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External Curries

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Abstract Curry's paradox is well known. The original version employed a conditional connective, and is not forthcoming if the conditional does not satisfy contraction. A newer version uses a validity predicate, instead of a conditional, and is not forthcoming if validity does not satisfy structural contraction. But there is a variation of the paradox which uses "external validity" (that is, essentially, preservation of theoremhood). And since external validity contracts, one might expect the appropriate version of the Curry paradox to be inescapable. In this paper we show that this is not the case. We consider two ways of formalising the notion of external validity, and show that in both of these the paradox is not forthcoming without the appropriate forms of contraction.

Keywords Curry paradox · Internal validity · External validity · Internal consequence relations · External consequence relations · Contraction · Higher-level sequent calculi

1 Introduction

The standard Curry's paradox is well known, cf. [2, 6]. It involves a sentence, σ, of the form $T(\overline{\sigma}) \rightarrow A$, for an arbitrary A. (We use 'T' s a truth predicate, and

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overlining as a name-forming device.) Assuming the T-schema, the naively correct principle about truth, one infers the arbitrary A. Call this the 'c-Curry'. Those who are wont to endorse the T-schema usually fault the paradox-argument by rejecting a principle of inference concerning the conditional, Contraction: $A \to (A \to B) \vdash A \to B$.[1]

Less well known is a version of the paradox concerning, not truth, but validity. Call this the 'v-Curry'. This involves a sentence, π, of the form $Val(\overline{\pi}, \overline{A})$. Here '$Val(x, y)$' is a validity predicate. Assuming the naively correct principle about validity, namely that $A \vdash B$ iff $\vdash Val(\overline{A}, \overline{B})$ one infers the arbitrary A. Those who are wont to endorse this natural principle about validity usually fault the paradoxical argument, by analogy with the truth-Curry, by rejecting the structural principle of Contraction, that is, in sequent calculus form:

$$X, A, A \Rightarrow B \quad \vdash \quad X, A \Rightarrow B$$

(where $\Rightarrow$ is the sequent operator). (See, e.g., [10].)

The notion of validity here, 'Val', is what one may call *internal*, recording the fact that the conclusion follows from the *assumption* of the premises, given the rules in operation. But there is a different, and perfectly legitimate, notion of validity, which we may call *external*. An inference is valid in this sense if the logical validity (theoremhood) of the conclusion follows from that of the premises, given the rules in operation. That is, given that $\vdash A$ and the rules in play, we can infer that $\vdash B$. This differentiation parallels the distinction between the internal and the external consequence relation defined over a given sequent calculus, see [1]: B is an *internal* consequence of $A_1, \ldots, A_n$ iff the sequent $A_1, \ldots, A_n \Rightarrow B$ is provable in the system; B is an *external* consequence of $A_1, \ldots, A_n$ iff the sequent $\Rightarrow B$ is provable in the sequent system that results from the given one by adding $\Rightarrow A_1, \ldots, \Rightarrow A_n$ as axioms (and cut as a primitive rule).[2] Whereas for internal consequence the structural rule of contraction may or may not be assumed, external validity appears to contract: theorems may be reused an arbitrary number of times, even if assumptions cannot. It might well be thought, then, that the validity Curry paradox will reappear for this notion of validity.

Is this, in fact, the case? What makes the issue hard is that to analyse the supposed paradox one needs to be very clear about how to formulate and reason about validity. In this paper we will give two natural approaches, and show that in each of them the Curry paradox for external validity *fails to be forthcoming*, if contraction, in the appropriate sense, is not assumed.

We will first recall the connective Curry paradox (Section 2) and then consider the v-Curry paradox as presented in natural deduction style by Beall and Murzi [3] and in sequent style by Mares and Paoli [9] (Section 3). We will then proceed to the two approaches to regimenting the notion of validity. In Section 4 we will present von Kutschera's higher-level sequent system $\overline{M}_\infty$. The next two sections then use this to

[1]Or, assuming that $\to$ detaches and satisfies the Deduction Theorem, the contraction axiom schema (Law of Absorption): $(A \to (A \to B)) \to (A \to B)$.

[2]External validity is sometimes called 'admissibility'. See, e.g., [4].

analyse both the internal and the external v-Curries. In the next Section 7, we turn to another way in which the external v-Curry may be formalised. This simply nests the Val predicate. We conclude the paper by drawing some of its threads together.

2 The Connective Curry Paradox

Let us start with the c-Curry paradox. An often emphasized remarkable feature of the c-Curry paradox is that it is *negation-free*; the only connective involved is implication. We assume a formal language $\mathcal{L}$ comprising a truth predicate T and suppose that Tarski's T-schema

$$A \leftrightarrow T(\overline{A})$$

holds. Moreover, we consider a self-referential $\mathcal{L}$-sentence σ such that[3]

$$\sigma \leftrightarrow (T(\overline{\sigma}) \rightarrow A)$$

holds and assume the principle of Contraction. The paradox is derived as follows:

1.	$T(\overline{\sigma}) \leftrightarrow (T(\overline{\sigma}) \rightarrow A)$	[the T-schema, $\sigma \leftrightarrow (T(\overline{\sigma}) \rightarrow A)$]
2.	$\sigma \leftrightarrow (\sigma \rightarrow A)$	[1., $\sigma \leftrightarrow T(\overline{\sigma})$, replacement]
3.	$\sigma \rightarrow A$	[principle of Contraction, 2. (left to right)]
4.	σ	[2. (right to left), 3., modus ponens]
5.	A	[3., 4., modus ponens]

The T-schema, self-reference, and contraction therefore lead to triviality. A simple solution is to jettison the principle of Contraction, in sequent form: $\Rightarrow ((A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B))$. If $\mathcal{L}$ contains an implication satisfying the standard left and right introduction rules of a sequent calculus in which cut is admissible, then, as is easy to check, this and the structural rule of contraction, $X, A, A \Rightarrow B \vdash X, A \Rightarrow B$ (where X is a finite, possibly empty multiset of formulas and X, A denotes the multiset $X \cup \{A\}$) are interderivable.

This observation brings us to the v-Curry paradox.

3 The v-Curry Paradox

Beall and Murzi's v-Curry Paradox [3] is meant to lift the c-Curry paradox from the level of connectives and axioms to the level of derivability statements. Instead of Tarski's disquotational T-schema, a disquotational "validity schema" (VS)

$$Val(\overline{A}, \overline{B}) \text{ iff } A \vdash B$$

is assumed[4] together with a self-referential sentence π satisfying

$$\pi \leftrightarrow Val(\overline{\pi}, \overline{A}).$$

[3] Or even $\sigma \equiv (T(\overline{\sigma}) \rightarrow A)$, where '$\equiv$' stands for syntactic identity.

[4] Beall and Murzi require

$$\vdash Val(\overline{A}, \overline{B}) \text{ iff } A \vdash B$$

because they assume that validity claims are themselves valid if true.

Intuitively, π says something like: The (arbitrary) sentence A is derivable from me. Note that whereas the T-schema in the c-Curry paradox is a first-order object-language formula and the derivation of the paradox is a derivation making use of an axiomatic proof system for that language, the above validity schema is a meta-language statement.

Beall and Murzi [3] then reason as follows in natural deduction style:[5]

1.	$\pi \leftrightarrow Val(\overline{\pi}, \overline{A})$	[self-referentiality]
2.	π	[assumption]
3.	$Val(\overline{\pi}, \overline{A})$	[1. (left to right), 2., modus ponens]
4.	A	[2., 3., (VS)]
5.	$Val(\overline{\pi}, \overline{A})$	[2. – 4., (VS), discharging 2.]
6.	π	[1. (right to left), 5., modus ponens]
7.	A	[5., 6., (VS)]

The assumption π is used twice, namely to detach $Val(\overline{\pi}, \overline{A})$ from $\pi \leftrightarrow Val(\overline{\pi}, \overline{A})$ and to derive A from $Val(\overline{\pi}, \overline{A})$. *Both* occurrences of π are then cancelled in the discharge step.

