmetic. Let N be the set of first-order sentences that hold in A. Even though N is complete, it has models other than A. These models, the non-standard models of arithmetic, have an interesting and well known structure.[1] If M is a set of sentences that properly contains N then M is inconsistent. This prevents it from having any classical models (i.e., models whose underlying logic is classical). It does not prevent it from having paraconsistent models, however. (Given a suitable paraconsistent logic, any set of sentences has a model.) As we will see, there are many paraconsistent structures whose sets of truths properly contain N, and which are therefore models of it. The aim of this paper is to discuss some of the properties of these. This part of the paper lays out general background considerations, and then provides a taxonomy of the finite models. The second part will look more closely at the general case.

In the next section I will define the paraconsistent logic that will be employed; I will state and prove the major metatheoretic fact about it that will be applied, the Collapsing Lemma. The next two sections will demonstrate the existence of a number of inconsistent models of arithmetic, many of which are finite. We will then be in a position to look at finite models in general, and in the next two sections we will do just that.

2. LOGICAL PRELIMINARIES

There are many paraconsistent logics.[2] One of the simplest and most tractable is the logic LP.[3] This, at any rate, is the logic that we will employ here. The language of logic, L, is that of classical first order logic, including function symbols and identity. An LP interpretation (henceforth, simply, 'interpretation'), is a pair $\langle D, I\rangle$, where D is a non-empty domain, and I assigns denotations to all the non-logical symbols of the language. For every constant, c, $I(c)$ is a member of D. For every n-place function symbol, f, $I(f)$ is an n-place function on D. For every n-place predicate, P, $I(P)$ is a pair comprising the extension and anti-extension of P. I will write these as $I^+(P)$ and $I^-(P)$, respectively. $I^+(P)$ and $I^-(P)$ may overlap, but their union must be the set of all n-tuples of D. The extension of the identity predicate, '$=$', is always the set $\{\langle x, x\rangle; x \in D\}$, though its anti-extension may overlap this.

To give the truth and falsity conditions of the language we employ the standard dodge of supposing that it is augmented with a name for every member of D. Without loss of generality, we will take the names to be the members of D themselves, and adopt the convention that for every $d \in D$, $I(d)$ is just d itself. If the interpretation is A, I will call the augmented language L_A. I can now be extended to assign every term of L_A a denotation, by the standard recursive clause: $I(ft_1 \cdots t_n) = I(f)(I(t_1) \cdots I(t_n))$, where f is any n-place function symbol, and the t_is are its arguments. Every formula, α, of L_A is now assigned a semantic value, $\nu_A(\alpha)$, in the set $\{\{1\}, \{1, 0\}, \{0\}\}$, by the following clauses. (Truth conditions are obtained by ignoring the material in square brackets; falsity conditions, by substituting it in the obvious way.) If α is atomic, $Pt_1 \cdots t_n$:

$$1[0] \in \nu_A(\alpha) \Leftrightarrow \langle I(t_1) \cdots I(t_n)\rangle \in I^{+[-]}(P)$$

The clauses for negation, conjunction and the universal quantifier are as follows:

$$\begin{aligned} 1[0] &\in \nu_A(\neg\alpha) \Leftrightarrow 0[1] \in \nu_A(\alpha) \\ 1[0] &\in \nu_A(\alpha \wedge \beta) \Leftrightarrow 1[0] \in \nu_A(\alpha) \text{ and [or] } 1[0] \in \nu_A(\beta) \\ 1[0] &\in \nu_A(\forall x\alpha) \Leftrightarrow 1[0] \in \nu_A\big(\alpha(x/d)\big) \text{ for all [some] } d \in D \end{aligned}$$

Disjunction and existential quantification have the natural dual truth/falsity conditions, or can simply be taken as defined by the standard clauses: $\alpha \vee \beta$ is $\neg(\alpha \wedge \neg\beta)$, $\exists x\alpha$ is $\neg\forall x\neg a$. $\alpha \supset \beta$ is defined, in the usual way, as $\neg\alpha \vee \beta$. If A is an interpretation, we will say that α is *true* [*false*] in A iff $1[0] \in \nu_A(a)$. If Σ is a set of sentences, A is a model for Σ, $A \vDash \alpha$, iff every member of Σ is true in A.

