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Abstract Niels Bohr's model of the hydrogen atom is widely cited as an example of an inconsistent scientific theory because of its reliance on classical electrodynamics (CED) together with assumptions about interactions between matter and electromagnetic radiation that could not be reconciled with CED. This view of Bohr's model is controversial, but we believe a recently proposed approach to reasoning with inconsistent commitments offers a promising formal reading of how Bohr's model worked. In this paper we present this new way of reasoning with inconsistent commitments and compare it with other approaches before applying it to Bohr's model and offering some suggestions for how it might be extended to account for subsequent developments in old quantum theory (OQT).

Keywords Niels Bohr · Hydrogen atom · Old quantum theory · Inconsistency · Paraconsistent logic · Weak aggregation

1 Chunk and permeate

One way to cope with inconsistency in a set of cognitive commitments is to divide the commitments up: if no contradictions are present, we can produce a consistent

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partition of the inconsistent premise set and reason in the old familiar way[1] within the cells of the partition. It seems to us that something like this is going on in old quantum theory (OQT).

Logics that weaken an underlying consequence relation by dividing a set of premises in this way are called *weakly aggregative* (Schotch and Jennings 1989). For Σ a set of sentences in a language L, an *n-covering* of Σ is an indexed set of sets of sentences, $\sigma_i, 0 \leq i \leq n$ such that $\bigcup_{0 \leq i \leq n}(\sigma_i) = \Sigma$. Such a covering of Σ, $C(\Sigma)$ is *consistent* relative to an underlying consequence relation $\vdash$ if and only if $\forall i, 0 \leq i \leq n, \sigma_i \nvdash \perp$. So long as Σ contains no contradictions there will be consistent n-coverings of Σ for some $n \leq \omega$; when Σ is inconsistent, we can avoid trivialization of Σ's consequences by closing each cell of such a covering under $\vdash$ separately.

If n is finite, coping with inconsistency in this way preserves a limited degree of *aggregation* for our premises: on pigeon-hole principles, if the number of sentences in Σ is greater than the number of cells in our covering, at least two sentences will end up in the same cell, and their conjunction (in general not a consequence of either individually) will follow in that cell. So if we identify the 'consequences' of Σ with the union of the logical closure of each σ_i, $\bigcup_{0 \leq i \leq n} Cl(\sigma_i, \vdash)$, there will be consequences of a commitment to Σ that do not follow from any individual sentence in Σ. But where $n \geq 2$, such commitments are not, in general, closed under conjunction.[2]

In their (2004), Priest and Brown applied a related strategy they dubbed *Chunk and Permeate* ($C\&P$) to give an account of reasoning in the old calculus of infinitesimals. The key innovation of $C\&P$ is the addition of a *permeation relation* to the divided commitments of weakly aggregative logic. A permeation relation permits specified sentences to be transferred from a cell of the initial partition where they have been derived to some other cell(s) of the partition. Here we apply $C\&P$ to Niels Bohr's 1913 model of the hydrogen atom.

We propose $C\&P$ as a formal reconstruction of Bohr's approach, leaving the historical question of how Bohr and other scientists understood Bohr's model aside. But our $C\&P$ account was directly inspired by Bohr's use of explicit restrictions on the contexts in which certain physical principles were to be applied when reasoning with his model.[3]

A $C\&P$ structure on Σ, $\wp$, is a 3-tuple $\langle P, \rho, i_0 \rangle$ where:

i. P is a consistent covering of Σ, with elements $\sigma_1, ...\sigma_n$

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[1] That is, classically or intuitionistically or indeed with any other logic we choose.

[2] Forcing, a weakly aggregative logic due to P.K. Scotch and R.E. Jennings, preserves a generalization of consistency called *level*: the level of Σ, $l(\Sigma)$ is the least n for which there is a consistent n-covering of Σ. For a level n, we can capture all the aggregation that follows from forcing using the rule $2/n + 1$: where $2/n + 1(\alpha_0, ...\alpha_n) = \bigvee_{0 \leq i \neq j \leq n}(\alpha_i \wedge \alpha_j), \Gamma \vdash \alpha_0, ...\Gamma \vdash \alpha_n / \Gamma \vdash 2/n + 1(\alpha_0, ...\alpha_n)$.Apostoli and Brown (1995) This is the strongest *formal* principle of aggregation we can apply to a set of level n without trivialization.

[3] When speaking of Bohr's account of the hydrogen atom we will use the word 'model', reserving 'theory' for a set of sentences closed under a consequence relation.

ii. ρ is a *permeability relation*, a function from pairs of integers $I \times I$ to subsets of L.
iii. i_0 is the label of the *designated chunk*, where we draw our conclusions.

The $C\&P$ consequences of Σ relative to a $C\&P$ structure $\wp = \langle P, \rho, i_0 \rangle$ are produced by a series of closure and permeation steps. Where σ_i is the i^{th} cell of P, σ_i^n is defined recursively:

$$\begin{aligned} \sigma_i^0 &= Cl(\sigma_i, \vdash) \\ \sigma_i^{k+1} &= Cl(\sigma_i^k \cup \bigcup_{j \in I} (\sigma_j^k \cap \rho(j, i))) \end{aligned}$$

Thus at each step, the cells are first closed under a consequence relation, after which any sentences in $\rho(j, i)$ that appear in cell j are added to cell i.

