19th Century German Logic

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Abstract

The 19th Century is the cite of one of the most significant transitions in the history of Western logic. The traditional logic of the syllogism was swept away and replaced by mathematical logic. This article traces the highly significant German contribution to this process. It starts with traditional logic, as found in Kant, and the flowering of dialectical logic in Hegel and the Engels. It then reviews debates about the nature of logic, and the decisive rejection of psychologism by Bolzano. The foundational work in mathematical logic provided by the algebraic techniques of Schröder and Frege's *Begriffsschrift* are then discussed.

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'Logic, by the way, has not gained much in *content* since Aristotle's times and indeed it cannot, due to its nature... In present times there has been no famous logician, and we do not need any new inventions in logic, because it contains merely the form of thinking.'

Immanuel Kant. From the introduction to his lectures on logic. [Hartman and Swartz, 1974: 24-25.]

1 Introduction

Let me start by saying how, for the purpose of this essay, I have chosen to interpret the words of its title: none is innocent. Let us work backwards.[1]

[1] There are some other preliminary remarks that need to be made. In a survey of this kind it is impossible to do justice to the richness and intricacies of the thought any one

'Logic' is used in many ways, and in different ways even by some of the thinkers we will meet. In an article of this length it would be impossible to take on board all these usages. I have chosen to interpret the word in the way that contemporary logicians understand it. That is, logic concerns what follows from what: which premises entail which conclusions, and why. Of course, this cannot be divorced from other important questions, such as: what sorts of things, exactly, are premises and conclusions? What sorts of things constitute them? And how do some of these things which are particularly important in the context of logic, such as negation, work.

Secondly, 'German'. Defining 'German' in terms of the geographical boundaries of modern Germany makes little intellectual sense. What is arguably the modern state of Germany did not, itself, come into existence until 1871. And there were significant thinkers who are clearly in the relevant intellectual community, but who lived outside its contemporary boundaries. (Kant was in Köningsberg, which is modern day Kaliningrad, in Russia; Bolzano was in Prague, in the modern Czech Republic.) It seems best, to me, to characterise the intellectual community we are dealing with by its common tongue. So I will take Germans to be people who were native German speakers.

'The 19th Century' might seem the least problematic of the words; but, in fact, it is the most problematic. It is silly to suppose that, intellectually, it came into existence at midnight of 1/1/1801. The 19th Century started before that; and the 18th Century ended after that. In exactly the same way, it is absurd to suppose that the 19th Century ended at exactly 1/1/1901, and that the 20th Century started then. So how best may one understand 'the 19th Century' in this context? To answer this question one needs to situate the century in the history of the deveopment of logic.

of the writers we will meet, let alone all of them. For the same reason, there are people who would have to be mentioned in a longer treatise, but for whom there is no space in this. I have had to select what seem to me to be the most significant people, and the most significant features of their work. This introduces an ineliminable subjectivity into the essay. A second source of subjectivity is the fact that history is not simply a catalogue of names and dates. It is a narrative which makes the names and dates meaningful. I would not wish to pretend that what I am doing here is anything other than telling a story about the history of logic as one contemporary logician see it—though for the most part, I do not think there is anything particularly idiosyncratic about it. At the end of each main section of this paper I will give references to places where the material covered in that section is discussed by others in greater detail.

2 The History of Western Logic

Broadly speaking, the history of Western logic falls into three major phases of growth, interspersed by two periods of stasis, and even decline. (The study of logic in the East has its own story to tell.)

The first phase of growth was in Ancient Greece. Aristotle developed the theory of the syllogism, and the Stoic logicians developed a somewhat different theory of logical consequence: a version of what we would now call propositional logic. With the decline of the Western part of the Roman Empire, the study of logic goes into decline in Christendom. Logic is still studied in the Islamic tradition, but mainly by way of writing commentaries, especially on Aristotle, rather than by the development of radically new ideas.