If the extension and anti-extension of a predicate are disjoint in an interpretation, I will call it *classical*. If all of the predicates of an interpretation are classical, I will call the interpretation itself classical. As should be clear, if an interpretation is classical the above truth/falsity conditions reduce to the conditions for those of classical logic (taking its truth values to be $\{1\}$, and $\{0\}$). We may therefore simply *identify* the interpretations of (classical) first order logic with the corresponding classical LP interpretations, an identification I make forthwith without further comment.

As usual, a sentence is an *(LP) logical truth* iff every interpretation is a model for it. In virtue of the above identification, it is clear that every LP logical truth is a logical truth of classical first order logic. The converse is not obvious but is, in fact, true.[4] Another important fact is as follows. Suppose that we have two interpretations. The first has interpretation function I; the second, J. I will say that the second is an *extension* of the first iff they are identical, except that for every predicate, P, $I^{+[-]}(P) \subseteq J^{+[-]}(P)$.

EXTENSION LEMMA. *If B is an extension of A then everything true [false] is A is true [false] in B.*

Proof. The definition of 'extension' is sufficient to secure this for atomic sentences. The result then follows by a straightforward recursion on sentence formation.[5] $\square$

3. THE COLLAPSING LEMMA

I now want to spell out the major metatheorem about LP that will be applied in what follows. To state it, we need a few preliminaries. Let A, $\langle D, I\rangle$, be any interpretation, and let $\sim$ be any equivalence relation on D, which is also a congruence relation on the interpretations of the function symbols in the language, i.e.: where f is an n-place function symbol in the language and $d_i, e_i \in D$ $(1 \leq i \leq n)$, if $d_i \sim e_i$ for all $1 \leq i \leq n$ then $I(f)(d_1 \cdots d_n) \sim I(f)(e_1 \cdots e_n)$. If $d \in D$, let $[d]$ be the equivalence class of d under $\sim$. We define a new interpretation, $A^{\sim} = \langle D^{\sim}, I^{\sim}\rangle$, called the *collapsed interpretation*, as follows. $D^{\sim} = \{[d]; d \in D\}$. For every constant, c, $I^{\sim}(c) = [I(c)]$. For every n-place function symbol, f, $I^{\sim}(f)([d_1] \cdots [d_n]) = [I(f)(d_1 \cdots d_n)]$. (This is well defined since $\sim$ is a congruence relation.) If P is an n-place predicate, $\langle [d_1] \cdots [d_n]\rangle$ is in its extension in $A^{\sim}$ iff there are $e_1 \sim d_1, \ldots, e_n \sim d_n$, such that $\langle e_1 \cdots e_n\rangle \in I^{+}(P)$. The anti-extension of P is defined similarly. The collapsed interpretation is, essentially, an interpretation that identifies certain members of D (namely, all those in an equivalence class), to produce a composite individual (the equivalence class), which has all the properties of its members (even if these are inconsistent).

COLLAPSING LEMMA. *For any formula, α, of L_A, $\nu_A(\alpha) \subseteq \nu_{A^{\sim}}(\alpha)$.*

Proof. To prove the lemma, we first show by a recursion on the formation of terms that for any function symbol, f, and terms $t_1, \ldots, t_n$, $I^{\sim}(ft_1 \cdots t_n) = [I(ft_1 \cdots t_n)]$. The lemma is then proved by a straightforward recursion on the formation of sentences.[6] □

The Collapsing Lemma tells us that in a process of collapse, truth values are never lost; anything true/false in the original interpretation is true/false in the collapsed interpretation. In particular, if $A \models \Sigma$ then $A^{\sim} \models \Sigma$: if we collapse a model of a theory, we therefore produce another model. In the next section we will apply these facts to models of arithmetic.