In order to make this contraction step clearly visible, to keep the v-Curry paradox free from an object-language implication, and to present it as a derivation within a proof system, the validity schema may be reformulated as a pair of natural deduction rules, namely rules of Val-introduction and Val-elimination:

$$\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{Val(\overline{A}, \overline{B})}\ (Val\text{–}I) \qquad \frac{\begin{array}{cc}\Delta & \Gamma \\ \vdots & \vdots \\ Val(\overline{A}, \overline{B}) & A\end{array}}{B}\ (Val\text{–}E)$$

and π may be assumed to satisfy the following π-introduction and π-elimination rules:

$$\frac{Val(\overline{\pi}, \overline{A})}{\pi}\ (\pi - I) \qquad \frac{\pi}{Val(\overline{\pi}, \overline{A})}\ (\pi\text{–}E)$$

The notation $[A]$ here indicates that occurrences of A as an assumption in the derivation of B must be cancelled in the derivation of $Val(\overline{A}, \overline{B})$; the rule $(Val - I)$ has no undischarged assumptions. The natural deduction rules for the binary *connective* $Val(\overline{\cdot}, \overline{\cdot})$ resemble the natural deduction rules for positive intuitionistic implication, and according to Mares and Paoli [9], $Val(\overline{\cdot}, \overline{\cdot})$ is as a kind of implication. Contraction is built into $(Val\text{–}I)$ because in deriving $Val(\overline{A}, \overline{B})$, more than one occurrence of the assumption A may be used to derive B, and hence it may come as no surprise that the v-Curry paradox arises.

[5] We slightly change the notation. In particular, Beall and Murzi use an absurdity constant $\perp$ instead of an arbitrary formula A.

Using the introduction and elimination rules for $Val(\overline{\cdot},\overline{\cdot})$ and π, the above derivation can now be represented in tree-format as follows:

$$\dfrac{\dfrac{\dfrac{[\pi]^2 \qquad \dfrac{[\pi]^2}{Val(\overline{\pi},\overline{A})}\,(\pi\text{–}E)}{A}\,(Val\text{–}E)}{Val(\overline{\pi},\overline{A})}\,(Val\text{–}I)^2 \qquad \dfrac{\dfrac{\dfrac{[\pi]^1 \qquad \dfrac{[\pi]^1}{Val(\overline{\pi},\overline{A})}\,(\pi\text{–}E)}{A}\,(Val\text{–}E)}{Val(\overline{\pi},\overline{A})}\,(Val\text{–}I)^1}{\pi}\,(\pi\text{–}I)}{A}\,(Val\text{–}E)$$

where the superscripts indicate contracting cancellation steps.[6]

In order to make the use of structural contraction clearly visible, Mares and Paoli [9] restate Beall and Murzi's argument in sequent style:[7]

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\pi \Rightarrow \pi \qquad \Rightarrow \pi \rightarrow Val(\overline{\pi},\overline{A})}{\pi \Rightarrow Val(\overline{\pi},\overline{A})} \qquad \pi \Rightarrow \pi}{\pi,\pi \Rightarrow A}}{\pi \Rightarrow A}\,\text{contraction}}{\Rightarrow Val(\overline{\pi},\overline{A})} \qquad \Rightarrow Val(\overline{\pi},\overline{A}) \rightarrow \pi}{\Rightarrow \pi} \qquad \Rightarrow Val(\overline{\pi},\overline{A})}{\Rightarrow A}$$

The subderivations

$$\dfrac{\pi \Rightarrow \pi \qquad \Rightarrow \pi \rightarrow Val(\overline{\pi},\overline{A})}{\pi \Rightarrow Val(\overline{\pi},\overline{A})} \qquad \dfrac{\Rightarrow Val(\overline{\pi},\overline{A}) \qquad \Rightarrow Val(\overline{\pi},\overline{A}) \rightarrow \pi}{\Rightarrow \pi}$$

compress some derivation steps, namely

$$\dfrac{\Rightarrow \pi \rightarrow Val(\overline{\pi},\overline{A}) \qquad \dfrac{\pi \Rightarrow \pi \qquad Val(\overline{\pi},\overline{A}) \Rightarrow Val(\overline{\pi},\overline{A})}{\pi, \pi \rightarrow Val(\overline{\pi},\overline{A}) \Rightarrow Val(\overline{\pi},\overline{A})}}{\pi \Rightarrow Val(\overline{\pi},\overline{A})}$$

and

$$\dfrac{\Rightarrow Val(\overline{\pi},\overline{A}) \qquad \dfrac{\Rightarrow Val(\overline{\pi},\overline{A}) \rightarrow \pi \qquad \dfrac{Val(\overline{\pi},\overline{A}) \Rightarrow Val(\overline{\pi},\overline{A}) \qquad \pi \Rightarrow \pi}{Val(\overline{\pi},\overline{A}), (Val(\overline{\pi},\overline{A}) \rightarrow \pi) \Rightarrow \pi}}{Val(\overline{\pi},\overline{A}) \Rightarrow \pi}}{\Rightarrow \pi}$$

More significantly, however, the subderivations

$$\dfrac{\pi \Rightarrow Val(\overline{\pi},\overline{A}) \qquad \pi \Rightarrow \pi}{\pi,\pi \Rightarrow A} \qquad \dfrac{\pi \Rightarrow A}{\Rightarrow Val(\overline{\pi},\overline{A})} \qquad \dfrac{\Rightarrow \pi \qquad \Rightarrow Val(\overline{\pi},\overline{A})}{\Rightarrow A}$$

are not derivations within a given proof system but incorporate meta-level reasoning, unless suitable sequent rules for $Val(\overline{A},\overline{B})$ and π are added. If cut is admissible,

[6] According to Roy Cook [5], the assumption of self-referentiality makes the v-Curry reasoning dependent on some extra-logical theory such as Peano Arithmetic, so that there is no paradox of strictly logical validity. In the present case the reasoning depends on the $(\pi\text{–}I)$ and $(\pi\text{–}E)$ rules.

[7] We use the sequent arrow $\Rightarrow$ instead of the turnstile $\vdash$.

then instead of right and left introduction rules the following so-called double-line rules may be used, rules that may be applied bottom-up and top-down (cf. [7]):

$$\frac{A \Rightarrow B}{\Rightarrow Val(\overline{A}, \overline{B})} \ (Val \updownarrow) \qquad \frac{\Delta \Rightarrow \pi}{\Delta \Rightarrow Val(\overline{\pi}, \overline{A})} \ (\pi \updownarrow)$$

We obtain:

$$\frac{\pi \Rightarrow Val(\overline{\pi}, \overline{A})}{\pi, \pi \Rightarrow A} \ (Val \uparrow) \qquad \frac{\pi \Rightarrow A}{\Rightarrow Val(\overline{\pi}, \overline{A})} \ (Val \downarrow) \qquad \frac{\Rightarrow Val(\overline{\pi}, \overline{A})}{\Rightarrow A} \ (\pi \uparrow)$$

Again $Val(\overline{\cdot}, \overline{\cdot})$ is similar to positive intuitionistic implication if the structural rules of contraction, monotonicity and permutation are assumed.

However, and most importantly, the argument, in whatever terms couched, uses the appropriate form of structural contraction, and falls if this is jettisoned.

Note, though, that If we add $\Rightarrow \pi$ as an axiom, one may easily observe that A is an external consequence of π, even if the underlying sequent calculus is contraction-free:

$$\dfrac{\Rightarrow \pi \qquad \dfrac{\Rightarrow \pi \qquad \dfrac{\dfrac{\pi \Rightarrow \pi}{\pi \Rightarrow Val(\overline{\pi}, \overline{A})} \ (\pi \downarrow)}{\pi, \pi \Rightarrow A} \ (Val \uparrow)}{\pi \Rightarrow A} \ (cut)}{\Rightarrow A} \ (cut)$$

Does this fact show that the external version of the v-Curry is forthcoming?