Finally, we define the $C\&P$ consequences of Σ:

$$\Sigma \Vdash_{\wp} \alpha \quad iff \quad \exists k, \alpha \in \sigma_{i_0}^k.^4$$

$C\&P$ achieves a fair degree of inferential unity: repeated permeation and closure operations ensure that premises available in other cells contribute to the conclusions drawn in $\sigma_{i_0}^n$. However, $C\&P$ structures take us a long way from the usual notion of a consequence relation: the set of sentences in $\bigcup \sigma_n^i$ doesn't determine the contents of $\sigma_{i_0}^n$. So the 'consequence relation' here is not a relation between sets of sentences and the sentences that follow from them. Instead, it's a relation between the $C\&P$ *structure* and some sentences: the structure as a whole determines the content of the output chunk σ_{i_0} at each step of the recursion. Finally, which covering of Σ and which permeation relation are the 'right ones' is not settled by formal criteria; instead, the covering and permeation relation are chosen in light of specific inferences we aim to preserve, and to avoid.

$C\&P$ has an advantage over other logically heterodox approaches here: it does not focus on identifying or proposing alternative logics that might lurk in the background of scientific reasoning. Instead, it focuses on a more directly observable feature of reasoning, viz. how and where different premises are invoked in the course of arguments. By focusing on how actual premises are deployed in the course of reasoning, $C\&P$ straddles Norton's distinction between logic-based and content-based approaches to inconsistency in science (Norton 2002, p. 191). But it also represents

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[4] It is tempting to allow every sentence *consistent* with σ_i^k to permeate from σ_j^k before closing under $\vdash$ to form σ_i^{k+1}. But with just two elements σ_1 and σ_2 such that $\sigma_1 \cup \sigma_2 \vdash \bot$, closing both under $\vdash$ and then allowing every sentence consistent with each to permeate in from the other is disastrous if the underlying logic is classical: for any sentence α consistent with σ_1^{k-1}, consider the sentence $\phi \to \alpha$, where ϕ follows from σ_1^{k-1} and $\neg\phi$ follows from σ_2^{k-1}. This is trouble– $\phi \to \alpha$ is a consequence of (and so contained in) σ_2^{k-1}, and it's consistent with σ_1^{k-1}. So every such sentence permeates into σ_{k-1} before we close under $\vdash$ to form Σ_1^k. But we already have ϕ in σ_1^k, so when we close again we find σ_1^k includes every α consistent with σ_1^{k-1}. But if $Cl(\sigma_1^{k-1})$ is not maximal consistent, then for some $\beta \in L$, both β and $\neg\beta$ are consistent with σ_1^{k-1}. So unless $Cl(\sigma_1^{k-1})$ is maximal consistent, σ_1^k is trivial.

a *general strategy* for avoiding disastrous consequences that could be inferred from some premises if they were freely combined.[5]

When a theory allows contradictory conclusions to be drawn, Norton describes his 'content driven control of logical anarchy' as an effort to determine "which of these conclusions to take seriously and which to ignore as spurious". Norton does worry that this seems entirely *ad hoc*, but his response is to suggest that we should regard such 'selective' responses as an attempt to capture, via "meta-level arguments applied to the inconsistent theory", the consequences of a *consistent* theory to which the inconsistent theory is assumed to be an approximation (Norton 2002, p. 193). This is elegantly accomplished, Norton reports, in Malament's subtle relativisation of Newtonian gravity (Malament 1995).

But this leaves the trouble with the original, inconsistent theory unresolved. We can't reason with it using classical logic, since its inconsistency renders it trivial. Yet scientists continued to write down and solve equations, apparently reasoning with the premises they actually had available to them, and finding some of the results convincing. $C\&P$ offers a systematic account of how this can be done and, unlike more narrowly logical approaches, it does this *without* drawing our attention away from the actual premises and arguments offered by scientists.

Furthermore, the meta-level arguments Norton invokes might also be taken to suggest the sort of restrictions on when to apply certain premises that $C\&P$ invokes. For example, Norton discusses Seeliger's efforts to apply Newtonian gravity in an infinite cosmos with a mean density larger than zero. Seeliger apparently accepted applications of the inverse square law to local systems, while refusing to apply it at the cosmological scale where it led to trouble. This left the cosmological side of things largely a matter of stipulating a uniform, non-zero and static average density of matter on the cosmological scale. But a simple $C\&P$ structure can capture this explicitly, by using a permeation relation that accepts Seeliger's refusal to apply the inverse square law to the universe as a whole: we allow motions calculated via Newton's theory for finite systems to permeate through to the cosmological scale (which we treat as our target chunk) while imposing a static cosmological mass distribution on that scale. If the permeation relation allows step-by-step *finite* extensions of the size of a local Newtonian model, the local model will remain consistent, while the cosmological chunk serves as a uniform, infinite framework in which the results of local Newtonian physics produced at each step can be embedded.

Another approach to coping with inconsistency in science has been proposed by Newton da Costa, Steven French and Otavio Bueno. Their general semantic framework for representing the content of inconsistent scientific theories and epistemic

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[5] A general plan for eliminating inconsistency in scientific theories was proposed in Norton (1987), where Norton emphasized the separation between quantum theory and CED in the course of showing that Planck's derivation of the black-body radiation law can be obtained using a sub-theory of CED that is *consistent* with quantum rules limiting the energy states of resonators and the radiation field. But in the same paper, Norton also invokes a weakly aggregative approach to Planck's original theory, remarking "...one could not derive any proposition within the theory because of the tacit introduction of a nonclassical device, the two domains of calculation with inarticulated restrictions on the exchange of results between them." Norton (1987, p. 348) $C\&P$ provides a systematic way of specifying such restrictions.