4. Inconsistent models: some examples

Henceforth, we will fix L to be the language of arithmetic. There is one binary predicate (identity), one constant symbol, **0**, and function symbols for successor, addition and multiplication, $'$, $+$ and $\times$, respectively. (I will boldface symbols of the language to distinguish them from the numbers and operations themselves.) As usual, the numeral $\mathbf{n}$ is $\mathbf{0}$ followed by n primes. N is the set of sentences in this language true in the standard model of arithmetic, defined as usual.

It is easy enough to construct models of supersets of N. Any extension of the standard model will be one such. For example, we might just add the pair $\langle 0, 0\rangle$ to the anti-extension of the identity symbol. By the Extension Lemma, the result is a model of N, as well as of $\mathbf{0} \neq \mathbf{0}$. (This shows, incidentally, that not all inconsistent models can be obtained by collapsing consistent models. For $\mathbf{1} \neq \mathbf{1}$ is not true in this model. Yet in any collapsed model where $\mathbf{0} \neq \mathbf{0}$ is true, this is so because 0 has been identified with some $x > 0$. But then, 1, its successor, must be identified with $x' > 1$ – since the equivalence relation is a congruence relation on successor. Hence, $\mathbf{1} \neq \mathbf{1}$ is true.)

Simple extended models are not terribly interesting, however. The Collapsing Lemma allows us to construct much more interesting models. Let us start with a couple of special cases, already to be found in the literature. These are both collapses of the standard model of arithmetic.

Given any number, n, identity modulo n is a congruence relation. Collapsing by this gives the model in which $\mathbf{i} = \mathbf{j}$ is true iff $i = j \pmod n$ and $\mathbf{i} \neq \mathbf{j}$ is true for every i and j (including when i is j). The behavior of the successor function in this model can be depicted as follows:

$$\begin{array}{ccccc} 0 & \rightarrow 1 \rightarrow & \cdots & \rightarrow & i \\ \uparrow & & & & \downarrow \\ n-1 & \leftarrow & \cdots & \leftarrow & i+1 \end{array}$$

Models of this kind are often called *cyclic* models, and were, in fact, the first inconsistent models of arithmetic to be discovered.[7]

Another congruence relation is one which, for some n, leaves everything less than n alone but identifies all numbers greater than or equal to n. The successor function in this interpretation can be depicted thus:

$$\begin{array}{ccccccccc} 0 & \to & 1 & \to & \cdots & \to & n & \to & \\ & & & & & & \uparrow & & \downarrow \\ & & & & & & & \leftarrow & \end{array}$$

These are sometimes called *heap* models, and have been used to make various philosophical points.[8]

5. LINEAR MODELS

With these initial examples in mind let us now look at a general construction for collapsing. In what follows, A will be any consistent model of N (standard or non-standard). The numbers referred to are numbers in A. Similarly, the operations and relations referred to are those of A. In particular, $\leq$, defined in the usual way – $i \leq j$ is $\exists x(i + x = j)$ – is the canonical ordering on the members of A. Since A is a model of N, $\leq$, is a linear ordering.

The following lemma of classical model-theory will be useful. If S is an initial section of the numbers of A closed under successor, addition and multiplication, I will call it a *slice*. In any model of arithmetic, the natural numbers clearly form a slice (in fact, this is the minimal slice), as does the set of all numbers. I will call a slice *proper* if it is neither of these.