4 Von Kutschera's Higher-Level Sequent Calculus $\overline{M}_\infty$

To investigate, we need a way of reasoning about higher-order consequence. We now explore the first way of doing this. This is a higher-order sequent calculus $\overline{M}_\infty$. It is defined for a given logical object-language $\mathcal{L}$, and the generalized, higher-level sequents are called *S-formulas* over $\mathcal{L}$. In presenting $\overline{M}_\infty$, we will closely follow von Kutschera's exposition for the sake of comparison with [8].

Definition 1 The set of S-formulas over $\mathcal{L}$ is the smallest set S-Form satisfying the following conditions:

1. Every formula of $\mathcal{L}$ belongs to S-Form.
2. $(\Rightarrow) \in S$-Form.
3. If $S_1, \ldots, S_n, T \in S$-Form, then $(\Rightarrow T) \in S$-Form, $(S_1, \ldots, S_n \Rightarrow T) \in S$-Form, and $(S_1, \ldots, S_n \Rightarrow) \in S$-Form.

The S-formulas $S_1, \ldots, S_n$, and T are said to be the antecedent formulas, respectively the succedent formula of the above S-formulas.

We use (i) the letters 'T', 'U', 'V', 'W', 'W'' to denote S-formulas over $\mathcal{L}$ and (ii) 'Γ', 'Δ', 'Π', 'Θ', 'Θ'' to denote finite, possibly empty sequences of S-formulas. Ω is always a sequence of S-formulas containing at most one S-formula. If $\Gamma \equiv T_1 \ldots T_n$, then T_i $(1 \leq i \leq n)$ is called a constituent of Γ. We sometimes omit outermost brackets in S-formulas and write $\Delta \Rightarrow \Gamma$ as an abbreviation of $\Delta \Rightarrow T_1$, $\Delta \Rightarrow T_2$, ..., $\Delta \Rightarrow T_n$ for $\Gamma \equiv T_1 \ldots T_n$.

Definition 2 Formulas from $\mathcal{L}$ are S-formulas of S-degree 0, and $(\Rightarrow)$ is an S-formula of S-degree 1. If n is the maximum of the S-degrees of the S-formulas from Δ, Ω, then the S-degree of $\Delta \Rightarrow \Omega$ is $n+1$. S-formulas of degree 1 are called (ordinary) sequents.

Every S-formula is an S-subformula of itself and the S-subformulas of Δ and Ω are S-subformulas of $\Delta \Rightarrow \Omega$. An S-formula T is said to be *positive* iff T does not contain any S-subformula that has no succedent formula. In what follows, we will be interested in positive S-formulas only, in particular the S-formula $(\Rightarrow)$ will play no role.

Definition 3 The higher-level sequent calculus $\overline{M}_\infty$ consists of the following axioms and rules restricted to positive S-formulas:

RF	$T \Rightarrow T$	(reflexivity)
TR′	$\Delta \Rightarrow T;\ \Gamma, T \Rightarrow U \vdash \Gamma, \Delta \Rightarrow U$	(cut)[8]
VV	$\Delta \Rightarrow T \vdash \Delta, U \Rightarrow T$	(monotonicity)
SK	$\Delta, T, T \Rightarrow U \vdash \Delta, T \Rightarrow U$	(contraction)
ST	$\Delta, T, U, \Gamma \Rightarrow V \vdash \Delta, U, T, \Gamma \Rightarrow V$	(permutation)
PB	$\Delta, \Gamma \Rightarrow T \vdash \Delta \Rightarrow (\Gamma \Rightarrow T)$	(premise removal)
PE′	$\Delta \Rightarrow (\Gamma \Rightarrow T) \vdash \Delta, \Gamma \Rightarrow T$	(premise introduction)

[8] Von Kutschera [8] uses another version of (cut), namely

$$TR \quad \Delta \Rightarrow T;\ \Delta, T \Rightarrow U \vdash \Delta \Rightarrow U,$$

which on the one hand builds in a contraction to a single occurrence of Δ and on the other hand is less general than TR′. TR is interderivable with TR′ in the presence of VV, SK and ST. Since we will consider dropping VV or SK, we will use TR′ instead of TR.

In addition to PE′, von Kutschera also uses another premise introduction rule and says that it is equivalent with PE′, namely

$$PE \quad \Delta \Rightarrow \Gamma;\ \Delta, T \Rightarrow U \vdash \Delta, (\Gamma \Rightarrow T) \Rightarrow U$$

where, due to the restriction to positive S-formulas, the sequence Γ must be non-empty. The rule PE also builds in a contraction to a single occurrence of Δ. Clearly, PE can be derived from PE′:

$$\dfrac{\dfrac{\dfrac{\Delta \Rightarrow \Gamma \quad \dfrac{(\Gamma \Rightarrow T) \Rightarrow (\Gamma \Rightarrow T)}{(\Gamma \Rightarrow T), \Gamma \Rightarrow T}\,PE'}{(\Gamma \Rightarrow T), \Delta \Rightarrow T}\,* \qquad \Delta, T \Rightarrow U}{(\Gamma \Rightarrow T), \Delta \Rightarrow U}\,*}{\Delta, (\Gamma \Rightarrow T) \Rightarrow U}\,ST$$

where $*$ indicates (possibly repeated) uses of TR′, SK and ST. The converse, however, is problematic, because von Kutschera derives $(\Rightarrow T) \Rightarrow T$ from $(\Rightarrow T) \Rightarrow (\Rightarrow T)$ by means of PE′ and thus allows Γ in PE′ to be empty. The obvious route to a derivation of PE′ from PE is blocked for empty Γ:

$$\dfrac{\Delta \Rightarrow (\Rightarrow T) \quad \dfrac{? \quad T \Rightarrow T}{(\Rightarrow T) \Rightarrow T}\,PE}{\Delta \Rightarrow T}\,TR'$$

We will therefore use PE′ instead of PE.

We here restate two important theorems from [8] together with their proofs.[9] These observations will be relevant for our suggested formalization of the external consequence Curry paradox.

Theorem 1 *(Internalization) If* $\Delta \vdash T$ *in* $\overline{M}_\infty$*, then* $\vdash \Delta \Rightarrow T$ *in* $\overline{M}_\infty$*.*

Proof The proof is by induction on the length l of derivations of T from Δ. If $l = 1$, then (i) T is an axiomatic S-formula or (ii) T belongs to Δ. In case (i), an application of PB gives $\Rightarrow T$ and by using VV, one obtains $\Delta \Rightarrow T$. In case (ii), we apply VV and ST to $T \Rightarrow T$. We now suppose that the claim holds for derivations of length at most n and consider $l = n + 1$. If T is an axiom or T belongs to Δ, the argument is as in the previous case. For the remaining cases we assume that the final step in the derivation of T is an application of one of the rules of $\overline{M}_\infty$.

TR′: $T \equiv \Pi, \Gamma \Rightarrow U$ and by the induction hypothesis, $\Delta \Rightarrow (\Gamma \Rightarrow V)$ and $\Delta \Rightarrow (\Pi, V \Rightarrow U)$ are provable in $\overline{M}_\infty$. We then obtain

$$\dfrac{\dfrac{\dfrac{\dfrac{\Delta \Rightarrow (\Gamma \Rightarrow V)}{\Delta, \Gamma \Rightarrow V}\,PE' \quad \dfrac{\Delta \Rightarrow (\Pi, V \Rightarrow U)}{\Delta, \Pi, V \Rightarrow U}\,PE'}{\Delta, \Pi, \Delta, \Gamma \Rightarrow U}\,TR'}{\Delta, \Pi, \Gamma \Rightarrow U}\,*}{\Delta \Rightarrow (\Pi, \Gamma \Rightarrow U)}\,PB$$

where $*$ indicates possibly repeated applications of SK and ST.

VV: $T \equiv \Gamma, U \Rightarrow V$ and by the induction hypothesis, $\Delta \Rightarrow (\Gamma \Rightarrow V)$ is provable in $\overline{M}_\infty$. Then

$$\dfrac{\dfrac{\dfrac{\Delta \Rightarrow (\Gamma \Rightarrow V)}{\Delta, \Gamma \Rightarrow V}\,PE'}{\Delta, \Gamma, U \Rightarrow V}\,VV}{\Delta \Rightarrow (\Gamma, U \Rightarrow V)}\,PB$$

ST, SK: Similar to the previous case.