commitments to them invokes the notions of 'partial structures' and 'quasi-truth' daCosta et al. (1998) and French (2003). In da Costa and French, (2003, p. 87) Da Costa and French criticize (Brown 1992)'s invocation of Schotch and Jennings' weakly aggregative logics as an account of Bohr's model of the hydrogen atom, because, they say, it involves a commitment, even if contextually limited, to the *truth* of both classical physics and the quantum principles Bohr relied on. The same could be said of $C\&P$, which also adopts contextual constraints on when particular premises involved in Bohr's model are invoked, but still relies on them in the course of arriving at his results. But this objection misreads both Brown and $C\&P$: the weakly aggregative strategies they employ preserve *level* and the consistency of our chunks, not *truth*. Being committed to a premise as a basis for *reasoning in some context* is not the same as taking that premise to be *true* in the context. The $C\&P$ approach, like Brown's, aims to systematize how scientists could *reason* with inconsistent premises while avoiding 'logical anarchy', not to explain how such premises could all be true or what would follow if they were.[6]

Thus the $C\&P$ strategy is general in two ways: it makes no assumptions about the 'underlying logic' of the chunks, and it makes no assumptions about the form of cognitive commitment that scientists reasoning in this way make to their premises. In connection with this point, it's worth noting that individual scientists appear to have adopted very different attitudes towards the conflicting premises at work in Bohr's model. Some, like Sommerfeld, seem to have taken the idea of orbiting electrons very seriously (see Section 5), while others, including Bohr himself, were more reserved. But all of these figures could have applied the $C\&P$ strategy to *reason* with Bohr's theory. Adding to the contrast between our approach and that of French, Bueno and da Costa, $C\&P$ does not rely on a philosophical account of representation. Formal semantic models are not central to the story we are telling. In our view, such accounts of the content of scientific models and theories take us too far from scientists' actual *use* of theoretical language, both in reasoning with it and in applying it to report observations of features of the world the language is applied to. But this is a debate for another occasion. Our aim here is just to show that the $C\&P$ approach provides a straightforward account of how to systematically distinguish inferences that were accepted from apparently available inferences that were rejected in Bohr's account of the hydrogen atom.

2 Planck and Bohr

We claim there is a close parallel between Bohr's model and the $C\&P$ structure we present in the following section. In this section we lay the groundwork for this claim with a review of the early history of old quantum theory, focusing on Bohr's model of the hydrogen atom.

Max Planck's work on black-body radiation quantized energy, limiting it to discrete units proportional to frequency with his h as the constant of proportionality.

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[6] While some paraconsistent logics do aim at this goal, $C\&P$ does not.

He initially intended his constant as a mere aide to calculation. But instead Planck found that his h could not be eliminated by taking a limit as it was reduced to zero. Instead, to capture the empirical black-body curve h had to be assigned a specific non-zero value. As a result, his derivation of the empirically established black body radiation curve was regarded with some skepticism. In the discussion at the end of the first Solvay conference (October 30-November 3, 1911) Poincaré remarked (perhaps somewhat tongue-in-cheek) "In this context one must keep in mind that one can probably prove every theorem without too much effort if one bases the proof on two mutually contradictory premises" (Poincaré, in Eucken (1914, 364), Mehra and Rechenberg (1982, p. 135)).

Neils Bohr was clearly worried about the inconsistency of Planck's theoretical apparatus, remarking:

> In formal respects Planck's theory leaves much to be desired; in certain calculations the ordinary electrodynamics is used, while in others assumptions directly at variance with it are introduced without any attempt being made to show that it is possible to give a consistent explanation of the procedure used.

But he went on to say,

> It is... hardly too early to express the opinion that whatever the final explanation will be, the discovery of "energy quanta" must be considered as one of the most important results arrived at in physics, and must be taken into consideration in investigations of the properties of atoms (Bohr 1922, p. 6).

Bohr's work on the hydrogen atom began with the aim of combining quantum considerations with the Rutherford atom. Rutherford's model conflicted so starkly with CED that Bohr hoped a quantum treatment of it might illuminate the difficult puzzle of how classical physics and quantization were related.

In 1913 Bohr published his model of the hydrogen atom in the first of a series of 3 papers (Bohr 1913a, b, c). He had not planned to give an account of the hydrogen spectrum, but realized late in preparing the papers that such an account was possible, and quickly incorporated the results in the first paper of the series. Like the early calculus, Bohr's model invoked inconsistent assumptions. The use of classical electrodynamics (CED) to describe the light emitted and its interaction with various instruments is taken for granted. But Bohr's mechanical model of the atom's "stationary states" includes an accelerating charged particle that does not radiate. This description cannot be reconciled with CED; to cope with this, Bohr proposed rules dictating what bits of theoretical apparatus were to be applied where. Three types of contexts are involved in Bohr's account. In Bohr, 1913a, these are specified in two numbered assumptions:

(1) That the dynamical equilibrium of the systems in the stationary states can be discussed by the help of the ordinary mechanics, while the passing of the system between different stationary states cannot be treated on that basis.
(2) That the latter process is followed by the emission of a *homogeneous* radiation, for which the relation between the frequency and the amount of energy emitted is the one given by Planck's theory. Bohr (1913a, p.7)

A later presentation is more detailed:

i. That an atomic system can, and can only, exist permanently in a certain series of states corresponding to a discontinuous series of values for its energy, and that consequently any change of the energy of the system, including emission and absorption of electromagnetic radiation, must take place by a complete transition between two such states. These states will be denoted as the 'stationary states' of the system.
ii. That the radiation absorbed or emitted during a transition between two stationary states is 'unifrequentic' and possesses a frequency ν , given by the relation

$$E' - E'' = h\nu,$$

where h is Planck's constant and where E' and E'' are the values of the energy in the two states under consideration (Bohr 1918, p. 97–8)

