What is the Specificity of Classical Mathematics?

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Abstract: This paper addresses the question of what is distinctive about classical mathematics. The answer given is that it depends on a certain notion of conditionality, which is best understood as a telling us something about the structure of the mathematics in question, and not something about the logical particle 'if'.

Key Words: Classical mathematics, mathematical pluralism, material conditional, distributive lattice logic, Frege

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In this paper, I want to give an answer to the question which is its title. Exactly how I intend the question to be understood, will not, however, be immediately apparent. In the first section of the paper, I will give the background which makes the question intelligible. In the third section of the paper I will answer it. The answer depends on a technical observation. In the second part of the paper, I will explain what this is.

Let me make it clear, at the start, that I am concerned with pure mathematics, mathematics *an sich*. We apply mathematics to many things: physics, economics, biology. The how and why of this poses many interesting questions, but none of these is my concern here.

1 The Question and its Background

To explain the question, I will approach matters historically.

In mathematics, the 19th Century may be fairly thought of as the age of rigor. With the work of Weierstrass, Dedekind, and others, the mathematical notions of the day—and especially the various kinds of number—were set on a more secure footing than hithertofore. Theories not before given an axiomatic treatment were axiomatised, the different kinds of number were defined, their properties were deduced from these definitions, and so on.[1]

In the process, a new canon of logic was developed—so called classical logic. This could do justice to the deductions in a way in which the traditional logic of the time could not. The prime architect of this development was, of course, Frege, though Russell was to take most of the credit for it in the first

[1] See Priest (1998).

half of the 20th Century.

Frege and Russell developed their logic to try to establish their logicist thesis: that mathematics (or, for Frege, number theory—including the theory of real numbers) was part of logic. The logicist programme famously crashed. However, this was due to the problems that the programme exposed in set theory. The logic itself (or at least, its first-order fragment) soon became established as orthodox. And, it must be said, when set theory was set on a more secure footing by Zermelo and others, the logic worked well for what it was intended to do. Classical logic plus ZF set theory did a pretty good job of regimenting mathematics *circa* 1900—that is, classical mathematics, I shall call it.

The new logic became so entrenched amongst logicians that it quickly came to be *assumed* that it was *the* logic for reasoning about anything. Arguments were rarely given for this assumption, however.[2] And it is by no means obvious—actually, it seems rather implausible—whatever Frege and Russell themselves believed, that a canon that was developed for the purpose of handling a certain kind of mathematical reasoning must be applicable equally to other subject matters, such as time and tense, vagueness, truth, fiction (to name but a few areas), where various principles of classical logic (such as the law of excluded middle), might well appear suspect. It is more natural to suppose that these topics require *different* canons. If not, a careful

[2]Thus, it was clearly Frege's view that his logic provided the *most general* principles of truth. (See, e.g., Goldfarb (2001) for discussion and references.) However, I am unaware of anything like a systematic attempt to argue for this in his literary corpus.

justification is required.

This diversity is compatible with logical pluralism;[3] but, note, it does not require it. It could well be that there is a core logic to which various principles may be added for different domains, justified by the nature of those domains—in the same way that an intuitionist might add the Law of Excluded Middle to intuitionist logic when reasoning about decidable domains.

Now, some of the areas to which one might *not* want to apply classical logic come from mathematics itself, notably some of the areas of mathematics that have been developed since 1900. In fact, we now know that there are things such as intuitionist mathematics and paraconsistent mathematics,[4] where one can apply classical reasoning only with disaster. Indeed, it is clear that there are many pure logics, just as there are many pure geometries. *Qua* mathematical structure (proof procedures *cum* semantics *cum* algebras) these are all equally good—just as all pure geometries are equally legitimate.[5] And one may build more sophisticated mathematical structures atop of these, in the way that smooth infinitesimal analysis is built on intuitionist logic, or dialetheic set theory is built on paraconsistent logic.[6] This does not imply that all such mathematical structures are equal—in their mathematical interest, beauty, applicability, and so on. That is another matter, and a subject for further investigation. But at any rate, whatever one makes of *logical*

[3]On which, see Russell (2013).