PB: $T \equiv \Gamma \Rightarrow (\Pi \Rightarrow U)$ and by the induction hypothesis, $\Delta \Rightarrow (\Gamma, \Pi \Rightarrow U)$ is provable in $\overline{M}_\infty$. Then

$$\dfrac{\dfrac{\dfrac{\Delta \Rightarrow (\Gamma, \Pi \Rightarrow U)}{\Delta, \Gamma, \Pi \Rightarrow U}\,PE'}{\Delta, \Gamma \Rightarrow (\Pi \Rightarrow U)}\,PB}{\Delta \Rightarrow (\Gamma \Rightarrow (\Pi \Rightarrow U))}\,PB$$

PE′: Inverse to the previous case. □

Let now V_T be an S-formula that contains a certain occurrence of T as an S-subformula and let V_U be the result of replacing this occurrence of T in V_T by U.

[9] Note that in the TR′-step of Theorem 1 we use SK, whereas the use of TR can be handled without any appeal to SK.

Theorem 2 *(Replacement) In $\overline{M}_\infty$ it holds that $T \Leftrightarrow U \vdash V_T \Leftrightarrow V_U$.*

Proof Let g be the S-degree of V_T minus the S-degree of T. The proof is by induction on g. If $g = 0$, then $V_T \equiv T$ and the claim is trivial. We now suppose that the claim holds for $g \leq n$ and consider $g = n + 1$. Then V_T has the shape $\Delta, W_T \Rightarrow W'$ or $\Delta \Rightarrow W_T$. By the induction hypothesis, we have $T \Leftrightarrow U \vdash W_T \Leftrightarrow W_U$. Using TR$'$, we obtain

$$\dfrac{\dfrac{T \Leftrightarrow U}{W_T \Leftrightarrow W_U}\,ind.h. \qquad \Delta, W_T \Rightarrow W'}{\Delta, W_U \Rightarrow W'}\,TR' \qquad \dfrac{\dfrac{T \Leftrightarrow U}{W_T \Leftrightarrow W_U}\,ind.h. \qquad \Delta, W_U \Rightarrow W'}{\Delta, W_T \Rightarrow W'}\,TR'$$

$$\dfrac{\dfrac{T \Leftrightarrow U}{W_T \Leftrightarrow W_U}\,ind.h. \qquad \Delta \Rightarrow W_T}{\Delta \Rightarrow W_U}\,TR' \qquad \dfrac{\dfrac{T \Leftrightarrow U}{W_T \Leftrightarrow W_U}\,ind.h. \qquad \Delta \Rightarrow W_U}{\Delta \Rightarrow W_T}\,TR'$$

Thus, $T \Leftrightarrow U, V_T \vdash V_U$ and $T \Leftrightarrow U, V_U \vdash V_T$. By Theorem 1, $\vdash T \Leftrightarrow U, V_T \Rightarrow V_U$ and $\vdash T \Leftrightarrow U, V_U \Rightarrow V_T$; therefore $T \Leftrightarrow U \vdash V_T \Leftrightarrow V_U$. □

As von Kutschera points out, since by PB and PE $'$, $(\Rightarrow T) \Leftrightarrow T$ is provable, the above replacement theorem reveals that $\overline{M}_\infty$ is a system of positive intuitionistic implicational logic.

Note that internal and external consequence defined with respect to $\overline{M}_\infty$ coincide. If we have $S_1, \ldots, S_n \Rightarrow T$ and assume $\Rightarrow S_1, \ldots, \Rightarrow S_n$, then by applying TR$'$ n times, we obtain $\Rightarrow T$. For the converse direction, note that by applying PB to the axiomatic $T \Rightarrow T$, we get $T \Rightarrow (\Rightarrow T)$ and by applying PE$'$ to $(\Rightarrow T) \Rightarrow (\Rightarrow T)$ we obtain $(\Rightarrow T) \Rightarrow T$. Theorem 2 tells us that replacing an S-formula $\Rightarrow T$ by T (or vice versa) within S-formulas results in mutually interderivable S-formulas. If now in $\overline{M}_\infty$ we have $(\Rightarrow S_1), \ldots, (\Rightarrow S_n) \vdash (\Rightarrow T)$, then by Theorem 1, $\vdash (\Rightarrow S_1), \ldots, (\Rightarrow S_n) \Rightarrow (\Rightarrow T)$. From the observed substitutability it follows that $\vdash S_1, \ldots, S_n \Rightarrow T$.

5 The Internal Consequence Curry Paradox Reformulated

Let us now look at the internal v-Curry given this machinery. As we just observed, internal and external consequence defined with respect to $\overline{M}_\infty$ coincide. A sequent calculus with respect to which internal and external consequence come apart can be obtained by dropping either monotonicity or contraction (or both). In order to deal with a framework in which the two definitions of consequence give rise to different relations, we first consider the result of removing the monotonicity rule VV from $\overline{M}_\infty$ and refer to this sequent system as $\overline{R}_\infty$.

The validity predicate of Beall and Murzi can be reformulated as a disquotational internal derivability predicate $Der(\cdot, \cdot)$ satisfying the following schema:

$$Der(\overline{A}, \overline{B}) \Leftrightarrow (A \Rightarrow B).$$

We add the four axiomatic S-formulas $Der(\overline{A}, \overline{B}) \Leftrightarrow (A \Rightarrow B)$ and $\pi \Leftrightarrow Der(\overline{\pi}, \overline{A})$ to $\overline{R}_\infty$, refer to this system as $\overline{R}^{\bullet}_\infty$, and refer to the result of adding the four axioms to $\overline{M}_\infty$ as $\overline{M}^\bullet_\infty$. The sequent $\pi \Rightarrow A$ can be proved using the contraction rule SK:

$$\dfrac{\dfrac{\dfrac{\pi \Rightarrow Der(\overline{\pi},\overline{A}) \quad Der(\overline{\pi},\overline{A}) \Rightarrow (\pi \Rightarrow A)}{\pi \Rightarrow (\pi \Rightarrow A)}\,TR' \quad \dfrac{(\pi \Rightarrow A) \Rightarrow (\pi \Rightarrow A)}{(\pi \Rightarrow A), \pi \Rightarrow A}\,PE'}{\pi, \pi \Rightarrow A}\,TR'}{\pi \Rightarrow A}\,SK$$

Let d be the above derivation. We then proceed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{d}{\Rightarrow (\pi \Rightarrow A)}\,PB \quad (\pi \Rightarrow A) \Rightarrow Der(\overline{\pi},\overline{A})}{\Rightarrow Der(\overline{\pi},\overline{A})}\,TR' \quad Der(\overline{\pi},\overline{A}) \Rightarrow \pi}{(\Rightarrow \pi)}\,TR' \quad d}{\Rightarrow A}\,TR'$$

Obviously, this trivialization proof is blocked if SK is given up and derivation d is no longer available.

The formalization of Beall and Murzi's v-Curry paradox using (Val–I) and (Val–E) in natural deduction, the sequent rules ($Val\ \Updownarrow$) in an ordinary sequent calculus, and the axiomatic sequents $Der(\overline{A}, \overline{B}) \Leftrightarrow (A \Rightarrow B)$ in $\overline{R}^\bullet_\infty$ result in an *internal consequence* (or *internal validity*) Curry paradox. We now come back to the problem of distinguishing between an internal and an external consequence version of the v-Curry paradox. Since $\overline{M}_\infty$ and its substructural subsystems allow a nesting of the sequent arrow, it seems quite natural to consider a disquotational and supposedly external derivability predicate $Prov(\cdot, \cdot)$ satisfying $Prov(\overline{A}, \overline{B}) \Leftrightarrow ((\Rightarrow A) \Rightarrow (\Rightarrow B))$.