LEMMA. *Let A be non-standard. If S be any slice except that of all numbers there is a proper slice that extends S.*

Proof. Let a be any number not in S. (a must therefore be non-standard.) Consider the set $M = \{x;$ for some natural number $n,\ x < a^n\}$. It is easy to check that M is closed under successor, addition and multiplication, and so is a slice. It is proper since it extends the natural numbers, and every member of M is less than a^a. (For further details, see Kaye (1991), 6.1.) □

Now, given A and $0 \leq \eta < \omega$, let $\{B_i,\ i \leq \eta\}$ be a chain of strictly increasing initial sections of the numbers in A, such that B_η is the set of all numbers, and if $0 < i \leq \eta$, B_i is a slice. Note that B_0 need *not* be a slice. (If A is the standard model, then η is at most 1. Otherwise, by the lemma, η may be any finite size.[9])

Let $C_0 = B_0$; and for $0 < i \leq \eta, C_i = B_i - B_{i-1}$. For $0 < i \leq \eta$ let p_i be a non-zero number (possibly non-standard) such that $p_1 \in B_1$, and if $i < j$, p_i is a multiple of p_j.[10] We define a relation $\sim$ on numbers as follows. $x \sim y$ iff:

$$(x, y \in C_0 \text{ and } x = y) \text{ or}$$
$$(\text{for some } i > 0,\ x, y \in C_i \text{ and } x = y \pmod{p_i})$$

THEOREM. *$\sim$ is an equivalence relation on numbers, and is a congruence relation with respect to arithmetic operations.*

Proof. It is not difficult to see that $\sim$ is an equivalence relation. We have therefore to check only the congruence.

(Successor) Let $x \sim y$. (i) Suppose that $x, y \in C_0$. Then $x = y$. Hence, $x' = y'$, and so whatever block (i.e., C_i) x' is in, $x' \sim y'$. (ii) Next, suppose that $i > 0$, and $x, y \in C_i$. Then $x = y \pmod{p_i}$. Hence $x' = y' \pmod{p_i}$. And since $x', y' \in C_i,\ x' \sim y'$.

(Addition) Let $x_1 \sim x_2$ and $y_1 \sim y_2$. (i) Suppose that either the x's or y's are in C_0, without loss of generality, the xs. Then $x_1 = x_2$. Thus, wherever block $x_1 + y_1$ and $x_2 + y_2$ are, they are in the same block (since if $k > 0,\ C_k$ is closed under finite addition) and, whatever that is, $x_1 + y_1 \sim x_2 + y_2$. (ii) Next, suppose that the xs are in C_i and the ys are in C_j where $0 < i \leq j$. Then $x_1 = x_2 \pmod{p_i}$ and $y_1 = y_2 \pmod{p_j}$. Now, since p_i is a multiple of p_j, $x_1 = x_2 \pmod{p_j}$. Hence $x_1 + y_1 = x_2 + y_2 \pmod{p_j}$. But the sums are in C_j since this is closed under arithmetic operations. Hence $x_1 + y_1 \sim x_2 + y_2$.

(Multiplication) The argument for multiplication is essentially the same as that for addition. □

Any model obtained by collapsing a model under a congruence relation of this kind I will call a *linear* collapsed model (for reasons that will become clear later). Linear models have a tail, T, comprising the members of C_0 (or, strictly speaking, their singletons), and then η cycles, where the period of cycle i is p_i. (Since $p_1 \in B_1$, each $C_i (i > 0)$ has length greater than p_i.)

If the original model was the standard model, then η must be 0 or 1. In the first case, collapsing just reproduces the standard model (or, strictly speaking, its type lift). If $\eta = 1$ we have a model with a single cycle and finite period. In fact, equivalence relations of this kind are the *only* non-trivial equivalence relations on the standard model that are congruence relations for successor, as can easily be seen. (Any non-trivial equivalence relation must identify two distinct numbers. Let j be the least number that is identified with another number, and k the least number greater than j that is identified with it. It is a simple exercise to show that $C_0 = \{m; m < j\}$ and $p_1 = k - j$). Hence, all collapses of the standard model are linear models. In particular, cyclic models and heap models are special cases of linear models. In the first, C_0 is the empty set; in the second $p_1 = 1$.