Despite combining a classical account of emitted radiation with a radically unclassical treatment of the 'stationary states,' and the lack of any account of how the atom actually emits radiation, Bohr's model was a breakthrough. Bohr's 'reconciliation' of Rutherford's atomic model, a tiny positively charged nucleus containing almost all the mass of a neutral atom, with the lack of any stable classical model for such a system, was achieved by a kind of brute force: he simply refused to apply the ordinary principles of electrodynamics to the stationary states. In Kragh (2012, p. 91), Holge Kragh remarks, "Bohr's atom sat like a baroque tower upon the Gothic base of classical electrodynamics." Its reception was marred by harsh comments from figures including Ehrenfest, who called Bohr's model "completely monstrous," in a letter to Sommerfeld from the spring of 1916 (cited in (Kragh 2012, p. 91)) even though Erhenfest went on to contribute substantially to *OQT* with his 'adiabatic principle'. Still, Bohr's model provided a systematic derivation of the hydrogen spectrum–one that made successful predictions, connected the empirical Rydberg constant to fundamental constants, and was subsequently refined and extended to capture some other spectra as well.[7]

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[7] The question of whether Bohr's account was *logically* inconsistent is difficult to answer directly: From a purely logical point of view, Bohr's description of the atom combined with CED implied that his atom could not have a stable ground state. Since Bohr's model included a stable ground state, the *sentences* used in the course of applying his theory to account for the hydrogen spectra were inconsistent. However, Bohr's personal views could still have been consistent: for example, he might have accepted CED instrumentally for purposes of observing light on macroscopic scales, while regarding it as unreliable on the atomic scale. But such interpretive questions about personal beliefs are not our concern here. Applying CED to the stationary states was obviously disastrous– the atom would rapidly collapse, radiating at increasing frequencies along the way. This consequence is not *logically inconsistent*, but while rapidly collapsing hydrogen atoms seem consistent enough in themselves, they are clearly inconsistent with observation. To stay consistent with observation, Bohr had to avoid applying CED to his stationary states. But classical electrodynamics was the only available way to model the radiation emitted by his atoms. Bohr apparently dealt with this tension simply by assuming 'for now' that no radiation occurs while the atom is in a stationary state– an uneasy kind of stipulation. Our $C\&P$ structure allows us to retain CED for purposes of interpreting spectral data while *systematically* avoiding the disastrous collapse of the stationary states by confining CED and Bohr's account of the stationary states in separate cells of our proposed $C\&P$ structure.

We propose a very conventional view of Bohr's model; it includes:

- Models of the stationary states of a hydrogen atom, combining classical mechanics, the Coulomb force attraction between the electron and the nucleus, and imposing quantum restrictions to determine the allowed orbits.
- Transitions between stationary states. These transitions are not modeled in any detail– Bohr explicitly denies that a mechanical model of the transitions is possible, and offers no other kind of model.
- Classical radiation emitted or absorbed by the atom in transitions from one stable orbit to another; the energy and frequency of the radiation are determined by Planck's relation $E = h\nu$, where E is the energy difference between the initial and final electron orbits.

Bohr's first model of the stationary states used circular orbits, with energy W (negative because the zero energy state is defined with the electron at rest at an infinite distance from the nucleus), radius a, frequency of revolution ω, angular momentum of the electron L and charges on the electron of $-e$ and the nucleus of $+Ze$. The boldness of Bohr's approach is striking: rather than struggle with the issue of stability for such an atom, Bohr simply says "(l)et us at first assume that there is no energy radiation" (Bohr 1913a, p. 3).

A further assumption is that the energy emitted in the capture of a free electron by the nucleus will be at $\frac{1}{2}$ the frequency of the electron's resulting orbit. This marks the first appearance of what later developed into the correspondence principle, as well as the idea that *averages* of classical quantities can guide the development of the quantum theory. To justify this claim, Bohr merely remarks, "(i)f we assume that the radiation emitted is homogeneous, (this) assumption suggests itself, since the frequency of revolution of the electron at the beginning of the emission is 0." (Bohr 1913a, p. 5).[8]

3 A simple *C&P* structure for the Bohr hydrogen atom

Applying *C&P* to reconstruct Bohr's model neatly captures the separation between Bohr's description of the atom's stationary states and the use of classical electrodynamics to describe the radiation emitted in transitions between the states. Since Bohr's model gives no account of transitions between stationary states, our *C&P* structure ignores transition contexts, including just two cells, one for the quantized treatment of stationary states and one for the description of the radiation emitted/absorbed by the atoms. Our *C&P* structure is:

$$\langle \{\sigma_Q, \sigma_C\}, \rho, \sigma_C \rangle$$

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[8] Bohr's initial treatment approximated by treating the ratio of the proton's mass to the electron's as infinite. After Fowler claimed that Bohr's calculation of the Pickering lines fell outside the bounds of experimental data, Bohr wrote a letter to *Nature* in which this assumption was dropped. The corrected calculation gave improved agreement with the Pickering lines and predicted several as yet unobserved lines (see Pais (1991, p. 149)) and Mehra and Rechenberg (1982, p. 192)

σ_Q includes classical mechanics, the Coulomb attraction, Bohr's quantum restriction on the allowed 'stationary states,' and Planck's frequency relation in the form $\Delta(E) = h\nu$, where ΔE is the difference between the energies of two stationary states.[9]

σ_C includes Maxwell's equations (CED) together with standard accounts of optical instruments including (of course) spectroscopes, applying Maxwell's equations (with accepted approximations) to observations of spectra, along with Planck's frequency relation $\Delta(E) = h\nu$.[10]

As indicated, σ_C is the output chunk, where we draw conclusions about spectra that can be experimentally tested; the accounts of instruments and their interaction with light included here are essential to the empirical testing of Bohr's model. Because the available accounts of the instruments' interaction with light relied on CED (as, indeed, standard contemporary treatments of such instruments often do), we include both CED and specific accounts of the instruments interactions with light of different frequencies in σ_C.