External consequence satisfies contraction and therefore one might expect that there exists an external consequence Curry paradox resulting from a disquotational external consequence predicate used in combination not only with monotonicity-free subsystems of $\overline{M}_\infty$, but also in combination with *contraction-free* subsystems of $\overline{M}_\infty$.

However, the attempt to obtain an external consequence Curry paradox by using $Prov(\cdot, \cdot)$ in combination with monotonicity-free or contraction-free subsystems of $\overline{M}_\infty$, seems to fail. If we assume a self-referential sentence ρ satisfying $\rho \Leftrightarrow Prov(\overline{\rho}, \overline{A})$ and try to obtain a Curry argument for $Prov(\cdot, \cdot)$ and ρ, we need a provable sequent $(\Rightarrow \rho) \Rightarrow (\Rightarrow A)$, for arbitrary A. As we will see, for this purpose we need a derivability relation between sequences of sequents and single sequents that contracts together with an analogue of the internalization result for $\overline{M}_\infty$.

We can define contraction-free derivability relations between sequences of sequents and single sequents for contraction-free subsystems of $\overline{M}_\infty$, and for these relations we obtain internalization. Let us first consider the contraction-free "linear" version $\overline{L}_\infty$ of $\overline{R}_\infty$, i.e., the result of dropping SK from $\overline{R}_\infty$, and let $\overline{L}^\star_\infty$ be the result of adding the axioms $Prov(\overline{A}, \overline{B}) \Leftrightarrow ((\Rightarrow A) \Rightarrow (\Rightarrow B))$ and $\rho \Leftrightarrow Prov(\overline{\rho}, \overline{A})$ to $\overline{L}_\infty$.

Let a certain higher-level sequent system be given. We write $\Delta \vdash_l T$ if in that system there exists a derivation of T from Δ in which each constituent of the sequence

Δ is used *exactly once* as a leaf in the derivation tree. Such a derivation is called a linear derivation of T from Δ in the system.

Theorem 3 *If $\Delta \vdash_l T$ in $\overline{L}^{\star}_{\infty}$, then $\vdash \Delta \Rightarrow T$ in $\overline{L}^{\star}_{\infty}$.*

Proof The proof is by induction on the length l of linear derivations of T from Δ. If $l = 1$, then T is an axiomatic S-formula and Δ is the empty sequence. Since T is a possibly higher-level sequent, $\Rightarrow T$ follows with PB. We now suppose that the claim holds for derivations of length at most n and consider $l = n + 1$. We may assume that the final step in the derivation of T is an application of one of the rules of $\overline{L}^{\star}_{\infty}$.

The cases of PB, PE′, and ST are as in the proof of Theorem 1.

The only remaining case is that of TR′. $T \equiv \Pi, \Gamma \Rightarrow U$. Since each constituent of the sequence Δ is used exactly once as a premise, we may represent Δ as a sequence Θ, Θ' and assume by the induction hypothesis that $\Theta \Rightarrow (\Gamma \Rightarrow V)$ and $\Theta' \Rightarrow (\Pi, V \Rightarrow U)$ are provable in $\overline{L}^{\star}_{\infty}$. We then obtain

$$\dfrac{\dfrac{\dfrac{\dfrac{\Theta \Rightarrow (\Gamma \Rightarrow V)}{\Theta, \Gamma \Rightarrow V}\,PE' \quad \dfrac{\Theta' \Rightarrow (\Pi, V \Rightarrow U)}{\Theta', \Pi, V \Rightarrow U}\,PE'}{\Theta, \Pi, \Theta', \Gamma \Rightarrow U}\,TR'}{\Delta, \Pi, \Gamma \Rightarrow U}\,*}{\Delta \Rightarrow (\Pi, \Gamma \Rightarrow U)}\,PB$$

where $*$ indicates possibly repeated applications of ST. □

Let $\overline{A}_{\infty}$ be the result of removing SK from $\overline{M}_{\infty}$ to obtain an "affine" linear logic and let $\overline{A}^{\star}_{\infty}$ be the result of adding $Prov(\overline{A}, \overline{B}) \Leftrightarrow ((\Rightarrow A) \Rightarrow (\Rightarrow B))$ and $\rho \Leftrightarrow Prov(\overline{\rho}, \overline{A})$ to $\overline{A}_{\infty}$. Let a certain higher-level sequent system be given. We write $\Delta \vdash_a T$ if in that system there exists a derivation of T from Δ in which each constituent of the sequence Δ is used *at most once* as a leaf in the derivation tree. Such a derivation is called an affine derivation of T from Δ in the system.

Theorem 4 *If $\Delta \vdash_a T$ in $\overline{A}^{\star}_{\infty}$, then $\vdash \Delta \Rightarrow T$ in $\overline{A}^{\star}_{\infty}$.*

Proof The proof is by induction on the length l of affine derivations of T from Δ. If $l = 1$, then (i) T is an axiomatic S-formula or (ii) T belongs to Δ. In case (i), an application of PB gives $\Rightarrow T$ and by using VV, one obtains $\Delta \Rightarrow T$. In case (ii), we apply VV and ST to $T \Rightarrow T$. We now suppose that the claim holds for derivations of length at most n and consider $l = n + 1$. If T is an axiom or T belongs to Δ, the argument is as in the previous case. For the remaining cases we assume that the final step in the derivation of T is an application of one of the rules of $\overline{A}^{\star}_{\infty}$.

The cases of PB, PE′, and ST are as in the proof of Theorem 1.

TR′: We may reason as in the TR′ case of Theorem 3. □

Let us consider also $\overline{R}^{\star}_{\infty}$, the result of adding $Prov(\overline{A}, \overline{B}) \Leftrightarrow ((\Rightarrow A) \Rightarrow (\Rightarrow B))$ and $\rho \Leftrightarrow Prov(\overline{\rho}, \overline{A})$ to $\overline{R}_{\infty}$. Let a certain higher-level sequent system be given. We write $\Delta \vdash_r T$ if in that system there exists a derivation of T from Δ in which each constituent of the sequence Δ is used *at least once* as a leaf in the derivation tree. Such a derivation is called a relevant derivation of T from Δ in the system.

Theorem 5 *If $\Delta \vdash_r T$ in $\overline{R}^{\star}_{\infty}$, then $\vdash \Delta \Rightarrow T$ in $\overline{R}^{\star}_{\infty}$.*

Proof The proof is by induction on the length l of relevant derivations of T from Δ. If $l = 1$, then T is an axiomatic S-formula and Δ is the empty sequence. Since T is a possibly higher-level sequent, $\Rightarrow T$ follows with PB. We suppose that the claim holds for derivations of length at most n and consider $l = n + 1$. We may assume that the final step in the derivation of T is an application of one of the rules of $\overline{R}^{\star}_{\infty}$. The cases of PB, PE′, ST *and* TR′ are as in the proof of Theorem 1. In particular, we apply SK in the case for TR′. □

Theorem 6 *(Replacement) In $\overline{L}^{\star}_{\infty}$ it holds that $T \Leftrightarrow U \vdash_l V_T \Leftrightarrow V_U$, in $\overline{R}^{\star}_{\infty}$ we have $T \Leftrightarrow U \vdash_r V_T \Leftrightarrow V_U$, and in $\overline{A}^{\star}_{\infty}$ it holds that $T \Leftrightarrow U \vdash_a V_T \Leftrightarrow V_U$.*

Proof Analogous to the proof of Theorem 2. □

If we want to re-do the Curry argument, we may in a first step derive $\Rightarrow A$ from $\Rightarrow \rho$, using the latter S-formula twice:

$$\dfrac{\dfrac{\dfrac{\Rightarrow \rho}{\Rightarrow (\Rightarrow \rho)}\,PB \qquad \dfrac{\dfrac{\Rightarrow \rho \qquad \dfrac{\rho \Rightarrow \mathit{Prov}(\overline{\rho}, \overline{A}) \quad \mathit{Prov}(\overline{\rho}, \overline{A}) \Rightarrow ((\Rightarrow \rho) \Rightarrow (\Rightarrow A))}{\rho \Rightarrow ((\Rightarrow \rho) \Rightarrow (\Rightarrow A))}\,TR'}{\Rightarrow ((\Rightarrow \rho) \Rightarrow (\Rightarrow A))}\,TR'}{(\Rightarrow \rho) \Rightarrow (\Rightarrow A)}\,PE'}{\Rightarrow (\Rightarrow A)}\,TR'}{\Rightarrow A}\,PE'$$