The second major growth phase of logic in the West was in the great Medieval universities, such as Oxford and Paris. The high period of this was the development of the *logica nova* (new logic, term logic) in the 14th Century. The Medieval logicians developed the Aristotelian theory of the syllogism, blended it with Stoic propositional logic, and developed many novel theories of, amongst other things, *consequentiae*, *suppositiones*, *obligationes*, *insolubiles* (as a first cut: logical consequence, truth conditions, rules of debate, logical paradoxes).

With the rise of Humanism, much of this sophistication fell into oblivion under the general attack on Scholasticism. (In fact, it was only in the second half of the 20th Century that the depth of Medieval logic was rediscovered.) There is then a somewhat dull period in logic until the commencement of the third great period of growth, the main bright spot being Leibniz, whose attempt to articulate a *characteristica universalis* and *calculus ratiocinator* (a sort of proto-formal language, with rules for calculating in it) arguably provided a premature anticipation of later developments.

The third great period of the development of logic commences in the second half of the 19th Century, and continues through today (with no sign of coming to an end). This period was inaugurated by logicians applying mathematical techniques to logic—such as those of axiomatization, model theory, abstract algebra—as well as the heightened standards of mathematical rigour being developed in the contemporary mathematics. In the 20th Century, this has produced metamathematics, the foundations of computational theory, the panoply of non-classical logics, and all the standard fare of the contemporary logic curriculum

Now, the period which is our special concern in this essay, the 19th Century, is the site of the rupture into this third great period. It starts with the rump of logic that was left after the decline of Medieval logic, and ends with the creation of mathematical logic. German logicians are not the only significant players in this period. However, Germany certainly produced some of the most significant.

Against this background, let us now turn to details.[2]

3 Kant and Logic

Let us start with Immanuel Kant (1724-1804).[3] Logic was singularly important for Kant. It provided the tectonic framework for the first of his three great Critiques, *Critique of Pure Reason*.[4] However, Kant is not a significant figure in the history of logic. Indeed, his reading of logic was singularly wrong-headed. He took it, not only that there had been no significant developments since Aristotle, but that there could not be. (See the quotation with which this article starts.)

He did, however, lecture on logic; and some of his lecture notes were subsequently edited and published by Gottlob Jäsche in 1800.[5] The notes paint a fairly clear picture of the logic of his day (which I will call, henceforth, *traditional logic*). The main part of this comprises the *Doctrine of Elements*. There is also a short second part called the *Doctrine of Method*, which contains a few miscellaneous remarks, mainly about definition. The *Doctrine of Elements* has three parts: *Concepts*, *Judgements*, and *Inferences*. *Inferences* contains a discussion of what inferences are valid. *Judgements* contains a discussion of the parts of inferences, the statements that make up the premises

[2]It is hard to find a good book that covers the whole history of logic. Between them, Kneale and Kneale (1962) and Haaparanta (2009) give quite good coverage. The encyclopedic Gabbay and Woods (2004-2012) contains detailed essays on most aspects of the history of logic. Lenzen (2004) can be consulted for an account of Leibniz' views on logic.

[3]When I reference books or articles that appeared in German, I shall give their original publication details, and then an accessible English translation if and where one exists. When dealing with symbolism, I have decided to write in the notation of modern logic. This is not because the notations actually used are without historical interest. And there is also a certain danger in this. One should not take it for granted that the writers we will meet meant by their symbols exactly what the modern logician means by theirs. However, the use of modern symbolism makes it easier to tell a uniform story, and one that is more intelligible for non-specialists. (Not to mention one that makes typesetting easier!) It should go without saying that, for someone who wants a detailed understanding of thinkers, their ideas, and their symbolism, there is no substitute for reading the primary texts.

[4]*Kritik der reinen Vernunft*, Riga: Johann Friedrich Hartnoch. 1st ed. 1781; 2nd ed. 1786. There are several accessible translations. Kemp Smith (1923) is an old standard. Guyer and Wood (1998) is a good more recent translation.