Another linear collapse is also worth noting *en passant*. Take any nonstandard consistent model. Take C_0 to be the finite (i.e., standard) numbers, let $\eta = 1$ and let $p_1 = 1$. This produces a model with a single point at infinity, Ω, formed by identifying all the nonstandard numbers, and depicted thus:

$$\begin{array}{ccccccccc} 0 & \to & 1 & \to & 2 & \to & \cdots\ \Omega & \to & \\ & & & & & & \uparrow & & \downarrow \\ & & & & & & & \leftarrow & \end{array}$$

6. FINITE MODELS I: THE BASIC STRUCTURE

We have seen that there are a number of inconsistent models of arithmetic, N. In the rest of this part of the paper we will have a look at the finite ones. It will prove convenient to classify them in terms of the graphs of their successor functions.[11]

Any model must have denotations for the numerals, $\mathbf{0},\ \mathbf{1},\ \mathbf{2}, \ldots$. And because we have a model of arithmetic, the successor function must start off with a structure depicted as follows:

$$0 \to 1 \to \cdots$$

(In a natural way, I will write the denotation of $\mathbf{n}$ as n.) Since the model is finite, at some stage two numerals must have an identical denotation. Let $\mathbf{n}$ be the least of these, and let $\mathbf{k}$ be the least numeral greater than this such that $\mathbf{n} = \mathbf{k}$ is true. (Since it is a model of arithmetic, $\mathbf{n} \neq \mathbf{k}$ is also true.) Suppose that $k = n + m$. Then the successor graph must be as follows:

$$\begin{array}{ccccccccc} 0 & \to & 1 & \to & \cdots & \to & n & \to & n+1 \\ & & & & & & \uparrow & & \downarrow \\ & & & & & & n+m-1 & \cdots & n+2 \end{array}$$

In particular, let $\mathbf{i}$ be the numeral that is $\mathbf{n}$ followed by j primes. Then the denotation of $\mathbf{i}$ is $n + j \pmod m$.

So far, then, we have a structure which gives a denotation to every numeral, and so is of minimum size. Must there be more elements in the model? Not necessarily. We know that there are models where there are no other elements. (E.g., linear collapses of the standard model.)

Not all finite models are of this kind, however. For example, take a linear collapse of a nonstandard model where C_0 is finite, $\eta = 2$, and p_1 and p_2 are finite. This gives us a structure of the kind we have just met, followed by a second cycle. What can therefore be said about other finite models? Let us call the objects which are denotations of the numerals *regular* numbers. For the rest of this section, I shall use the letters n, m, p, for regular numbers, and i, j, k, for irregular numbers.

Let i be any irregular number and n any regular number. Since we have a model of arithmetic, we know that $\forall \mathbf{x}(\mathbf{x} = \mathbf{0} \vee \cdots \vee \mathbf{x} = \mathbf{n} \vee \mathbf{x} \geq \mathbf{n})$ is true. Since i is distinct from $0, \ldots, n$ in the model, we must have $i \geq n$.

Now, i must have a successor, which must, in turn, have a successor, etc. (Note that, for all we know so far, the successors may be regular numbers.) Moreover, since $\forall \mathbf{x}(\mathbf{x} = \mathbf{0} \vee \exists \mathbf{y}\mathbf{y}' = \mathbf{x})$ is true, and i is distinct from 0, it must have a predecessor (at least one – if more than one, pick one, for the moment). This must itself be non-zero (since i is not 1) and so must have a predecessor, and so on. Hence we have a structure of the following shape:

$$\cdots \to i-1 \to i \to i+1 \to \cdots$$

Now let n and m be as depicted in the previous diagram, and let k be any member of this sequence. Then $k \geq n$. We know this is true for i. If k follows i in the sequence, then $k \geq i$, and the result follows, since $i \geq n$. ($n + x = i$ and $i + y = k$, so $n + (x + y) = k$.) If k precedes i in the sequence, k must be irregular (since the successors of regular numbers are regular); so, again, as we have already noted, $k \geq n$. Hence, whatever k is, for some j, $k = n + j$. Thus, $k + m = (n + j) + m = (n + m) + j = n + j = k$. The structure is therefore a cycle, and it has a period that is a divisor of m. In particular, each cycle comprises numbers of the form $i + p$, where p is any regular number (though these need not all be distinct). Moreover, every member of the cycle must be irregular, since if any were regular, the others, being its successors, would be also.