We thus obtain (i) $\Rightarrow \rho, \Rightarrow \rho \vdash_l \Rightarrow A$, (ii) $\Rightarrow \rho, \Rightarrow \rho \vdash_a \Rightarrow A$, and (iii) $\Rightarrow \rho, \Rightarrow \rho \vdash_r \Rightarrow A$. An application of the contraction rule

$$\Delta, T, T \vdash_r U / \Delta, T \vdash_r U$$

for $\vdash_r$ gives $\Rightarrow \rho \vdash_r \Rightarrow A$ in $\overline{R}^{\star}_{\infty}$. Such a move is, however, not available in $\overline{L}^{\star}_{\infty}$ with respect to $\vdash_l$ or in $\overline{A}^{\star}_{\infty}$ with respect to $\vdash_a$. Moreover, if we apply Theorems 3 and 5 to prove $(\Rightarrow \rho), (\Rightarrow \rho) \Rightarrow (\Rightarrow A)$ in $\overline{L}^{\star}_{\infty}$ and in $\overline{A}^{\star}_{\infty}$, respectively, we cannot contract to the sequent $(\Rightarrow \rho) \Rightarrow (\Rightarrow A)$ which we need to continue the Curry argument.

If we consider $\overline{R}^{\star}_{\infty}$ and relevant derivations, we have $\Rightarrow \rho \vdash_r \Rightarrow A$ but we face another problem. In a second step we apply Theorem 4 to obtain $(\Rightarrow \rho) \Rightarrow (\Rightarrow A)$ as an S-formula provable in $\overline{R}^{\star}_{\infty}$. We may use this S-formula twice to prove $\Rightarrow A$ in $\overline{R}^{\star}_{\infty}$. Let d' be the following derivation:

$$\dfrac{\dfrac{(\Rightarrow \rho) \Rightarrow (\Rightarrow A)}{\Rightarrow ((\Rightarrow \rho) \Rightarrow (\Rightarrow A))}\,PB \qquad ((\Rightarrow \rho) \Rightarrow (\Rightarrow A)) \Rightarrow \mathit{Prov}(\overline{\rho}, \overline{A})}{\Rightarrow \mathit{Prov}(\overline{\rho}, \overline{A})}\,TR'$$

Then

$$\dfrac{\dfrac{\dfrac{d' \quad Prov(\overline{\rho}, \overline{A}) \Rightarrow \rho}{\Rightarrow \rho}\ TR'}{\Rightarrow (\Rightarrow \rho)}\ PB \qquad (\Rightarrow \rho) \Rightarrow (\Rightarrow A)}{\Rightarrow A}\ TR'$$

The displayed derivations of the two steps in fact do not make use of the contraction rule SK (or the monotonicity rule VV), but we nevertheless fail to obtain an SK-free derivation of $\Rightarrow A$ in $\overline{R}^{\star}_{\infty}$ because the reasoning appeals to Theorem 4 and SK was used in the proof of Theorem 4.

The derivability relations $\vdash_l$, $\vdash_a$, and $\vdash_r$ for $\overline{L}^{\star}_{\infty}$, $\overline{A}^{\star}_{\infty}$, and $\overline{R}^{\star}_{\infty}$, respectively, have the additional drawback that internal and external consequence coincide in each case because we have both internalization and replacement. (This fact does not contradict the earlier observation that internal and external consequence may come apart with respect to sequent calculi that lack either weakening, or contraction, or both. Merely adding axiomatic sequents to a given sequent system does not impose any restrictions on how often such an axiomatic sequent may be used as a leaf in a sequent calculus proof tree.)

Considering ordinary or relevant derivations for $\overline{L}^{\star}_{\infty}$ or $\overline{A}^{\star}_{\infty}$ does not help because the internalization proof has to deal with the cut rule TR′, and the treatment of TR′ for such derivations requires the use of SK.[10]

We are thus still confronted with the *internal consequence* Curry paradox with respect to π for $\overline{R}^{\bullet}_{\infty}$ and, *a fortiori*, $\overline{M}^{\bullet}_{\infty}$, and do not obtain a trivialization for a subsystem of $\overline{M}^{\bullet}_{\infty}$ lacking SK. Indeed, in addition to the earlier trivialization with respect to $Der(\cdot,\cdot)$ and π we may use the $Der(\cdot,\cdot)$ predicate and the self-referential sentence π in a way analogous to the way we used the $Prov(\cdot,\cdot)$ predicate and the self-referential sentence ρ:

$$\dfrac{\Rightarrow \pi \qquad \dfrac{\dfrac{\dfrac{\pi \Rightarrow Der(\overline{\pi}, \overline{A}) \qquad Der(\overline{\pi}, \overline{A}) \Rightarrow (\pi \Rightarrow A)}{\Rightarrow \pi \quad \pi \Rightarrow (\pi \Rightarrow A)}\ TR'}{\pi \Rightarrow A}\ TR'}{\Rightarrow A}\ TR'$$

We thereby obtain $\Rightarrow \pi, \Rightarrow \pi \vdash_r \Rightarrow A$ in $\overline{R}^{\star}_{\infty}$, and an application of the contraction rule $\Delta, T, T \vdash_r U / \Delta, T \vdash_r U$ gives $\Rightarrow \pi \vdash_r \Rightarrow A$ in $\overline{R}^{\star}_{\infty}$. We apply

[10] If TR′ were eliminable, the problem would disappear. But our use of PE′ instead of PE stands in the way of cut-elimination. A counter example to cut-elimination is the sequent $(U \Rightarrow V), (T \Rightarrow U), T \Rightarrow V$ for $\mathcal{L}$-formulas T, U, V. It can be proved with the aid of TR′:

$$\dfrac{\dfrac{(T \Rightarrow U) \Rightarrow (T \Rightarrow U)}{(T \Rightarrow U), T \Rightarrow U} \qquad \dfrac{(U \Rightarrow V) \Rightarrow (U \Rightarrow V)}{(U \Rightarrow V), U \Rightarrow V}}{(U \Rightarrow V), (T \Rightarrow U), T \Rightarrow V}$$

but has no cut-free proof.

Theorem 4 to obtain $(\Rightarrow \pi) \Rightarrow (\Rightarrow A)$ as an S-formula provable in $\overline{R}^{\star}_{\infty}$. By Theorem 6, $\pi \Rightarrow A$ is provable in $\overline{R}^{\star}_{\infty}$ and we can continue as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\pi \Rightarrow A}{\Rightarrow (\pi \Rightarrow A)}\ PB \quad (\pi \Rightarrow A) \Rightarrow Der(\overline{\pi}, \overline{A})}{\Rightarrow Der(\overline{\pi}, \overline{A})}\ TR' \quad Der(\overline{\pi}, \overline{A}) \Rightarrow \pi}{\Rightarrow \pi}\ TR' \quad \pi \Rightarrow A}{\Rightarrow A}\ TR'$$

In this section we have seen two things. First, the internal v-Curry paradox reappears in the presence of the contraction rule SK and it falls with SK's dismissal. Second, the obvious way of introducing an external, Curry-prone validity predicate $Prov$ by assuming $Prov(\overline{A}, \overline{B}) \Leftrightarrow ((\Rightarrow A) \Rightarrow (\Rightarrow B))$ runs into problems. One might expect the predicate to lead to trivialization even for contraction-free subsystems of $\overline{M}_{\infty}$, where the internal v-Curry paradox fails. The internalization step that is needed to obtain the trivialization in the case of $Prov$ requires, however, the use of SK, and using SK, returns the trivializing internal v-Curry paradox with respect to Der. Nevertheless, von Kutschera's approach to formalising higher-level consequence opens a route to an external v-Curry paradox, to which we now turn.