Having the Planck/Einstein frequency relation $\Delta(E) = h\nu$ in σ_Q allows us to calculate the frequencies of light emitted or absorbed in transitions between various states as Bohr did, assigning the energy released or absorbed by a transition $\Delta(E) = E_i - E_f$ to a particular frequency ν. Thus, when such a transition is modeled in σ_Q, we infer $\Delta E_\nu = E_i - E_f$ where $\nu = \frac{E_i - E_f}{h}$.

Finally, $\rho(Q, C) = \rho(C, Q)$ is the set of equations $\Delta(E) = h\nu$, where ΔE_ν is the difference in energy between two stationary states. This ensures that equations specifying the frequencies of light that the Bohr model predicts will be emitted or absorbed when a hydrogen atom changes state arrive in σ_C. It also ensures that observations of spectral lines reported in σ_C get added to σ_Q, where they are taken to represent transitions between (known or unknown) stationary states separated by the requisite ΔE. Thus our chosen $\rho(Q, C)$ and $\rho(C, Q)$ allow the $C\&P$ structure to capture both how conclusions about the quantized atom's states were used to make predictions about spectral lines and how observations of spectral lines motivated extensions and modifications of σ_Q.

To see how this works, it's helpful to think through the results of the first two $C\&P$ steps: closure of the two sides under consequence, followed by permeation.

First Closure Step: In this step, reasoning goes on *within* the two chunks. In σ_Q, this leads us to infer the conclusions about the hydrogen spectrum Bohr drew from his model in Bohr (1913a). In σ_C, this step captures the empirical data on the hydrogen spectrum and its interpretation in terms of classical electrodynamic radiation. Having Plank's equation $\Delta E_\nu = h\nu$ in both chunks creates a two-way link between

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[9] The Appendix provides a standard account of Bohr's original model of the hydrogen atom, specifying the stationary states, deriving the energy differences between them and calculating the frequencies of the resulting radiation (or the radiation absorbed) using Planck's rule. The equations used in these calculations can be read as a detailed list of the contents of σ_Q.

[10] As an example of such treatments of instruments and their interaction with light, the location of brightness maxima for various wavelengths λ of plane wave light diffracted from a grating are determined by calculating the difference of path lengths from each line of the grating to each point on the illuminated surface: maxima appear when the difference of path lengths to the surface equals λ.

observation reports about frequencies of light observed in σ_C and transitions between states of the atoms represented in σ_Q.

First Permeation Step: $\rho(Q, C)$ allows information about the frequencies predicted by Bohr's model to permeate into σ_C, while $\rho(C, Q)$ allows information about the energy differences between stationary states corresponding to observed frequencies to permeate into σ_Q. In this step we get two kinds of empirical test of Bohr's proposal: do the frequencies predicted by Bohr's model appear in the observed spectra, and do observed frequencies emitted or absorbed by samples of hydrogen gas match the frequencies predicted by Bohr's model and the Planck equation?

The novelty of Bohr's model lay strictly in σ_Q and $\rho(Q, C)$ and the exclusion of classical electrodynamics from calculations in σ_Q. Thus Bohr did not need to include an account of the familiar calculations that go on in σ_C in his presentation of the model. The derivation of energy levels of the states of Bohr's atom presented in the Appendix turns on premises available in σ_Q. But at the end of the derivation, Planck's equation $E = h\nu$ is used to calculate the frequency of the radiation emitted when the atom moves from a higher to a lower energy state. Since Bohr's only model of that radiation and its interactions with instruments such as spectroscopes is based on classical electrodynamics, we chose $\rho(Q, C)$ to allow equations representing changes in energy of a classical radiation field surrounding the atom to be added to a classical description of the field, producing predictions about the spectrum emitted by a sample of excited hydrogen gas but shielding the earlier parts of the derivation from CED's drastic implications for the stationary states.

We close this section with an objection and a reply: our $C\&P$ model draws its conclusions in σ_C. But not all the observational implications of Bohr's model require a shift from σ_Q to σ_C. For example, Bohr derived a satisfactory figure for the characteristic radius of a hydrogen atom in its ground state, and he invoked the much larger radii of higher energy states to explain the absence of some lines of his spectrum in samples of hydrogen gas at atmospheric pressure. Thus our $C\&P$ model does not yet allow all the empirically tested implications of Bohr's model to appear in our output chunk, σ_C. Our reply is simple: the equations stating these results can be added to $\rho(Q, C)$ without creating problems for σ_C. More generally, extensions of $\rho(Q, C)$ allowing further results obtained in σ_Q to permeate into σ_C play an important role in producing $C\&P$ models of later extensions of Bohr's model.