Next, any two cycles of irregular numbers are disjoint. For let i be a number in both cycles; then its successor, *its* successor, and so on are unique. This is so for all regular addends, and so both cycles are identical. Since every two cycles are disjoint, it follows that no irregular number has two predecessors; or otherwise, it would have to be in distinct cycles.

The general structure of a finite model is therefore a tail, plus a cycle of regular numbers, plus a collection of cycles of irregular numbers, such that the period of each is a divisor (not necessarily proper) of the cycle of regular numbers. In a finite model the number of cycles must obviously be finite. But there can be any finite number of cycles. Take any non-standard model of arithmetic, and consider the linear collapse where C_0 and η are finite, and for each i each p_i is finite. This gives a finite model with η cycles.

It is important to note that the behavior of the successor function in a model does not determine all aspects of the model. In particular, it does not determine the behavior of the other functions, or of the identity predicate (the only predicate). There will, therefore, be different models with the same successor 'shape'.

Take identity to start with. Let i be some number (regular or irregular) in a cycle. Then we have $i = i$. But since $i = i'^{\cdots\prime}$ for some finite sequence of primes, and it is true that $\forall\mathbf{x}\mathbf{x} \neq \mathbf{x}'^{\cdots\prime}$, we have $i \neq i$ also. In any collapsed model the members of the tail behave consistently (in particular they are identical with themselves but not distinct), since they have not been identified with anything else. In an extension of a collapsed model, however, any one of these can be made to behave inconsistently.

Next, consider addition. The recursive equations for addition, true in any model of arithmetic, determine the behavior of addition for any regular addend, since this can then be reduced by a finite number of steps to that of the successor function; but for irregular numbers, this is not so. In particular, then, given two models with the same successor graph, addition may behave differently for irregular numbers in each. I will give an example of this later in the paper. Similar comments apply to multiplication.

7. Finite models ii: cycle ordering

We have seen that a finite model comprises a number of disjoint blocks, the tail, T, of regular numbers (which may be empty), a cycle of regular numbers, and cycles of irregular ones. The ordering of numbers imposes relationships of an ordering kind on the cycles themselves. A natural one is the following. If B and C are cycles, we will say that $B \preceq C$ iff for every x in B and y in C, $x \leq y$.

$\preceq$ is obviously reflexive, and can easily be seen to be transitive. Suppose that $B \preceq C \preceq D$. Then for any x, y, z in the respective blocks $x \leq y \leq z$. So there are an i, j, such that $x + i = y$ and $y + j = z$. But then $x + i + j = z$. Hence, $B \preceq D$.[12]

$\preceq$ is also connected. Let B and C be any distinct cycles, and let i and j be in B and C, respectively. Then since $\forall\mathbf{x}\forall\mathbf{y}(\mathbf{x} \leq \mathbf{y} \vee \mathbf{y} \geq \mathbf{x})$ is true, $i \leq j$ of $j \geq i$ (or both). Without loss of generality, suppose the former. Now, every member of C is of the form $j + p$ for some regular p. Since $i \leq j + p$, i is less than or equal to every member of B. Conversely, every member of B is of the form $i + p$. Now $i + p \leq j + p$, and since C is a cycle, $j + p \leq j$. Hence, every member of B is less than or equal to j.

As we see, then, $\preceq$ is a relation on a finite domain, which is connected, reflexive and transitive. One might call this a linear preorder.