[5]*Immanuel Kants Logik, ein Handbuch zu Vorlesungen*, Königsberg: F. Nicolovius. English translation, Hartman and Scholtz (1974).

and conclusions; *Concepts* contains a discussion of the parts of judgements, namely, concepts.

The most striking thing about what Kant has to say about concepts, from a contemporary perspective, is that they are clearly mental, psychological notions. In *Judgements*, we find, likewise, that judgements, being composed of concepts are psychological acts: they are propositions endorsed as true. (A modern logician is likely to point out that in an inference the premises do not have to be endorsed as true: logic itself need have no concern with the truth or otherwise of the premises.)

According to Kant every judgement has a quality, quantity, relation, and modality. There are three possibilities in each case, which we may tabulate as follows (where the glosses are those of a modern logician, not Kant):

Quantity		
	Singular	The subject of the sentence is a noun phrase
	Particular	The subject of the sentence is of the form 'Some As'
	Universal	The subject of the sentence is of the form 'All As'
Quality		
	Affirmative	The predicate of the sentence is 'is (are) $B(s)$'
	Negative	The predicate of the sentence is 'is (are) not $B(s)$
	Infinitive	The predicate of the sentence is 'is (are) non-$B(s)$
Relation		
	Categorial	The sentence contains no propositional connective
	Hypothetical	The sentence is of the form 'If Athen B'
	Disjunctive	The sentence is of the form 'A(exclusively) or B'.
Modality		
	Problematic	The sentence is stated as possibly true
	Assertoric	The sentence is stated as actually true
	Apodictic	The sentence is stated as necessarily true

Oddly, Kant does not observe that only categorical judgements can have a quality or quantity, as such.

Kant's treatment of modality is also worth noting. Unlike the other categories, which are purely syntactic, modality concerns the attitude one has when one judges a sentence: whether one takes the content to be possible, actual, or necessary. Hence, nothing like modal logic in the contemporary, medieval (or even Aristotelian) sense is possible. In such logics, the modal operator is taken to be part of the content of the sentence, not one concerning the attitude of the person who judges.

In *Inferences*, we find a fairly standard account of Aristotelian syllogistic, that is inferences of the form:

$$\frac{\begin{array}{c}\text{All/some/no } S \text{is/are } M \\ \text{All/some/no } M \text{is/are } P\end{array}}{\text{All/some/no } S \text{is/are } P}$$

—though it is worth noting that this includes syllogisms of the fourth figure (where the middle term occurs, M, as the predicate of the major premise, and the subject of the minor premise). This is not to be found in Aristotle, but is a Medieval creation. Kant also claims that the conclusion of any syllogism has apodictic modality (i.e., holds of necessity). This seems to confuse the necessity of the conclusion with the necessity of the connection between premises and conclusion.

After the discussion of the Aristotelian syllogism, we find the simple cataloging of a few valid propositional inferences, such as *modus ponens* (A, if A then B; so B) and the disjunctive syllogism (A or B, it is not the case that A; so B).

The section ends, interestingly, with some comments on inductive inference. That topic hardly features in Medieval discussions of logic, which concerns itself mainly with deductive inference. By Kant's time, an awareness of the importance of non-deductive inference has been brought to logic by the "scientific revolution", and its novel conception of scientific methodology.[6]

4 Hegel and Dialectic

Georg Wilhelm Friedrich Hegel (1770-1831) took over much of Kant's thought, but changed it in very important ways. Notably, he added a dynamic element that was entirely absent in Kant. From the simplest and most elementary concept, that of *being,* a sequence of concepts develops in a zig-zag fashion until we reach the concept which is most adequate for characterising reality, the *absolute idea*. The concepts are no mere abstracta, however. They are embodied in human and natural history. The conceptual development is therefore embodied in the historical development of the world.

Hegel describes the evolution of concepts in his *Science of Logic*.[7] The matter is covered again more briefly in Part 1 of Hegel's *Encyclopedia of the Philosophical Sciences*.[8] This is often referred to as the *Lesser Logic*, as opposed to the *Logic* (*Science of Logic*); and it is often easier to understand

[6]For further discussion, see Tyles (2004) and Young (1992).

