A Note on the Axiom of Countability

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Abstract

The note discusses some considerations which speak to the plausibility of the axiom that all sets are countable. It then shows that there are contradictory but nontrivial theories of ZF set theory plus this axiom.

In this note, I will make a few comments on a principle concerning sets which I will call the Axiom of Countability. Like the Axiom of Choice, this comes in a weaker and a stronger form (local and global). The weaker form is a principle which says that every set is countable:

WAC $\forall z \exists f(f \text{ is a function with domain } \omega \wedge \forall x \in z \exists n \in \omega\, f(n) = x)$

(The variables range over pure sets—including natural numbers. ω is the set of all natural numbers.) The stronger form is that the totality of all sets is countable:

SAC $\exists f(f \text{ is a function with domain } \omega \wedge \forall x \exists n \in \omega\, f(n) = x)$

The stronger form implies the weaker. Any set, a, is a sub-totality of the totality of all sets. Hence, if the latter is countable, so is a. So I focus mainly on this.

Let us start by thinking about the so called Skolem Paradox. Take an axiomatization of set theory, say first-order classical ZF. This proves that some sets, and *a fortiori* the totality of all sets, are uncountable. Standard model theory assures us that there are models of this theory (in which '$\in$' really is the membership relation) where the domain of the model is countable. There *is* a function which enumerates the members of the domain. It is just one which has failed to get into the domain of the interpretation. Why should we not suppose, then, that the universe of sets

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really is countable? From the perspective of the metatheory, ZF^+ (ZF + 'There is a model of ZF'), the countable model is not the intended "interpretation". Our metatheory tells us that the domain of all sets is actually uncountable. But ZF^+ itself has a countable model, so the situation is exactly the same with this. We might suppose that the countable model of this tells us how things actually are. True, in the metatheory we are now working in, ZF^{++} (ZF^+ + 'There is a model of ZF^+'), that model will appear not to be the intended model. But we can reply in the exactly same way. Clearly, the situation repeats indefinitely. And at no stage are we *forced* to conclude that the universe of sets is really uncountable. We will always have a countable model at our disposal.

Indeed, it is not just the case that there is nothing that will force us to conclude that the universe of sets is really uncountable. There are certain conceptions of sethood which actually push us to that conclusion. Thus, suppose that one takes the not implausible view that sets are simply the extensions of predicates (or some predicates anyway).[1] Then, given that the language is countable, so it the universe of sets.

Now, imagine that the history of set theory had been slightly different. Suppose that set theory had been investigated for a few years before Cantor, and that those who investigated it took sets to be simply the extensions of predicates. Suppose also that the theory had actually been formalised, say by some mathematician, Zedeff. The (strong) Axiom of Countability, being an *a priori* truth about sets, was one of the axioms. Things were bubbling along nicely, until Cantor came along and showed that within the theory one could prove that some sets are uncountable. The theory was inconsistent. In this history, Cantor was playing Russell to Zedeff's Frege. We can imagine that the community was dismayed by this paradox, and started to try to amend the axiomatization in such a way as to avoid paradox. Perhaps, indeed, the hierarchy ZF, ZF^+, ZF^{++}, ... emerged—rather as the hierarchy of Tarski metalanguages emerged in our actual history.

In actual history, set theory was consistentized in response to Russell's paradox and related ones. However, as we now know, there is an alternative: maintain the naive comprehension schema—that is, the schema $\exists x \forall y (y \in x \leftrightarrow \psi)$, where ψ does not contain y—allow the paradoxes, and deploy a paraconsistent logic, which quarantines the paradoxes. The same was an option in our hypothetical history; maintain the Axiom of Countability, the paradoxes it generates, and deploy a paraconsistent logic.

Now back to reality. Is there such a theory? There is. Using the paraconsistent logic LP, we can show the existence of such a theory by applying a result called the

[1] See [2, ch. 10]. See also [1].