[4]See, respectively, Dummett (1977) and Mortensen (1995).

[5]For a discussion, see Priest (2014).

[6]See, respectively, Bell (2008) and Weber (2012).

pluralism, it is clear that we are faced with *mathematical* pluralism.[7]

From this perspective, it makes sense to ask what is distinctive about classical mathematics. The obvious and straightforward answer is that it can be developed employing classical logic. One can hardly gainsay this answer. But I think that it may mask a more profound answer: one to do with the nature of the mathematics itself, and not just its underlying logic. What follows is an attempt to unearth this.

2 Main Logical Observation

We may now turn to the technical observation which informs the answer. What follows concerns propositional logic (not necessarily classical propositional logic). The extension of the following observations to a full first-order logic with identity are routine.

Take some logic, L, whose vocabulary comprises $\vee, \wedge, \neg, \rightarrow$, with their usual interpretations. We make some minimal assumptions about L. First, we assume that the logic of $\vee$ and $\wedge$ is that of a distributive lattice. That is, L verifies the principles of Distributive Lattice Logic (DLL):[8]

- $A \wedge B \vdash A$ (and B)
- If $C \vdash A$ and $C \vdash B$ then $C \vdash A \wedge B$

[7]See Priest (2013).

[8]In DLL the inference from A to B is valid iff, when A and B are assigned values in distributive lattice, in the obvious way, the value of A is less than or equal to the value of B.

- A (and B) $\vdash A \vee B$
- If $A \vdash C$ and $B \vdash C$ then $A \vee B \vdash C$
- $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$

We also assume that the conditional satisfies Identity and *Modus Ponens*:

- $\vdash A \rightarrow A$
- $A, A \rightarrow B \vdash B$

One might think of these as generic principles of conditionality. Finally, we assume that $\vdash$ satisfies the usual structural rules, such as Weakening, Cut, etc.

Note that we make no assumptions about negation. Nearly all non-classical logics satisfy these conditions: intuitionist logic, relevant logics, most paraconsistent logics.[9]

Now, let is define $A \supset B$ as $\neg A \vee B$, and consider the following principle:

- $A \supset B = A \rightarrow B$

Let us call this the *Principle of Extensional Conditionality* (*PEC*) since it says that the conditional behaves extensionally, in a certain sense. (One may understand this as the bi-deducibility $A \supset B \dashv\vdash A \rightarrow B$; or equivalently in this context, inter-substitutivity.)

Notice that the *PEC* delivers Explosion and Implosion:

[9]Though by no means all. The obvious exceptions are quantum logics and non-adjunctive logics.

- $A \wedge \neg A \vdash B$
- $A \vdash B \vee \neg B$

The second is immediate, given Identity. The first follows from *modus ponens* by a familiar argument:

- $A \wedge \neg A \vdash \neg A$
- $A \wedge \neg A \vdash \neg A \vee B$
- $A \wedge \neg A \vdash A \rightarrow B$
- $A \wedge \neg A \vdash A$
- $A \wedge \neg A \vdash B$

the last step following by *modus ponens*.

And when Explosion and Implosion are added to the principles above, we obtain all of classical logic. The easiest way to see this is to note that we can define $\bot$ and $\top$ as $A \wedge \neg A$ and $A \vee \neg A$, respectively (for any A). Then Explosion and Implosion deliver:

- $\bot \vdash A$
- $B \vdash \top$

and also that:

- $A \wedge \neg A \vdash \bot$

- $\top \vdash B \vee \neg B$

So $\vdash$ satisfies the conditions of a complimented distributive lattice, that is, a Boolean algebra, the algebraic guise of classical logic.[10]

Before we move to the third and final section of the essay, I will make three technical observations:

Observation 1: If L has a propositional logical truth which translates, via the PEC, into something that is not a classical logical truth, then the addition of the Principle produces triviality, since classical logic is Post-complete. This can happen, for example if L is a connexive logic which validates Aristotle $\neg(A \to \neg A)$ and/or Boethius $(A \to B) \to \neg(A \to \neg B)$.[11]

Observation 2: If we were to formulate the PEC as $A \to B = \neg(A \wedge \neg B)$, this would not deliver classical logic. Here are matrices (due to Hitoshi Omori) which validate distributive lattice logic, identity and *modus ponens* (under this translation), but which do not validate, for example, Explosion.