6 The Higher-Order Sequent Calculus $\underline{M}_{\infty}$

In order to formalize an external consequence Curry Paradox, we may, instead, employ a higher-level sequent calculus defined over *a sequent language*. We shall refer to such a sequent system as a higher-order sequent calculus.

The sequent calculus $\overline{M}_{\infty}$ formalizes reasoning about nested sequents. We now consider a higher-order higher-level sequent calculus $\underline{M}_{\infty}$, which is defined for *positive S-formulas* over a given logical object-language $\mathcal{L}$ and which admits a nesting of the turnstile $\vdash$. The system $\underline{M}_{\infty}$ formalizes reasoning about nested statements about derivability between sequents. Its higher-order sequents are called $S^{\vdash}$*-formulas* over $\mathcal{L}$.

Definition 4 The set of $S^{\vdash}$-formulas over $\mathcal{L}$ is the smallest set $S^{\vdash}$-Form satisfying the following conditions:

1. Every positive S-formula over $\mathcal{L}$ belongs to $S^{\vdash}$-Form.
2. $(\vdash) \in S^{\vdash}$-Form.
3. If $\underline{S_1}, \ldots, \underline{S_n}, \underline{T} \in S^{\vdash}$-Form, then $(\vdash \underline{T}) \in S^{\vdash}$-Form, $(\underline{S_1}, \ldots, \underline{S_n} \vdash \underline{T}) \in S^{\vdash}$-Form, and $(\underline{S_1}, \ldots, \underline{S_n} \vdash) \in S^{\vdash}$-Form.

The $S^{\vdash}$-formulas $\underline{S_1}, \ldots, \underline{S_n}$, and $\underline{T}$ are said to be the antecedent formulas, respectively the succedent formula of the above $S^{\vdash}$-formulas.

We use (i) the letters '$\underline{T}$', '$\underline{U}$', '$\underline{V}$', '$\underline{W}$', '$\underline{W'}$' to denote $S^{\vdash}$-formulas over $\mathcal{L}$ and (ii) '$\underline{\Gamma}$', '$\underline{\Delta}$', '$\underline{\Pi}$', '$\underline{\Theta}$', '$\underline{\Theta'}$' to denote finite, possibly empty sequences of $S^{\vdash}$-formulas. $\underline{\Omega}$ is always a sequence of $S^{\vdash}$-formulas containing at most one $S^{\vdash}$-formula. We sometimes omit outermost brackets in $S^{\vdash}$-formulas.

Definition 5 Positive S-formulas over $\mathcal{L}$ are $S^{\vdash}$-formulas of $S^{\vdash}$-degree 0, and $(\vdash)$ is an $S^{\vdash}$-formula of $S^{\vdash}$-degree 1. If n is the maximum of the $S^{\vdash}$-degrees of the $S^{\vdash}$-formulas from $\underline{\Delta}$, $\underline{\Omega}$, then the $S^{\vdash}$-degree of $\underline{\Delta} \vdash \underline{\Omega}$ is $n + 1$.

Every $S^{\vdash}$-formula is an $S^{\vdash}$-subformula of itself and the $S^{\vdash}$-subformulas of $\underline{\Delta}$ and $\underline{\Omega}$ are $S^{\vdash}$-subformulas of $\underline{\Delta} \vdash \underline{\Omega}$. An $S^{\vdash}$-formula $\underline{T}$ is said to be *positive* iff $\underline{T}$ does not contain any $S^{\vdash}$-subformula that has no succedent formula. In what follows, we will be interested in positive $S^{\vdash}$-formulas only, in particular the $S^{\vdash}$-formula $(\vdash)$ will play no role.

Definition 6 The higher-order sequent calculus $\underline{M}_{\infty}$ consists of the following axioms and rules restricted to positive $S^{\vdash}$-formulas:

$\underline{\text{RF}}$	$\underline{T} \vdash \underline{T}$	(reflexivity)
$\underline{\text{TR}'}$	$\underline{\Delta} \vdash \underline{T};\ \underline{\Gamma}, \underline{T} \vdash \underline{U} \,/\, \underline{\Gamma}, \underline{\Delta} \vdash \underline{U}$	(cut)
$\underline{\text{VV}}$	$\underline{\Delta} \vdash \underline{T} \,/\, \underline{\Delta}, \underline{U} \vdash \underline{T}$	(monotonicity)
$\underline{\text{SK}}$	$\underline{\Delta}, \underline{T}, \underline{T} \vdash \underline{U} \,/\, \underline{\Delta}, \underline{T} \vdash \underline{U}$	(contraction)
$\underline{\text{ST}}$	$\underline{\Delta}, \underline{T}, \underline{U}, \underline{\Gamma} \vdash \underline{V} \,/\, \underline{\Delta}, \underline{U}, \underline{T}, \underline{\Gamma} \vdash \underline{V}$	(permutation)
$\underline{\text{PB}}$	$\underline{\Delta}, \underline{\Gamma} \vdash \underline{T} \,/\, \underline{\Delta} \vdash (\underline{\Gamma} \vdash \underline{T})$	(premise removal)
$\underline{\text{PE}'}$	$\underline{\Delta} \vdash (\underline{\Gamma} \vdash \underline{T}) \,/\, \underline{\Delta}, \underline{\Gamma} \vdash \underline{T}$	(premise introduction)

We may obtain analogues of Theorems 1–6, but for our purposes it is of primary interest to formalize the external consequence Curry paradox in complete analogy to the internal consequence Curry paradox. Let us therefore assume the following disquotational derivability schema:

$$Bew(\overline{\Rightarrow B}, \overline{\Rightarrow C}) \dashv\vdash ((\Rightarrow B) \vdash (\Rightarrow C))$$

and a self-referential S-formula $\Rightarrow \gamma$ satisfying

$$\Rightarrow \gamma \dashv\vdash Bew(\overline{\Rightarrow \gamma}, \overline{\Rightarrow A}).$$

Let $\underline{R}_{\infty}$ be the result of removing the monotonicity rule $\underline{\text{VV}}$ from $\underline{M}_{\infty}$. We use internal consequence in $\underline{R}_{\infty}$ to represent external consequence in some underlying *contraction-free* sequent calculus such as $\overline{L}_{\infty}$ or $\overline{A}_{\infty}$ and add the four $S^{\vdash}$-formulas $Bew(\overline{\Rightarrow B}, \overline{\Rightarrow C}) \dashv\vdash ((\Rightarrow B) \vdash (\Rightarrow C))$ and $\Rightarrow \gamma \dashv\vdash Bew(\overline{\Rightarrow \gamma}, \overline{\Rightarrow A})$ to $\underline{R}_{\infty}$. In a first step we prove the $S^{\vdash}$-formula $\Rightarrow \gamma \vdash \Rightarrow A$ using the contraction rule $\underline{\text{SK}}$:

$$\dfrac{\dfrac{\dfrac{\Rightarrow \gamma \vdash Bew(\overline{\Rightarrow \gamma}, \overline{\Rightarrow A}) \quad Bew(\overline{\Rightarrow \gamma}, \overline{\Rightarrow A}) \vdash (\Rightarrow \gamma \vdash \Rightarrow A)}{\Rightarrow \gamma \vdash \Rightarrow (\Rightarrow \gamma \vdash \Rightarrow A)} \quad \dfrac{(\Rightarrow \gamma \vdash \Rightarrow A) \vdash (\Rightarrow \gamma \vdash \Rightarrow A)}{(\Rightarrow \gamma \vdash \Rightarrow A), \Rightarrow \gamma \vdash \Rightarrow A}}{\Rightarrow \gamma, \Rightarrow \gamma \vdash \Rightarrow A}}{\Rightarrow \gamma \vdash \Rightarrow A}\ \underline{SK}$$

Let us refer to this derivation as d''. We then proceed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{d''}{\vdash (\Rightarrow \gamma \vdash \Rightarrow A)} \quad (\Rightarrow \gamma \vdash \Rightarrow A) \vdash Bew(\overline{\Rightarrow \gamma}, \overline{\Rightarrow A})}{\vdash Bew(\overline{\Rightarrow \gamma}, \overline{\Rightarrow A})} \quad Bew(\overline{\Rightarrow \gamma}, \overline{\Rightarrow A}) \vdash \Rightarrow \gamma}{\vdash \Rightarrow \gamma} \quad d''}{\vdash \Rightarrow A}$$

The whole argument still requires <u>SK</u>, however (in argument d''). Thus, even formalising external consequence using the higher-order von Kutschera sequent calculus, <u>SK</u> is still required to push through the paradox.