4 Consistency

The union of σ_Q and σ_C is clearly inconsistent: in particular, $sigma_Q$ specifies a stable ground state with a characteristic radius for a Rutherford hydrogen atom , while *CED*, which is included in σ_C, rules out a stable ground state for any Rutherford atom. Thus a definition of consistency for $C\&P$ structures which demands consistency of the union of all chunks would make our account of Bohr's model 'inconsistent.' However, an extension of Post's criterion provides a more useful notion of consistency for a $C\&P$ structure: $\langle P, \rho, i_0 \rangle$ is *consistent* iff $\exists \alpha : \langle P, \rho, i_0 \rangle \nvdash \alpha$, that is, iff the conclusion chunk is non-trivial. In this sense, the *consistency* of our $C\&P$ structure depends on whether every sentence in the language appears in some σ_{C^n}. Using σ_C as the output chunk and considering just $\rho(Q, C)$ focuses our attention on the flow of inference in the model from the models of quantum systems and their stationary states that appear in σ_Q to predictions about spectra observed in various circumstances. $\rho(C, Q)$ focuses our attention on inferences in the reverse direction, from observations of spectra to the models of quantum systems and the stationary states that appear in σ_Q. But neither type of inference adds other *kinds* of consequences in either chunk. Thus allowing only equations of the form $\Delta E_\nu = r$ to permeate from σ_Q to σ_C and vice-versa makes showing the non-triviality of our $C\&P$ structure, in the sense proposed above, fairly easy to do (Norton 1987, 2000).

σ_C begins with CED as its contents; when we extend it to model (say) the radiation field surrounding a collection of excited hydrogen atoms, we begin with an initial state of the classical radiation field surrounding the atoms, and then add (or subtract) the correct amounts of energy at frequencies determined by Planck's equation to (or from) the field. In the case of radiation absorbed by the atomic system in a transition to a higher energy state, we assume a model of the initial state of the field that includes sufficient energy in the right place and frequency to balance the books. At least from the point of view of the physics community of the day, there was no concern that inconsistency posed a threat here.

The consistency of σ_Q raises more interesting historical questions, since σ_Q imposed a strangely limited collection of stationary states described in terms of classical mechanics. These limited stationary states are at best *unnatural* from a classical point of view: nothing in classical mechanics distinguishes them from the many states that are excluded by σ_Q. Worse, the 'quantum leaps' from one allowed state to another that Bohr invoked to account for the addition or subtraction of energy at the corresponding frequency to the surrounding classical radiation field are handled essentially by fiat. However, this does not seem to be outright inconsistent, so long as any energy absorbed in a transition to a higher energy state is available in the surrounding field prior to the transition. Einstein argued in his elegant derivation of Planck's black body radiation law (Einstein 1967) that it is necessary to treat the radiation emitted or absorbed as a directed quantity, producing a change in momentum for the atom. But we ignore this complication for now (as did Bohr), since it is irrelevant to Bohr's calculations of spectral emission and absorption lines.

Finally, newly observed spectral lines can be added to σ_C by adding equations of the form $\Delta E_\nu = E_i - E_f$. Since such equations are included in $\rho(C, Q)$, they will permeate back into σ_Q. But this doesn't lead to inconsistency either: when σ_Q includes no pair of states whose energy difference satisfies the equation, it can be consistently extended to posit such pairs of states.

From a broader perspective, the consistency of σ_C is not trivial, as the arguments in Frisch (2005) for the inconsistency of CED show. But at the level of applied physics there was (and is) no recognized difficulty in adding energy at a given frequency to some region of a classical radiation field. Of course, the emission and absorption events receive no description in classical or other terms. To keep σ_Q consistent we need to limit the role of classical mechanics in σ_Q to modeling each allowed stationary state separately. If we think of this simply as specifying the class of acceptable classical models for states of a hydrogen atom, the result seems strange, but consistent. Further applications of classical ideas in σ_Q played an important heuristic role in efforts to extend old quantum theory– and these efforts were, in turn, important in the later emergence of quantum mechanics. But even these extended applications were tightly controlled. They never gave rise to a general quasi-classical theory of quantum systems; instead, they connected and systematized the stationary states of different systems and revealed more ways to link features of the classical descriptions of stationary states to claims that could then permeate to σ_C and be tested by various optical observations.

So long as CED itself is consistent, the consistency of σ_C as an applied account of electromagnetic radiation is clear enough.[11] On the other hand, if CED is inconsistent, Bohr's model of the atom is not the problem. More importantly, any consistent account of the physical reasonings that were conducted using CED would allow for a consistent $C\&P$ structure capturing Bohr's model of the hydrogen atom. Of course on the other hand, it's clear that *observations* could still conflict with the predictions about spectra we find in σ_C.

5 Enriching the model

Bohr's division of contexts and his focus on σ_Q and the stationary states (see the Appendix) in his initial derivation of the hydrogen spectrum (Bohr 1913a) make the application of $C\&P$ to his 1913 model straightforward. But a successful reconstruction of Bohr's model should do more than fit the model as Bohr first presented it– it should also make sense of subsequent modifications and extensions as *OQT* developed. In the next stage of this project we will examine some extensions of Bohr's model and show that they, too, can be reconstructed as extensions of our basic $C\&P$ model. Here we anticipate that work with a brief sketch of some extensions that were made and how, in general, a $C\&P$ structure could be extended to model them.