$\neg$	
t^*	f
b^*	b
f	t

$\wedge$	t	b	f
t	t	t	f
b	t	b	f
f	f	f	f

$\vee$	t	b	f
t	t	t	t
b	t	b	b
f	t	b	f

This is perhaps rather surprising. In many of our target logics, $\neg A \vee B$ and

[10]See Bell and Slomson (1969), ch. 1. A more flat-footed way of seeing that we now have classical logic is as follows. Take a rule system for classical logic—e.g., that in Tennant (1978), ch. 4—and show that the principles deliver all of its rules.

[11]See Priest (2008), 9.7a.

$\neg(A \wedge \neg B)$ are logically equivalent.[12] The reason is that, with the alternative definition, the proofs of Explosion and Implosion above would require principles involving negation not delivered by any of the machinery.

I note, though, that had we started, not from the logic of a distributive lattices, but from that of a De Morgan algebra, classical logic would be delivered by this version of the PEC, because of the negation principles of the algebra.[13]

Observation 3: The above construction starts by taking distributive lattice logic as the *Grundlogik*. One could make more minimal choices. For example, one might start by assuming that conjunction and disjunction are those of lattice logic (not distributive lattice logic). And certainly, there are mathematical theories based on non-distributive logics, with their own specificities. But the issue here is the specificity of classical mathematics, and starting with distributive lattice logic certainly recommends itself in this context. For a start, nearly all logicians (including myself) endorse the principles of DLL for the extensional conjunction and disjunction. Moreover, if one does start with lattice logic, the distribution principle hardly gives a characterisation of conjunction and/or disjunction. By contrast, the PEC appears to give a straightforward characterisation of a particular notion of conditionality.

[12] Though not in all; the equivalence fails in intuitionist logic, for example.

[13] A De Morgan algebra adds to the principles of distributive lattice logic, the negation principles $\overline{\overline{a}} = a$ and $a \leq b \Rightarrow \overline{b} \leq \overline{a}$.

3 The Answer and its Status

We may now turn to the answer I wish to offer to the question explained in the first section of the paper.

What we have seen is that, starting from any of the familiar sub-classical logics satisfying our simple conditions, what characterises classical logic, and so classical mathematics, assuming that this is deployed in it, is exactly the *PEC*.[14]

One can certainly hear the *PEC* as a principle concerning the natural language logical particle *if*, to the effect that this is the material conditional,[15] so that the principles concerning it are applicable whenever one uses this locution. This was the way that Russell understood it;[16] it was also the way that most logicians in the next 50 years thought of it, judging by several generations of text books.

I note, however, that it is not the way that Frege, more circumspectly, understood the material conditional. He is careful to gloss his conditional

[14]Given the above, one might wonder why it could not equally be characterised by Explosion and Implosion. The reason is that even if these things are added to DLL, we still need a definition of the conditional to do mathematics. Of course, we could add that as well; but, as we have seen, this makes the addition of Explosion and Implosion redundant.

[15]At least when this is the indicative conditional, as opposed to the subjunctive. That distinction is not on the agenda here. I note, also, that not all uses of 'if' are uses of the logical particle. 'If I may say so, you look stunning today' is just a polite way of saying 'You look stunning today'.