[7]*Wissenschaft der Logik*, Nürnberg: Schrag. Vol. 1, Pt. 1, 1812; Vol. 1, Pt. 2, 1813; Vol. 2, 1816. Translation, Miller (1969).

[8]*Enzyklopädie der philosophischen Wissenschaften*, Heidelberg: Oßwald. 1st ed., 1817; 2nd ed., 1827; 3rd ed., 1830. Translated as Wallace (1873).

than the *Logic*—in part because of the *Zusätze*, culled from Hegel's lectures, and added by Leopold von Henning.

The part of the *Logics* which is our major concern here is where Hegel discusses what I am calling logic: the theory of inference. This occurs in Sec. 1, Vol. 2 of the *Logic*, where the first three chapters are: the *Concept*, the *Judgement*, and the *Syllogism*. Hegel structures the general development of concepts as a sequence of triples—or better, triples of triples. Interestingly, the major exception to this is the chapter on Judgement, which is a quartet of triples, one member of the quartet dealing with each of Kant's quality, quantity, relation, and modality.[9]

In these three chapters, Hegel covers much the same ground as Kant covers in his lectures on logic. There are few technical novelties. Where the material mainly differs from Kant, it is in that Hegel dresses up the material in terms of the general story of conceptual dynamics he wishes to tell.

The material in question falls under the topic of what Hegel calls *Subjective Logic* ('subjective' because it deals with individual subjects reason). He contrasts this with what he calls *Objective Logic*, which is the dynamical evolution of the concepts. This (according to Hegel) has a certain pattern. Reflection on a concept produces an opposite concept. Thus, the first concept, *being*, delivers the concept *nothing*. These two then deliver a concept which is said to *aufhebt* the pair. This is a term that is virtually impossible to translate into English, since it can mean both *to preserve* and *to get rid of*. And Hegel means both of these things at once. The third term in the triad resolves the tension between the first two, so to say, by accepting it. Thus, *being* and *nothing* are *aufgehoben* by *becoming*. Things in a state of change both are and are not. At any rate, the new term produces its own opposite, and so the cycle starts anew.

Now, this process has absolutely nothing to do with inference, and so with the sense of logic in this essay. However, it is worth noting that when Hegel's thought was taken up in the Marxist tradition, this sort of development did come to be thought of as delivering a way of reasoning: dialectical logic. Thus, in *Anti-Dühring* and *Dialektik der Natur*[10] Friedrich Engels (1820-1895) argues that formal (Aristotelian) logic is alright as far as it goes; but

[9]It is clear to readers of Hegel that he often struggles to fit material into his procrustean structure. It would appear that, in this case, he just gave up!

[10]*Herrn Eugen Dührings Umwälzung der Wissenschaft*, Leipzig, 1878. Translated as *Anti-Dühring*: *Herr Eugen Dühring*'s *Revolution in Science*, Moscow: Progress Publishers, 1947. Notes for the *Dialektik* were compiled between about 1873 and 1883, but never completed. They were published posthumously (with a Russian translation) Moscow, 1935. This was translated into English as *Dialectics of Nature*, Moscow: Foreign Language Publishing House, 1954.

to reason properly about things in their dynamics, one requires dialectical logic. He even suggested some laws of dialectical logic, such as the *mutual penetration of opposites* (things produce their opposites) and the *negation of the negation* (when the opposite of the opposite arises, it is at a "higher level" than the original). These were never developed into anything like a logic in a sense that a contemporary logician would recognise, however.

Part of the problem was that, even to start to do this, one has allow for the possibility of contradictory situations. Now, the Principle of Non-Contradiction, which says that such things are impossible, has been high orthodoxy since Aristotle defended the view in *Metaphysics* Γ. For Aristotle, the Principle was one of metaphysics, not of logic, but it blocks the way of any attempt to reason about situations that are genuinely contradictory.