7 A Second Way of Representing the External v-Curry Paradox

We now turn to a second approach to formalising the external v-Curry paradox. As heretofore the Val predicate represents internal validity. In the literature, this is standardly taken to be a one-premise inference, but we need to generalise this to an arbitrary finite number of premises. To this end, we now take Val to be a binary predicate whose first argument is a term denoting a finite set of sentences, and whose second argument denotes a set containing a single sentence. (One may generalise this to arguments with finitely many conclusions, but this will not be necessary in what follows.) Thus, we may write things of the form $Val(\{A_1, ..., A_n\}, \{B\})$. Let us write α for $\{A_1, ..., A_n\}$, and β for $\{B\}$. Then the obvious natural deduction rules for Val are:

$$\dfrac{\begin{array}{c}[A_1], ..., [A_n] \\ \vdots \\ B\end{array}}{Val(\alpha, \beta)} \qquad \dfrac{\begin{array}{c}\Delta \\ \vdots \\ Val(\alpha, \beta)\end{array} \quad \begin{array}{c}\Gamma_1 \\ \vdots \\ A_1\end{array} \ldots \begin{array}{c}\Gamma_n \\ \vdots \\ A_n\end{array}}{B}$$

where the first rule contains no undischarged assumptions in the sub-deduction other than those shown, and these are all discharged. Call these rules Val-in and Val-out. Note that (for the moment), the arguments of Val denote sets (*not* multisets).

The special cases of the rules where $\alpha = \varnothing$ are:

$$\dfrac{\begin{array}{c}\vdots \\ B\end{array}}{Val(\varnothing, \beta)} \qquad \dfrac{Val(\varnothing, \beta)}{B}$$

The first inference has no undischarged assumptions at all in the sub-deduction.

The standard internal validity Curry paradox concerns a sentence, G, of the form: $Val(\{G\}, \{A\})$, where A is an arbitrary sentence. The core of the Curry argument is now a deduction of G, that is, $Val(\{G\}, \{A\})$, as follows:

$$\dfrac{\dfrac{[G] \qquad \dfrac{[G]}{Val(\{G\}, \{A\})}}{A}}{Val(\{G\}, \{A\})}$$

The second inference is Val-out. The last inference, Val-in, discharges both occurrences of the assumption G, and hence produces an unconditional proof of $Val(\{G\}, \{A\})$. But this is, of course, G, so Val-out gives a proof of A.

As already mentioned, a standard solution to the paradox is to deny contraction. In this form of proof, what this amounts to is taking the arguments of Val to denote, not sets, but multi-sets. The above argument then establishes only that $Val(\{G, G\}, \{A\})$, and the argument is broken.

Of course, if we have a proof of G, we can simply prepend it to each occurrence of G in the above proof, to produce an unconditioned proof of A. But this fact is not paradoxical, since we have (at least as far as anything so far goes) no contraction-free proof of G.

Now, can we obtain the validity-Curry paradox for external validity on this approach? The notion of external validity, Bew, can be defined in the obvious way, as:

$$Val(\{Val(\varnothing, \{A\})\}, \{Val(\varnothing, \{B\})\})$$

Note that this is quite different from internal validity: $Val(\{A\}, \{B\})$.

We may now form a Curry sentence, H, of the form $Bew(\{H\}, \{A\})$. (H is: The inference from [the validity of the inference from the empty set to this sentence] to [the validity of the inference from the empty set to A] is valid.) Paralleling the core part of the internal validity-Curry proof, we now have the following argument for H, that is $Bew(\{H\}, \{A\})$:

$$\cfrac{\cfrac{Val(\varnothing,\{H\}) \qquad \cfrac{\cfrac{\cfrac{Val(\varnothing,\{H\})}{H}}{Bew(\{H\},\{A\})}}{Val(\{Val(\varnothing,\{H\})\},\{Val(\varnothing,\{A\})\})}}{Val(\varnothing,\{A\})}}{\cfrac{Val(\{Val(\varnothing,\{H\})\},\{Val(\varnothing,\{A\}\})}{Bew(\{H\},\{A\})}}$$

The first inference is an application of Val-out; the fourth inference is another application of Val-out; the fifth inference is Val-in, discharging the two occurrences of the assumption $Val(\varnothing, \{H\})$; and the other inferences are just a matter of definition. As is clear, the argument will fail if Val-in is formulated in terms of multisets. For we will then establish only

$$Val(\{Val(\varnothing, \{H\}), Val(\varnothing, \{H\})\}, \{Val(\varnothing, \{A\}\})$$

at the penultimate line.

But as before, line five gives us a proof of $Val(\varnothing, \{A\})$ depending only on two occurrences of $Val(\varnothing, \{H\})$. Hence, if we had an unconditional proof of this, we would have a proof of A by Val-out. However, for a proof of $Val(\varnothing, \{H\})$, we need an unconditional proof of H; that is, an unconditional proof of

$$Val(\{Val(\varnothing, \{H\})\}, \{Val(\varnothing, \{A\})\}).$$

That is, we need an argument of the form:

$$\begin{array}{c} Val(\varnothing, \{H\}) \\ \vdots \\ Val(\varnothing, \{A\}) \end{array}$$

with only one undischarged occurrence of the assumption. And again as in the internal case, this is exactly what we do not have.

The internal and external validity Curry paradoxes are, then, on this analysis, exactly parallel.

8 Conclusion

Drawing on their distinction between connective paradoxes and structural paradoxes, Mares and Paoli [9, Section 5.3] make the following comment on the v-Curry paradox (where "**LL**" denotes linear logic with no exponentials and no additive constants):

> This typically structural paradox underscores the ambiguity of the classical concept of consequence. On the internal reading, as we have seen, Structural Contraction is utterly suspect, and in fact, it fails in the internal consequence relation of **LL**, which therefore avoids the difficulty. On the other hand, one could suggest to switch to an external reading, where such a rule is again available. Does that mean that we are back in trouble? No, for external consequence lacks anything like the deduction theorem, and therefore the disquotational validity schema – in particular its Val-In part, namely if $A \vdash B$ then $Val(\overline{A}, \overline{B})$ – becomes problematic. This, of course, if $Val(\overline{A}, \overline{B})$ is understood (correctly, we think) as a predicate appropriate for *internal* validity. Graham Priest suggested in conversation that we could regain the paradox if we had a corresponding predicate for external validity, for then an external reading of the turnstile would support all the ingredients needed for the argument to succeed. We think, however, that our theory should contain no such predicate, exactly as it contains no implication that residuates extensional conjunction – and Val, for us, *is* a form of implication.

In our discussion, we started by recalling that the c-Curry paradox is broken by giving up Contraction for the conditional $\rightarrow$ (or by giving up the Law of Absorption). Moreover, in the appropriate context, this principle is equivalent to structural contraction. Giving up structural contraction also breaks the v-Curry paradox for internal validity. We have seen that an external v-Curry paradox can be formalised with two different approaches; and in both of these—contra Mares and Paoli—there is an external validity predicate. The first uses a higher-order sequent calculus, the second nests the internal Val predicate. But in both of these, the Curry-reasoning involves contraction, either as a structural rule (in the first formalisation), or as reducing multisets to sets (in the second formalisation). Hence, giving up the appropriate forms of contraction solves all the versions of the Curry paradox.

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