Bohr said of his model, "I am by no means trying to give what might ordinarily be described as an explanation" (Pais 1991, p. 155) of the atom's emission and absorption of light. One reason for this reserved stance was that the frequency of light emitted or absorbed had nothing to do with any frequency of motion of the electron: although the frequency could be *calculated* using Planck's equation, the model offered no richer physical understanding of the radiation process or, in particular, of why the transition should produce that particular frequency of light. Further, Bohr's initial model of the hydrogen atom said nothing about selection principles restricting which state-to-state transitions could occur, or about the polarization of emitted radiation. Finally, though energy differences between states could be calculated using the classical description of the states, the standard approach to *defining* that energy difference, in terms of the energy required or released by an adiabatic process taking the atom smoothly from one state to the other, could not be used, since the allowed states

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[11] Of course if we include a rich mathematics in each cell of our structure, the possibility of proving consistency can't be ruled out, and this in turn would imply inconsistency; but that uncertainty runs deeper than the tensions in OQT, and deeper than we venture here.

of the atom did not include intermediate states. These questions were later addressed by drawing selectively on classical electrodynamics and thermodynamics. Modifications of Bohr's initial model of the hydrogen atom also emerged. These led to Bohr's treatment of the ionized Helium spectrum (which impressed Einstein greatly, (Rosenfeld 1963, p. xiii)), Sommerfeld's account of some fine structure of the hydrogen spectrum using relativistic corrections for the energy of highly elliptical electron orbits, (Mehra and Rechenberg 1982, p. 220f) Ehrenfest's adiabatic principle along with its subsequent extension (Mehra and Rechenberg 1982, p. 236f), further work on stationary states (Mehra and Rechenberg 1982, p. 214ff) and Bohr's *correspondence principle* (Mehra and Rechenberg 1982, p. 247f).

Over time Bohr's correspondence principle was extended: in Bohr (1913a) Bohr picked out quantum restrictions on stationary states by insisting that for transitions between adjacent quantum states approaching the energy of a free electron, the frequency of light emitted must correspond to the result of a classical treatment, identifying the frequencies of the light emitted from a collection of atoms with frequencies of the electron's orbital motion. At this point Bohr interpreted this as an analogy between classical and quantum physics. But this relation between the classical and quantum was later extended to link the coefficients of the Fourier series representing the electron's motion in different quantum states to the probabilities of various transitions, allowing explanations of the relative intensities and polarizations of spectral lines (Janert 2013, p. 154f). In the end, Bohr came to regard the correspondence principle as a fundamental law of quantum mechanics rather than an analogy between the classical and the quantum.

A detailed treatment of these developments in terms of $C\&P$ is a goal for further work. Here we confine ourselves to three brief suggestions of how extensions of OQT could be captured by extensions of our initial $C\&P$ structure: we believe a $C\&P$ account of the development of CED would treat the decade of work following Bohr's 1913 papers by

- Adding further apparatus from classical physics to σ_Q, as physicists learned how to integrate that apparatus and its results with quantum principles. Examples include the adiabatic principle, and the treatment of samples of atoms in electrical and magnetic fields.
- Extending $\rho(Q, C)$ to allow more information about the allowed states and transitions between them in σ_Q to permeate into σ_C, for example, information about polarization.
- Applying $\rho C, Q$ to identify differences between the energies of quantum states in new systems or systems under new conditions, guiding the search for quantum models of these systems, for example, motivating Sommerfeld's appeal to relativity in his model of fine structure in the hydrogen spectrum.

The development of OQT continued through to the mid-1920's. Serious trouble emerged when Pauli found that the hydrogen atom in crossed electric and magnetic fields admitted a periodic classical model of the stationary states: when the adiabatic principle was applied to the model, allowed states could be converted into forbidden ones (Vickers 2013, p. 65f). This time central parts of the program that had contributed substantially to its success led to untenable results, even when applied with the accepted contextual restrictions. The upshot from a $C\&P$ perspective is clear: either a finer division of contexts would be required, or some commitments of OQT would have to be surrendered. As things turned out, though, neither option was pursued. A new, consistent but very strange theory superseded OQT: Quantum Mechanics.[12]

6 Conclusion

Like $C\&P$, paraconsistent logics allow negation-inconsistent but non-trivial theories; they can also allow theories that avoid other undesirable consequences that their premises would otherwise seem to imply. In some cases features of how reasoning is conducted within a particular theory fit fairly neatly with features of one paraconsistent logic or another. For example, in Brown (1992), Bryson Brown proposed a weakly aggregative approach to OQT. Brown's proposal was based on the observation that premises from CED relied on in the interpretation of spectroscopic data were incompatible with Bohr's assumption of the stability of his 'stationary states'. However, this pure 'divide and conquer' approach leaves the actual *reasoning* Bohr did unaccounted for. $C\&P$ recognizes both the division and the restricted flow of information across the divide. Other paraconsistent logics provide a weaker consequence relation that would not trivialize the inconsistent union of our σ_Q and σ_C. But as Nuel Belnap once pointed out to us,[13] it's difficult to identify detailed *logical* commitments in the arguments scientists accept or reject: different logics typically agree on most inference rules, and where they differ, logics that reject a particular rule often accept inferences that appear to use the rule as reasonable enthymemes in particular cases. Further, failure to make a relevant inference whose premises seem to be available need not be due to logical heterodoxy; the attitudes scientists take towards the premises they use are often more reserved than outright *belief*, and sometimes involve limits on the contexts in which some premises can reasonably be relied on. It's only when premises are taken to be true *tout court* that we could expect scientists to be committed to whatever conclusions can be logically derived from them. Finally, as John Norton has remarked (2002, p. 191), scientists have rarely if ever proposed changes in logic, even when dealing with apparent inconsistencies in their theories. We believe the $C\&P$ approach strikes a good balance between the need to keep our logical models close to the actual phenomena (how scientists actually reasoned) and the *philosophical* aim of providing a systematic and workable reconstruction of how scientists reached their conclusions without falling into either arbitrariness or triviality.

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[12] The resolution of Pauli's difficulty emerged from the re-interpretation of stationary states in quantum mechanics, which allowed states corresponding to these states forbidden by old quantum theory; see Vickers (2013, p. 67f).