Unsurprisingly, in virtue of his views about conceptual development, Hegel criticises and rejects the Principle in the *Logic* (Vol. 1, Bk. 2, Sect. 1): something can be both P and not-P. He was, in fact one of the few (and certainly the most significant) thinkers post-Aristotle and before the present day, to challenge the Principle. One significant feature of contemporary logic is the development of paraconsistent logics. These are logics which, in a certain sense, do not accept the Principle of Non-Contradiction, and which allow for contradictory states of affairs in a non-trivial fashion.[11] Such logics are hardly dialectical logics. They have nothing, as such, to do with zig-zag dialectical developments. However, one might certainly attempt to use the techniques of paraconsistent logic to produce something that is recognisably a dialectical logic—though how one might best do this is moot.[12]

5 No Man's Land

In the decades that followed Hegel's death, German philosophy was in something of a state of turmoil: the influence of Hegel waned, or morphed into the materialism of Feuerbach and Marx; under the influence of science, empiricism and naturalism became highly significant, perhaps threatening make philosophy obsolete; this, in turn, prompted a resurgence of Kantianism. Somewhere in this turmoil was the *Logische Frage* (Question of Logic). The question was, roughly, what to make of logic in a post-Hegelean environment.

The question was put on the table by the person who coined the term in his essay of 1842, 'On the History of Hegel's Logic and Dialectical Method.

[11] See Priest and Tanaka (2009).

[12] On Hegel's logic, see Burbridge (2004). For some steps towards dialectical logic, see Priest (1982) and (1990).

The Logal Question in Hegel's System'[13], Friedrich Adolph Trendelenberg (1802-1872), who followed Hegel in Berlin. In his *Logical Investigations*[14] he argued that Hegel had been right to criticise formal logic for being useless. Logic must always concern itself with content as well as form (a view which, strangely enough, he claimed to find in Aristotle). However, Hegel's panlogical metaphysics could not provide what is required in this regard. How, then to turn this trick? Trendelenberg looked to Leibniz for an answer. (His essay of 1857, 'On Leibniz' Outline of a general Characteristic'[15] may, in fact, be credited with bringing Leibniz back into the purview of German philosophers.) Though he was critical of many of the details of Leibniz' *characteristica universalis*, he argued that what is required for the job at hand is a language which can express our concepts with a precision that natural languages do not do, a *Begriffsschrift* (concept script).

Language was also important for another of Hegel's critics, Otto Friedrich Gruppe (1804-1876), though for him it was natural language that was important. Gruppe rejected all *a priori* philosophy entirely; science had shown that naturalism was the path to progress. This did not mean that logic had to be given up, but it had to be approached in a novel way, via how people use natural language. In his *Turning Point of Philosophy in the Nineteenth Century*,[16] Gruppe argued as follows. Traditionally, logicians had taken concepts to be foundational, and judgements to be made up of thereof. However, this gets things the wrong way around: it is judgements that are primary; concepts are abstracted from these. And what is one to make of the inferences which comprise judgements? An answer to that was given by another naturalist, Heinrich Czolbe (1819-1873). In his *New Account of Sensualism*[17] Czolbe argued that inference (like other facets of language use) were simply matters of empirical psychology—and in the last instance, the laws of physiology.

The philosophical naturalism of writers such as Gruppe and Czolbe generated a reaction, a resurgence of Kantianism. The most important of the neo-Kantians, and arguably the most influential of the writers on logic in these interregnum years, was Rudolph Hermann Lotze (1817-1881). In his two

[13]'Zur Geschichte von Hegels Logik and dialektischer Methode. Die logische Frage in Hegels Systeme', *Neue Jenaische Allgemeine Literatur-Zeitung* 1, 97: 405-8, 98: 409-12, 99: 413-4.

[14]*Logische Untersuchungen*, Berlin: Bethge. 1st ed., 1840; 2nd ed., 1862; 3rd ed., 1870.

