HYPER-CONTRADICTIONS

Graham PRIEST

> '... if we take seriously both true and false and neither true nor false separately, what is to prevent our taking them seriously conjunctively? As in "It is both true and false and neither true nor false that snow is white". That way, in the end, lies madness.'
>
> Bob Meyer(0)

1. Contradictions

There is growing evidence that the logical paradoxes (and perhaps some other kinds of assertions) are both true and false. (1) Orthodox semantics, therefore, need to be modified to allow for this possibility. A natural way of doing this is as follows. (2)

Let $S_0 = \{1, 0\}$, the set of orthodox truth values.

We now want to allow for the possibility that a sentence may have more than one of these values. Thus the value of a sentence may be any member of

$$S_1 = P(S_0) - \{\varphi\}.$$

We remove the φ since a sentence must have *some* value at least. Now let $\wedge_0, \vee_0, \neg_0$ be the orthodox truth functions on S_0 (thus $\neg_0 1 = 0$ $1 \wedge_0 0 = 0$ etc.). What are the corresponding truth functors on S_1? The obvious thing to do is to define them point-wise. Thus if $x, y \in S_1$

i) $x \wedge_1 y = \{z \mid \exists x_1 \varepsilon x\ \exists y_1 \varepsilon y\ z = x_1 \wedge_0 y_1\}$
ii) $x \vee_1 y = \{z \mid \exists x_1 \varepsilon x\ \exists y_1 \varepsilon y\ z = x_1 \vee_0 y_1\}$
iii) $\neg_1 x = \{z \mid \exists x_1 \varepsilon x\ z = \neg_0 x_1\}$

(0) [1978], p. 19.
(1) See G. Priest [1984].
(2) See G. Priest [1979]

Thus, for example, if θ has the value true and ψ has the value both true and false, $\theta \wedge \psi$ will be true, since both ψ and θ are true, but it will also be false, since at least one conjunct is false. Hence $\theta \wedge \psi$ will be both true and false. I will call these functors LP functors.

Should we allow sentences to take a value from $S_0 \cup S_1$? We could, but this is unnecessary. For let σ be the map such that $\sigma(x) = \{x\}$. Then

i) σ is an injection of S_0 into S_1
ii) $\sigma(\neg_0 x) = \neg_1 \sigma(x)$
iii) $\sigma(x \vee_0 y) = \sigma(x) \vee_1 \sigma(y)$
iv) $\sigma(x \wedge_0 y) = \sigma(x) \wedge_1 \sigma(y)$

Thus S_0 is isomorphic to a subset of S_1.

$$< S_0, \wedge_0, \vee_0, \neg_0 > \cong < \sigma[S_0], \wedge_1 \restriction \sigma[S_0], \vee_1 \restriction \sigma[S_0], \neg_1 \restriction \sigma[S_0] >$$

Thus we do not need to add S_0 to S_1 explicitly.

Having defined the truth values in this way, we can now define the semantic consequence relation in the usual way. Let v_i be any map from the propositional parameters of a (propositional) language to S_i which is extended to a map of all formulas of the language by the truth functions with subscript i. (For i = 1 or 2.)
Let $D_0 = \{1\}$, $D_1 = \{\{1\}\{1,0\}\}$
These are the sets of designated truth values of S_0 and S_1 respectively. Any sentence which is at least true is designated. We can now define:
$\Sigma \models_i A$ iff all v_i, $\exists B \varepsilon \Sigma\, v(B) \not\varepsilon D_i$ or $v(A) \varepsilon D_i$ (i = 1 or 2)
It is easy enough to show(3) that

i) $\models_0 \neq \models_1$ but
ii) $\phi \models_0 A$ iff $\phi \models_1 A$.

2. *Hypercontradictions*

But now a natural question is posed: if there are sentences which are so twisted as to take impossible values, such as both true and false, might there not be sentences which are so contradictory as to

(3) See G. Priest [1979] Part III.

take impossible values such as both true and false ({1,0}) and true only ({1})? Indeed, there are, as a simple application of the extended liar paradox shows.

Consider the sentence: this sentence is false only. It is either true or false. If it is true, it is either true and false ({1,0}) or true only ({1}). But since it is true, it is false only ({0}). Hence, it takes impossible values. If, on the other hand, it is false, then it is not false only ({0}). Hence, it is true, which we have already seen to lead to impossible values. (4).

Does the possibility of these hyper-contradictions materially affect the logic we need to use? To answer this question we need to make the suggestion more precise. The obvious way to do this is to extend S_1 in a way analogous to the way in which we extended S_0 in the previous section. Of course, if we do this there is no reason why we should stop there. Similar considerations will force us to iterate the construction indefinitely. Let us therefore generalize the construction thus:

$S_0, \wedge_0, \vee_0, \neg_0$ are as before:

$S_{n+1} = P(S_n) - \{\varphi\}$ for all $n \varepsilon \omega$ (5)

$x \wedge_{n+1} y = \{z \mid \exists x_1 \varepsilon x\, \exists y_1 \varepsilon y\, z = x_1 \wedge_n y_1\}$

$x \vee_{n+1} y = \{z \mid \exists x_1 \varepsilon x\, \exists y_1 \varepsilon y\, z = x_1 \vee_n y_1\}$

$\neg_{n+1} x = \{z \mid \exists x_1 \varepsilon x\, z = \neg_n x_1\}$

As before let $\sigma(x) = \{x\}$. Then σ is an isomorphism between S_n and $\sigma[S_n]$.