Collapsing Lemma. Take a first-order language (without function symbols) for LP.[2] Let $M = \langle D, \delta \rangle$ be any interpretation for this. Let $\sim$ be any equivalence relation on D.[3] If $d \in D$, let $[d]$ be its equivalence class under $\sim$. We define a new interpretation (the collapsed interpretation), $M^{\sim} = \langle D^{\sim}, \delta^{\sim} \rangle$, as follows. $D^{\sim} = \{[d] : d \in D\}$. For any constant, c, $\delta^{\sim}(c) = [\delta(c)]$. For any n-place predicate, P, $\langle a_1, ..., a_n \rangle$ is in the extension of P in $M^{\sim}$ iff there are $d_1 \in a_1$, ..., $d_n \in a_n$, such that $\langle d_1, ..., d_n \rangle$ is in the extension of P in M. Similarly for the anti-extension of P. The collapse, in effect, simply identifies all the members of an equivalence class, producing an object with the properties of each of its members. The Collapsing Lemma tells us that any sentence in the language of M (i.e., the language augmented with a name for each member of D) which is true in M is true in $M^{\sim}$; and any sentence false in M is false in $M^{\sim}$.[4]

To apply this: let the language be the language of first-order ZF (without set abstracts). Take a (classical) interpretation of this, M, which is a model of ZF. Let k be any countable set in D. (Here, and in what follows, I mean countable—or uncountable—in the sense of M.) Consider the equivalence relation on D which identifies all uncountable sets with k, and otherwise leaves everything alone. That is, $x \sim y$ iff in M:

- x and y are uncountable
- or (x is uncountable and y is k)
- or (y is uncountable and x is k)
- or (x and y are both k)
- or (x and y are countable sets distinct from k, and $x = y$).

Now consider the collapsed model obtained with $\sim$. By the Collapsing Lemma, this is a model of ZF. But in $M^{\sim}$ every set is countable. For every constant, c, that denotes a countable set in M :

- $\exists f(f$ is a function with domain $\omega \wedge \forall x \in c \exists n \in \omega\, f(n) = x)$

is true in M, and so by the Collapsing Lemma, in $M^{\sim}$. Since every member of $D^{\sim}$ has such a name in $M^{\sim}$, we have the WAC in $M^{\sim}$:

- $\forall z \exists f(f$ is a function with domain $\omega \wedge \forall x \in z \exists n \in \omega\, f(n) = x)$.

[2] For a presentation of the semantics of LP, see [2, sec. 16.3].

[3] If the language were to contain function symbols, $\sim$ would also have to be a congruence on their interpretations.

[4] For full details, including the proof, see [2, sec. 16.8].

A slightly different equivalence relation delivers an interpretation which verifies SAC. Let k now be the object which is V_ω (the sets of rank ω) in M. Consider the equivalence relation which identifies all things of rank greater than ω with V_ω, and leaves everything else alone. That is, $x \sim y$ iff in M:

- $x, y \in V_\omega$ and $x = y$
- or $x, y \notin V_\omega$.

Again, this is a model of ZF. $k \cup \{k\}$ is countable in M. Let i be the name of the function that enumerates it, and let e be the name of any member of $k \cup \{k\}$. Then in M it is true that:

- i is a function with domain $\omega \wedge \exists n \in \omega\, i(n) = e$.

Hence this is true in $M^\sim$. But since every member of $D^\sim$ is named by some e of this kind, we have in $M^\sim$:

- i is a function with domain $\omega \wedge \forall x \exists n \in \omega\, i(n) = x$.

Hence we have the SAC in $M^\sim$:

- $\exists f(f$ is a function with domain $\omega \wedge \forall x \exists n \in \omega\, f(n) = x$.

For good measure, $M^\sim$ is also a model of the naive comprehension schema, $\exists z \forall x(x \in z \equiv A)$, too.[5] If sets just are the extensions of predicates, one would expect this schema to hold. I note also that both of the models we have constructed are non-trivial. Thus, if c and d refer to two distinct objects in D that are not involved in the collapse, $c = d$ is not true in the collapsed model.

What we see, then, is that there are (non-trivial) theories that contain the (strong or weak) Axiom of Countability, plus ZF (plus, in one case, the naive comprehension schema). If T is the set of things true in either of the collapsed models we have constructed, T is one such theory. Within such a theory, every set is countable; but, because of Cantor's Theorem, some sets are uncountable as well. It is Cantor's Theorem that generates the hierarchy of different sizes of infinity. And as seen from the perspective of one of these theories, the Theorem is recognizably paradoxical. The whole hierarchy of infinities is therefore a consequence of the paradox. The transfinite, then, is generated by the transconsistent.[6]

In a nutshell: the Axiom of Countability makes perfectly good paraconsistent sense, even within the context of ZF. And it provides a radically new possible perspective on the universe of sets.

[5]For details see [2, sec. 18.4].

[6]It was Zach Weber who first suggested to me that one might see the transfinite in this way. See [3, sec. 8].

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