[15]'Über Leibnizens Entwurf einer allgemeinen Charakteristik', *Philosophische Abhandlungen der Königlichen Akademie Wissenschaften zu Berlin. Aus dem Jahr 1856*: 36-69.

[16]*Wendepunkt der Philosophie im neunzehnten Jahrhundert*, Berlin: Reimer, 1834

[17]*Neue Darstellung des Sensualismus*, Leipzig, 1855.

books called *Logic*[18] Lotze defended Aristotelian logic on *a priori* grounds. However, he insisted on the distinction between psychological acts of thought, and their objective contents.[19] Logic concerns the latter.[20]

6 Bolzano, a Lone Voice

None of these post-Hegelean developments produced any really novel developments in logic itself, though they certainty created an atmosphere of uncertainty in which new ideas could flourish. And flourish they did. In fact, even in the earlier part of the century such ideas were developing.

Perhaps the most important person in the early such development was Bernard Bolzano (1781-1848). Bolzano was a remarkable person. Working almost entirely in isolation, he developed notably new ideas in logic, mathematics, and philosophy. As far as logic goes, his most significant publication was his *Theory of Science*.[21] As the title of the book indicates, Bolzano was interested in knowledge quite generally, its constitution, ground, and structure. But logic plays the core role in this.

Knowledge is expressed in propositions. But these are not subjective judgements. Rather, propositions are, essentially, the sorts of things that can be the objective contents of declarative statements. And a proposition is true or false, also objectively, depending on whether the world is as it says it to be. Thus, both propositions and their truth depend in no way on actual thinkers, though thinkers may understand them and grasp their truth.

Propositions are made up of ideas. But the ideas are just as objective as propositions. In particular, they are nothing to do with particular thinkers—so *concept* might be a better word for what is intended here. Concepts are the sort of thing that apply to the objects in their extensions. (So *city* applies to New York, Melbourne, Berlin...) We are still working, note, within an Aristotelian framework, so that, e.g., *Aristotle the Stagyrite* is a concept that applies to just one object.

Using the notion of extension, Bolzano characterised a number of important logical relations between concepts. For example:

- *A is compatible with B* just if there are objects which are in the exten-

[18] *Logik*, Leipzig, 1843 and 1874.

[19] He also anticipates two more Fregean themes, if somewhat inconsistently. One is the priority of the judgement over the concept; the other is the similarity between conceptual application and functional application in mathematics.

[20] Further discussion of the matters in this section can be found in Peckhaus (2009) and Sluga (1980), chs. 1 and 2.

[21] *Wissenschaftlehre*, Sulzbach: Seidel, 1837. Translated as George (1972).

sion of both A and B

- *A is included in B* iff A and B are compatible, and the extension of A is contained in the extension of B.

It is worth noting that one might expect a modal element to be present in some of these relations. Thus, one might expect: A is compatible with B if *it is possible that* there are objects which are... . Such an element is, however, absent in Bolzano.

Arguably, Bolzano's most novel contribution to logic was his definition of logical consequence. First, given any proposition, P, fix on some of the concepts which occur in it, $\overrightarrow{a} = a_1, ..., a_n$. Call these *parameters.* Let $\overrightarrow{b} = b_1, ..., b_n$ be a corresponding string of concepts, where each b_i is of the same kind as the corresponding parameter a_i. We can form the proposition $P_{\overrightarrow{a}}(\overrightarrow{b})$ which is obtained by replacing each parameter, a_i, in P with the corresponding b_i. Relative to a bunch of parameters, $\overrightarrow{a}$, we can now mirror the logical relations between concepts with relations between propositions. Thus:

- *P is compatible with Q* just if there is a $\overrightarrow{b}$ such that $P_{\overrightarrow{a}}(\overrightarrow{b})$ and $Q_{\overrightarrow{a}}(\overrightarrow{b})$ are both true.