There is an important connection between $\preceq$ and the periodicity of the cycles. Suppose that $B \preceq C$, that B and C are cycles with periods p and q, respectively, and that x and y are in B and C, respectively. Then for some i, $x + i = y$. Hence, $y + p = (x + i) + p = (x + p) + i = x + i = y$. Hence, q must be a divisor or p. (We have already noted the special case of this when B is the cycle of regular numbers.) In particular, then, if the cycles form a linear ordering under $\preceq$, their periods must form a sequence of (not necessarily proper) divisors.

The only models we have met so far are models where the cycles have a simple linear order. (This is why I called them linear models.) We may establish antisymmetry for these models as follows. Suppose that in a linear collapsed model $C \preceq D$ and $D \preceq C$. Let x and y be in C and D. Then $x \leq y$ and $y \leq x$. C and D are cycles obtained by collapsing two blocks, say, C_i and C_j, respectively. Suppose, for *reductio*, that i and j are distinct. Say, without loss of generality, that $i < j$. Let x, y, be $[a]$, $[b]$, respectively. Since $y \leq x$, for some c, $[a] = [b] + [c] = [b + c]$. So $b + c$ is in C_i, which is impossible, since b is in C_j. Thus, $i = j$ and so $C = D$.

It is not difficult, however, to construct non-linear collapsed models. For example, take a set-up as in the theorem of Section 5. Let $1 < i+1 < j \leq \eta$; suppose that $p_i = p_j = p$, say. Let $\sim$ be defined as before. But now define a relation, $\approx$, on the numbers as follows. $x \approx y$ iff:

(i) $x \sim y$ or

(ii) one of $x, y,$ is in $C_i,$ the other is in $C_j,$
and $x = y \pmod p$

THEOREM. $\approx$ *is an equivalence relation and also a congruence relation on arithmetic operations.*

Proof. Given the properties of $\sim$, it is clear that $\approx$ is reflexive and symmetric. Transitivity is hardly more demanding. Suppose that $x \approx y$ and $y \approx z$. If both arise because of clause (i), then $x \approx z$ since $\sim$ is transitive. If both arise because of clause (ii) then x and z are in the same block and $x = z \pmod p$. Hence $x \approx z$. If one (say the first) arises because of clause (i) and the other arises because of clause (ii) $x \approx z$ because of clause (ii).

It remains to check that $\approx$ is a congruence. Successor is trivial. Multiplication is essentially the same as addition, which is as follows. Suppose that $x_1 \approx x_2$ and $y_1 \approx y_2$. If both arise because of clause (i) then $x_1 + y_1 \approx x_2 + y_2$ by the properties of $\sim$. If one (say the first) arises because of clause (i) and the other arises because of clause (ii) then $x_1 + y_1 \approx x_2 + y_2$ by clause (ii). If both arise because of clause (ii) then there are two cases. In the first, x_1 and y_1 occur in the same block, say C_i, whilst x_2 and y_2 occur in C_j. In this case $x_1 + y_1$ is in C_i, whilst $x_2 + y_2$ is in C_j, since the blocks are slices. In this case, the result holds by (ii). In the second case x_1 and y_2 occur in the same block, say C_i, whilst x_2 and y_1 occur in C_j. In this case $x_1 + y_1$ and $x_2 + y_2$ are both in C_j, and the result holds by (i). $\square$

Now consider a model of arithmetic collapsed by the relation $\approx$. It has, in general, a tail and a number of cycles; but now the cycles that result from the collapse of C_i and C_j are identical. (Every member of C_i is identified with some member of C_j and vice versa, since $\forall x \forall y \exists z (y \leq z \leq y + p \wedge x = z \pmod p)$ and the blocks have length greater than p.) Call this cycle C. If D is any cycle formed by collapsing a block between C_i and C_j, we therefore have $C \preceq D \preceq C$. And hence if E is any other such cycle, we have $D \preceq E$ and $E \preceq D$.