In virtue of this isomorphism, and to ease notation, we can simply *identify* S_n with $\sigma[S_n]$, $\wedge_n$ with $\wedge_{n+1} \restriction \sigma[S_n]$ etc. To finish things off we now need to collect up.

Let $S = \bigcup_{n \varepsilon \omega} S_n$

$\wedge = \bigcup_{n \varepsilon \omega} \wedge_n$

$\vee = \bigcup_{n \varepsilon \omega} \vee_n$

$\neg = \bigcup_{n \varepsilon \omega} \neg_n$

(4) See G. Priest [1979] § V.3.

(5) If we allow a sentence to take two values which are mutually exclusive, there seems to be no reason why we should not allow them to take an arbitrary number.

So much for the truth values. Which of them are in the set of designated values D? The obvious thing is to let a value be designated just if it contains *some* truth, i.e. it contains 1 at some depth of membership. To make this precise, define a map $\eta: S_0 \rightarrow S_1$ thus:

$$\eta(0) = \{0\}$$
$$\eta(1) = \{1\}$$

and extend it to a map $\eta: S \rightarrow S_1$ by the recursion clause:

$$\text{if } x \varepsilon S_{n+1} \quad \eta(x) = \cup\{\eta(z) \mid z \varepsilon x\}$$

(Note that η respects the embedding σ. $\eta(\{x\}) = \eta(x)$.)
For all $x \varepsilon S$, $\eta(x) \subseteq S_0$, and we let $x \varepsilon \dot{D}$ iff $1 \varepsilon \eta(x)$.
(In the case at hand, D is simply $S - \{0\}$. However when we generalize the construction in § 3 to allow for the empty set, matters are not that simple.)
Logical consequence can now be defined in the usual way. If v is an evaluation whose range is S, respecting $\wedge$, $\vee$ and $\neg$:

$$\Sigma \models A \text{ iff all } v, \ \exists B \varepsilon \Sigma \ v(B) \notin D, \text{ or } v(A) \varepsilon D$$

What is the relation $\models$? The answer is that $\models = \models_1$ as we will now see. It is obvious that $\models \subseteq \models_1$ since $S_1 \subseteq S$. We need only prove the converse. This follows from the following central fact.

Lemma

η is a homomorphism.

Proof

To show this we need to prove that

i) $\neg\eta(x) = \eta(\neg x)$
ii) $\eta(x) \vee \eta(y) = \eta(x \vee y)$
iii) $\eta(x \wedge y) = \eta(x) \wedge \eta(y)$

For i), note first that $\neg\neg x = x$. (The proof is a simple one by induction on the construction.) i) is itself now proved by an induction on the construction. The case for S_0 is trivial. Suppose it holds for S_n and that $x \varepsilon S_{n+1}$:

$$
\begin{aligned}
\neg\eta(x) &= \neg \bigcup \{\eta(y) \mid y \varepsilon x\} && \text{(Def. of } \eta\text{)}\\
&= \neg\{z \mid \exists y \varepsilon x\, z \varepsilon \eta(y)\}\\
&= \{\neg z \mid \exists y \varepsilon x\, z \varepsilon \eta(y)\} && \text{(Def. of } \neg\text{)}\\
&= \{z \mid \exists y \varepsilon x \neg z \varepsilon \eta(y)\} && \text{(Double negation)}\\
&= \{z \mid \exists y \varepsilon x\, z \varepsilon \neg\eta(y)\} && \text{(Def. of } \neg\text{)}\\
&= \{z \mid \exists y \varepsilon x\, z \varepsilon \eta(\neg y)\} && \text{(Induction hypothesis)}\\
&= \{z \mid \exists y \varepsilon \neg x\, z \varepsilon \eta(y)\} && \text{(Def. of } \neg\text{)}\\
&= \bigcup \{\eta(y) \mid y \varepsilon \neg x\}\\
&= \eta(\neg x) && \text{(Def. of } \eta\text{)}
\end{aligned}
$$

ii) is also proved by an induction on the construction. The case for S_0 is trivial. Suppose it holds for S_n and that $x, y \varepsilon S_{n+1}$:

$$
\begin{aligned}
\eta(x \vee y) &= \bigcup\{\eta(z) \mid z \varepsilon x \vee y\} && \text{(Def. of } \eta\text{)}\\
&= \bigcup\{\eta(x_1 \vee y_1) \mid x_1 \varepsilon x,\ y_1 \varepsilon y) && \text{(Def. of } \vee\text{)}\\
&= \bigcup\{\eta(x_1) \vee \eta(y_1) \mid x_1 \varepsilon x,\ y_1 \varepsilon y\} && \text{(Induction hypothesis)}\\
&= \{z \mid \exists x_1 \varepsilon x\ \exists y_1 \varepsilon y\ z \varepsilon \eta(x_1) \vee \eta(y_1)\}\\
&= \{z_1 \vee z_2 \mid \exists x_1 \varepsilon x\ \exists y_1 \varepsilon y\ z_1 \varepsilon \eta(x_1) \text{ and } z_2 \varepsilon \eta(y_1)\} && \text{(Def. of } \vee\text{)}\\
&= \{z_1 \mid \exists x_1 \varepsilon x\ z_1 \varepsilon \eta(x_1)\} \vee \{z_2 \mid \exists y_1 \varepsilon y\ z_2 \varepsilon \eta(y_1)\} && \text{(Def. of } \vee\text{)}\\
&= \bigcup\{\eta(x_1) \mid x_1 \varepsilon x\} \vee \bigcup\{\eta(y_1) \mid y_1 \varepsilon y\}\\
&= \eta(x) \vee \eta(y) && \text{(def. of } \eta\text{)}
\end{aligned}
$$

For iii) the proof is similar.

That $\models \supseteq \models_1$ now follows simply. For suppose $\Sigma \not\models A$. Let $v(B) \varepsilon D$ for all $B \varepsilon \Sigma$ and $v(A) \notin D$. By the lemma, ηv is an evaluation with range S_1. Furthermore $x \varepsilon D$ iff $1 \varepsilon \eta(x)$ iff $\eta(x) \varepsilon D_1$. Hence $\Sigma \not\models_1 A$.

Thus, hyper-contradictions make no difference: the first contradiction $\{1,0\}$ of S_1 changes the consequence relation (but not the set of logical truths). Subsequent contradictions have no effect.

3. Ringing the Changes

This main result extends naturally to a variety of modifications of the basic semantical construction.

i) We can allow the empty set to enter at every stage of the construction. Thus, we simply let

$$S_{n+1} = P(S_n).$$

All the definitions and results of the previous section go through in exactly the same way. In particular $\models$ (so defined) is still identical to $\models_{\overline{1}}$ (so defined). This logic is not, what it might at first be thought to be, first degree entailment.(6) Rather the point-wise definitions of the truth functors give the extension of the LP functors according to the rule: gap-in, gap-out.(7) As far as I know, this logic has not been characterized proof-theoretically.

ii) A second way the construction could be altered is by symmetrising the notion of logical consequence. Thus, we define a domain, D′, of anti-designated values. The obvious way to do this is to define $x \varepsilon D'$ iff $0 \varepsilon \eta(x)$. Semantic consequence is now taken to require both preservation of designation forward and preservation of antidesignation backwards. This clearly has no effect on $\models_{\overline{0}}$. $\models_{\overline{1}}$ does change but, as the proof shows, $\models = \models_{\overline{1}}$ still.

iii) S is closed under finite subsets, i.e. if x is finite and $x \subseteq S$, $x \varepsilon S$. Thus, that S may take any finite number of incompatible values is accomodated by the construction. It may well be thought natural to allow sentences to take infinite numbers of incompatible values. An obvious way of allowing for this is to iterate the construction into the transfinite in the obvious way, collecting up a limit ordinals. We thus obtain a set S_∞, closed under arbitrary subsets. Moreover, if we define a consequence relation, $\models_{\overline{\infty}}$, over this set of values, it is straightforward enough to extend the result of the previous section to show that $\models_{\overline{\infty}} = \models_{\overline{1}}$.

The hypercontradiction construction can clearly be applied in a wide variety of cases. The above results indicate that it is, in general, not a destabilising construction. Finally, the construction shows that the notion of hypercontradictions is a quite coherent and intelligible one, Bob Meyer notwithstanding.(8)

University of Western Australia Graham PRIEST
Dept. of Philosophy
NEDLANDS – Western Australia

(6) In virtue of the four-valued semantics of J. M. Dunn [1976].
(7) Essentially as in T. Smiley [1960]
(8) See the quotation with which this paper starts.

REFERENCES

Dunn J. M., [1976] "Intuitive Semantics for First Degee Entaiments and Coupled Trees", *Philosophical Studies* 29, 136-52.

Meyer R., [1978] *Why I am not a Relevantist*, Research Paper #1, Logic Group, Research School of Social Sciences, Australian National University.

Priest G., [1979] "Logic of Paradox", *Journal of Philosophical Logic* 8, 219-241.

Priest G., [1984] "Logic of Paradox Revisited", *Journal of Philosophical Logic*, 13, 153-79.

Smiley T., [1960] "Sense without Denotation", *Analysis* 20, 125-135.