[13] In a conversation about OQT.

Appendix: Bohr's Hydrogen Atom

In this appendix we follow through a standard contemporary treatment of the main features of Bohr's account of the hydrogen atom with our proposed $C\&P$ structure in mind. Like Bohr's, this account is focused on the properties of stationary states and inferring from them the energy and frequency of light emitted or absorbed in transitions between states (Eisberg 1961, p. 115f). We begin with Bohr's key postulates:

1. $L = \frac{nh}{2\pi} = n\hbar$[14]
2. $E_i - E_f = h\nu_{ab}$, or $\nu = \frac{E_i - E_f}{h}$

1 picks out the quantized orbits of the electron from the continuum of circular orbits allowed by classical mechanics, while 2 applies the Planck/Einstein relation between the energy and frequency of a quantum of light to the light emitted in a transition between stationary states. The most radical element in Bohr's approach was his simple postulation that these states are stable. In the end, a new electrodynamics would be required to provide a satisfactory account of these states, but Bohr's approach was simply to set that problem aside and attempt to characterize the states and the consequences of transitions between them as best he could without a new electrodynamics.

The next step applies the quantization rule, 1, to the classical models of the stationary states:

3. $\frac{Ze^2}{r^2} = \frac{mv^2}{r}$
4. $L = mvr = \frac{nh}{2\pi} = n\hbar$[15]

Combining 3 and 4 leads to helpful results regarding the radius and velocity of the orbiting electron in the stationary states:

5. $r = \frac{n^2\hbar^2}{mZe^2}, n = 1, 2, 3...$
6. $v = \frac{n\hbar mZe^2}{mn^2h^2} = \frac{Ze^2}{n\hbar}, n = 1, 2, 3...$

Given experimental values for m, e and $\hbar$ 5 predicts $r_1 = 5.3x10^{-9}cm$, compatible with evidence suggesting the order of magnitude of atomic radii was about $10^{-8}cm$. Further, v_1 (the highest velocity for an orbiting electron) is $2.2x10^8 cm/sec$, which is less than 1% of c, indicating that special relativity need not be brought into the calculations.

Next, the total energy of the stationary states is obtained and applied to give the key results, Bohr's calculation of the Rydberg constant and the general law of the hydrogen spectrum.

7. $V = \int_{\infty}^{r} \frac{Ze^2}{r^2} dr = -\frac{Ze^2}{2r}$
8. $T = \frac{1}{2}mv^2 = \frac{Ze^2}{2r}$, giving
9. $E = \frac{Ze^2}{2r} = -T$

Where V is the potential energy of the electron (with the 0 of potential energy defined as the state in which the electron is at rest, infinitely far from the nucleus), T is the kinetic energy and E the total energy.[16]

Applying 2 to the energy levels given by 9 allows us to derive Bohr's formula for the hydrogen spectrum and to determine the Rydberg constant for a nuclear charge Z:

10. $\nu = \frac{mZ^2e^4}{4\pi\hbar^3}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$

Re-writing 10 to follow Rydberg's formula for the hydrogen spectrum, written in terms of wave number, $k = 1/\lambda = \nu/c$, we get an expression for the Rydberg constant for hydrogen:

11. $R_H = \frac{me^4}{4\pi c\hbar^3}$

Up to this point the principles of CED have played no role in the derivation. From the $C\&P$ point of view, the entire argument has been conducted within σ_Q. However, identifying this value for the Rydberg constant with the quantity known empirically from spectral observations requires a link to CED, in which frequencies, wavelengths and spacings of spectroscopic gratings interact in ways that, at the time, only CED captured. So the *application* of this theory to data on the hydrogen spectrum implicitly invokes the principles contained in σ_C.[17]

Relying on CED in this way does not require inconsistent beliefs: some, including Bohr, hoped that quantum constraints did not apply to the radiation field, but only to its interaction with matter. Other physicists accepted that CED would need to be replaced by a quantum theory of electrodynamics. But at the time no other theory provided an account of the interactions between light and instruments that underlie the observational practice of spectroscopy: for the purpose of *reasoning* about spectra, CED was indispensable.

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[14] Bohr's original expression for the quantization condition was given in terms of the electron's kinetic energy, $W = \tau h \frac{\omega}{2}$, where ω is the angular frequency of the electron's orbit and τ is the quantum number Bohr (1913a, p. 5). Since $W = \pi\omega L$ this is equivalent to our 1.

[15] Note that in Bohr (1913a) Bohr omits the Z in 3., using instead e for the charge of the electron and E for the charge of the nucleus, a for the radius of the orbit and τ for the quantum number. Most contemporary texts use r for the radius and n for the quantum number; here we follow contemporary usage.

[16] Note that Bohr used 'W' for energy.

[17] While the experimental data of the time did not provide precise values for e/m and h, Bohr applied the available values to 11, obtaining a value of 3.1×10^{15}, which compared well to the spectrally determined value of 3.29×10^{15} (Bohr 1913a, p. 9). Programmatically, Bohr's proposal provided a general account of the relation between quantized energy levels of atomic systems and frequencies of their spectral emissions, drawing a direct link between physical models of atoms and spectroscopic observations for the first time. A further significant empirical advance was Bohr's explanation for the absence of certain hydrogen lines in terrestrial observations, as due to the high ambient pressure at the earth's surface and the size of electron orbits at higher energy levels: "According to the theory the necessary condition for the appearance of a great number of lines is therefore a very small density of the gas" (Bohr 1913a, p. 9).

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