In the language of graph theory, these cycles form a *clique*, that is, each bears the relation $\preceq$ to all the others. The general structure of a linear preorder is, essentially, that of a linear order, with some or all points expanded to non-unit cliques.[13] The above construction produces only one non-unit clique, of size $j - i - 1$. But it is clear that, in combination with an appropriate choice of η and its family of p's, it could be used to produce any finite number of cliques of any finite size. In other words, the order-type of the blocks in a finite model can be any (finite) linear preorder.

Given the connection between cycle ordering and periods, all the cycles in a clique have the same period. Thus we may speak of the period of a clique itself to mean the period of all the cycles in it. Moreover, again, it is clear that by a judicious choice of η and its family of p's, we can construct a collapsed model in which the sequence of cliques has periods of any non-ascending sequence of divisors.

Let me conclude this section by providing an example of models with the same successor graph, but with different addition functions. Take a linear collapse with, say, three cycles, and consider the last of these. Because this is obtained from the collapse of the end-section of the model, it is not difficult to see that the addition of any number to a number in that cycle gives a number in the same cycle. Now take a non-linear model with the same tail and cycles, but in which the cycles form a clique. Let x be any member of a cycle, and y a member of a different cycle. Then since $x \leq y$, there must be a z such that $x + z = y$. Hence, the addition functions in these two models behave differently.

8. Conclusion

We have now charted the structure of the finite models. For summary, let me state what that is. Any finite model has a tail and cycle of regular numbers. The tail may be empty, and the cycle's period can be any positive finite number. There is then some finite number (including zero) of cycles of irregular numbers. The set of all cycles is linearly preordered. Any linear preorder is possible. Finally, the periods of the cliques of the preorder can be any non-ascending sequence of divisors.

I finish this part of the paper with a couple of open questions about finite models.

1. It is clear that the number of models of each finite cardinality is finite. Hence, the total number of finite models is countable. For each finite cardinal, n, how many models of cardinal n are there?

2. All the finite models that we have seen are constructed by collapsing classical models – or at least, by collapsing them and then extending the collapse. Are all the finite models to be obtained in this way?

In the second part of this paper, we will turn to inconsistent models in general, and look at their structure.[14]

NOTES

[1] See, for example, Boolos and Jeffrey (1974), Ch. 17, or Kaye (1991), Ch. 6.

[2] See Priest, Routley and Norman (1989), especially the introduction to Part 2.

[3] See Priest (1987), Ch. 5.

[4] For a proof, see Priest (1987), Ch. 5.

[5] A similar result is proved as Proposition 2.10 of Mortensen (1995).

[6] For details, see Priest (1991), Sec. 7. A very similar result is proved in Dunn (1979).

[7] See Meyer (1978). See, further, Meyer and Mortensen (1984), and Mortensen (1995), Ch. 2. Their models also model various properties of a non-extensional connective, $\rightarrow$, but this is not pertinent here.

[8] See van Bendegem (1993) and Priest (1994).

[9] The following theorem clearly generalises to infinite ηs. However, this generalisation is unnecessary here.

[10] I use 'p' here because these numbers are going to be periods, not because they are primes: they need not be.

[11] The material in this section owes much to a number of interchanges with Greg Restall. His ideas are certainly present in it.

[12] For the record, if T is the tail and C any cycle, then we also have $T \preceq C$. T is obviously $\preceq$ the cycle of regular numbers; and we already know that for any regular m and irregular i, $i \geq m$. That is, if B is a cycle of irregular numbers $T \preceq B$. However, we obviously do not have $T \preceq T$ unless $T = \{0\}$.

[13] For if $\preceq$ is a linear preorder, consider the relation $C \equiv D$, defined as $C \preceq D$ and $D \preceq C$. This is an equivalence relation; the order inherited by the equivalence classes is a linear order, and the classes are cliques.

[14] This part of the paper was read at the logic seminar of Indiana University. I am grateful for many interesting comments to those present, including Jon Barwise, Mike Dunn, Anil Gupta, David McCarty, Larry Moss and, especially, Jerry Seligman. I am also grateful to Greg Restall for written comments.

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