- *Q is deducible from P* iff P and Q are compatible, and for every $\overrightarrow{b}$ such that $P_{\overrightarrow{a}}(\overrightarrow{b})$ is true, $Q_{\overrightarrow{a}}(\overrightarrow{b})$ is true.

It is to be noted that deducibility holds with respect to a bunch of parameters (so that 'Fred is red' is deducible from 'Fred is coloured' with respect to the parameter *Fred*, since if b is any concept referring to a physical object, if it is true that b is red, then it is true that b is coloured). Bolzano does appear to accept the distinction between what would now be called logical constants (like *if*, *not*) and non-(logical constants) (like *Fred* and *red*)—or to give them their medieval names *syncategorematic terms* and *categorematic* terms—though he offers no principled account of the distinction. But given this distinction, he can frame an absolute notion of consequence, viz. deducibility where the parameters are the non-(logical constants).

Note also that for Q to be deducible from P, P and Q must be compatible. Now, with respect to the parameters which are the non-(logical constants), P is not compatible with 'it is not the case that P'. *A fortiori*, no Q is compatible with 'P and it is not the case that P'. Hence, according to this conception of consequence, contradictions do not entail everything; in fact they entail nothing. The account of consequence was therefore paraconsistent. In fact, though there is probably no way he could have known this, Bolzano was reinventing the connexive notion of logical consequence endorsed by medieval logicians such as Abelard.[22] This account is quite different from contemporary explosive logics, according to which a contradiction entails everything, and even from most contemporary paraconsistent logics, according to which contradictions entail some things but not others.

A pleasing feature of Bolzano's notion of logical consequence is that it allowed him to extend his account of consequence to a non-inductive one. Fix the parameters, $\overrightarrow{a}$, and assume that the possible replacements for each parameter are finite in number. We can define the conditional probability of P given Q, $Pr(Q/P)$, as the number of true things of the form $(P\&Q)_{\overrightarrow{a}}(\overrightarrow{b})$ divided by the total number of true things of the form $P_{\overrightarrow{a}}(\overrightarrow{b})$. Given Bolzano's account, if Q is a consequence of P, then $Pr(Q/P) = 1$. (And this can hold in general only because P and Q are compatible. In particular, then, substituting for *some* parameters makes P true. Hence, the divisor is non-zero.) But the value $Pr(Q/P)$ can, in principle, be any rational number between 0 and 1. So a proposition P may offer some lesser degree of support (or unsupport) for another.

Because of his isolation, Bolzano's work had very little immediate effect on the developments in logic. It first appears to have been noticed late in the century by Franz Brentano and his school. When Brentano's student Kazimierz Twardowski founded what was to become the Lvov-Warsaw school, this knowledge moved there, though developments made by logicians such as Alfred Tarski (né Teitelbaum) were already overtaking it. That story belongs to the history of the 20th Century, however.[23]

7 Schröder and the Algebra of Logic

When one reads Bolzano, it is striking that, though the ideas he is expressing are quite complex, beyond the occasional use of letters for quantities, he makes no use of mathematical symbolism. Matters are quite different with the next two people in our story, Schröder and Frege. Both were professional mathematicians; both used mathematical symbolism freely.

The branch of mathematics called abstract algebra started to blossom towards the end of the 18th century, and developed throughout the 19th. Ernst Schröder (1841-1902) worked squarely in this tradition. In abstract algebras, we are concerned with a bunch of objects and operations on them.

[22] See Priest (1999).

[23] Further discussion of Bolzano and his logic can be found in Sebestic (2011) and Rusnock and George (2004).

Thus, if a, b, and c are objects of our concern, and $+$ and $\times$ are binary operations on the objects, we may form objects such as $(a + b)c$ and $ac + bc$.[24] Relationships between objects are typically expressed by equations, such as $(a + b)c = ac + bc$, and the algebra seeks to determine which relationships of this kind obtain, via a manipulation of these equations (of a kind now familiar from highschool algebra). It is characteristic of an algebra, note, that the objects of the algebra can be thought of as different kinds of things. In other words, the algebra may have more than one natural interpretation. (In the language of modern logic, the algebras are not intended to be *categorical.*) The point, indeed, is to chart the commonalaties of structure between different domains.