[16]Russell, however, confuses the conditional (a connective) with implication (a relation). Thus: '... the proposition $\sim p \vee q$ will be quoted as saying that p implies q. ... The symbol may also be read as "if p, then q" '. (Whitehead and Russell (1927), p. 7.) He uses *both* locutions when he glosses the axioms of *Principia* (*ibid.* p. 96).

notation (in modern notation, $A \supset B$) simply as a truth function ruling out just one possibility: 'A is affirmed and B is denied'.[17] (Of course, in the context in which Frege was working, this is equivalent to: either A is denied of B is affirmed.) Frege himself points out that there is more to a natural language conditional than this.[18]

Rather than taking the *PEC* to be a principle of logic, concerning the particle, *if*, it is better, I suggest, to understand it as one which tells us something about the relation of conditionality that is operative within mathematics of a certain kind.[19] And different kinds of mathematics may operate with quite different kinds of conditional relation.[20]

Why is it more appropriate to hear the *PEC* in this way, rather than as something to do with the natural language *if*? Here are two reasons (though I do not suggest that they are definitive).

First reason: If one hears it as a claim about the meaning of a natural language conditional 'if', it would seem to be false. Few, now, would suppose that 'if'—even the so called indicative conditional—is material. Better not

[17] See Bynum (1972), p. 114 ff.

[18] Thus: '... we can translate $A \supset B$ with the aid of the conjunction "if": "If the moon is in the quadrature [with the sun], it appears as a semicircle". The causal connections implicit in the word "if", however, is not expressed in our symbols...'. (Bynum (1972), p. 115 f.)

[19] Note that I am not denying the possibility that this notion of conditionality is operative in other sorts of discourse as well.

[20] One might essay the claim that other kinds of mathematics, such as intuitionist mathematics and relevant mathematics, are also be characterised by the kind of conditional they employ. That suggestion is not on the agenda here, however. My concern is solely with classical mathematics.

to lumber classical mathematics with this mistake. To illustrate:[21] let a be some planar quadrilateral, and consider the inference:

[P] If a has equal sides and equal angles then it is a square.

[C] So: Either, if a has equal sides and unequal angles it is a square, or if a has equal angles and unequal sides it is a square.

No mathematician in their right mind (classical or otherwise) would reason in this way. However, it is valid if 'if ... then' is replaced by $\supset$, and intelligibly so: if the consequent of [P] is true, so are both disjuncts of [C]; and if its antecedent is false, so is the antecedent of one of the disjuncts of [C], as, then, is its corresponding conditional.

Second reason: suppose that one takes some non-classical logic, such as intuitionist or paraconsistent, to be correct. Then classical reasoning is invalid. One can no longer, then, accept the *truth* of those results of classical mathematics for which no acceptable proof can be found. If one supposes that the PEC simply characterises the relation of conditionality in the mathematics in question, one can avoid this wholesale decimation.

One may look at matters as follows. When one investigates a mathematical theory, one is discovering things about what holds in the structure or structures that realise it. Thus, when one investigates group theory, one is determining the things that hold in all its models. When one adds the commutativity postulate, one is simply cutting down the class of models

[21] For a fuller discussion of the matter, see Priest (2008), ch. 1.

in question. Similarly, when one investigates ZF, one is investigating the things that hold in all of its models. If one then adds the Axiom of Choice to the theory, or some large cardinal axiom, one is simply cutting down to a distinctive class of structures or models of a certain kind. In the same way, when one investigates some mathematical theory based on one of our target logics, one is determining what holds in the structures or models that realise it. Adding the PEC then narrows down the class of structures under investigation.

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The title of this essay was 'What is the specificity of classical mathematics?' The answer I propose should now be clear. What is distinctive about classical mathematics? The Principle of Extensional Conditionality.[22]

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[22] Versions of this paper were given in 2015 at the conferences *Pluralism*, Inter-University Centre, Dubrovnik; *Logic in Bochum 1*, University of Bochum; *Mathematical Pluralism*, Universidad Nacional Autonoma de Mexico; *Midwest Conference on the Philosophy of Mathematics*, University of Notre Dame. I am grateful to the audiences there for their helpful suggestions; also to a referee from this journal.

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