Schröder framed the project early in his life of developing an algebra that charted the commonalities of structure between all mathematical quantities, very generally understood—a universal algebra—and applying it to various areas of mathematics and physics. He then came under the influence of two brothers with similar sympathies, Herman Günther Graßman (1809-1877) and Robert Graßman (1815-1901). Soon after this, he discovered the work of the English logician/algebraist George Boole (1779-1848), and a little later, that of the polymath from the United States of America, Charles Sanders Peirce (1839-1914), both of whom made significant contribution to the algebraicisation of logic.

Schröder's first main foray into the area was his *The Circle of Operations of the Logical Calculus*.[25] This was followed by his mammoth *Lectures on the Algebra of Logic*, in three volumes.[26] The second part of Vol. 3 was published posthumously, edited by Karl Eugen Müller.

Volumes 1 and 2 contain an exposition of what would now be called Boolean algebra. In Volume 1, the objects concerned are thought of as classes; in Volume 2, they are thought of as propositions. (A proposition, Schröder notes, following Boole, may be identified with the set of times at which it is true.) There are two special objects, 1 and 0. 1 represents the set of all the objects (times) in the domain of inquiry; 0 represents the empty collection. There are three main operations, union (disjunction), $+$; intersection (conjunction), $\times$; and, complementation (negation), now standardly indicated by an overline: $\overline{a}$ is whatever it is that remains when the members of a are taken away from those in 1. Schröder discusses the relations between

[24]I will often, as is standard in algebra, write things of the form $a \times b$ as ab.

[25]*Der Operationskreis der Logikkalculus*, Leipzig: Teubner, 1877.

[26]*Vorlesungen über die Algebra der Logik,* Leipzig: Teubner. Vol. 1, 1890; Vol. 2 1891; Vol. 3, Pt. 1, 1895; Vol. 3, Pt. 2, 1905. The work has not been translated into English as far as I know. But a modern German version was published with New York, NY: Chelsea, 1966.

these various notions, such as $a + \overline{a} = 1$, $a\overline{a} = 0$. He proves that there is no way of deducing the distribution law, $a(b+c) = ab+ac$, from other standard principles concerning $+$ and $\times$, by showing that the other principles, but not distribution, hold in a structure which would now be called a non-distributive lattice. This may be the first appearance of both such a lattice, and an independence proof in logic. (Independence proofs of this kind had been known in geometry for some time.)

Of special importance is the relation of subsethood (subsumption), $a \leq b$—which Schröder takes as primitive, but which may be defined as $ab = a$. Using this, one may algebraicize standard logical reasoning. Thus, take the syllogism (Barbara): All *a*s are *b*s; all *b*s are *c*s; hence all *as* are *c*s. The premises may be written as $a \leq b$ and $b \leq c$. Operating on these equations by algebraic rules, one may deduce the conclusion, $a \leq c$. Thus, we are given that $ab = a$ and $bc = b$. Hence, $ac = (ab)c = a(bc) = ab = a$. That is, $a \leq c$.

Schröder departs from Boole in small but significant ways. Notably, he interprets $+$ as inclusive. For Boole, $a + b$ is defined only if a and b are disjoint (that is, $ab = 0$). This causes a number of unnecessary complexities. Secondly, Boole needed a way to express the thought that a and b are not disjoint. To do this, he introduced a special symbol, ν, where νa is to be interpreted as some non-deterministically determined non-empty subset of b. The fact that a and b overlap can then be expressed by $\nu a = \nu b$. The notion ν is both of dubious intelligibility and complex to operate with. Schröder does not dispense with ν, but does not need it. Unlike Boole, he operates with inequalities as well as equalities. He can therefore express overlap simply as: $ab \neq 0$.

There are inelegancies in Schröder's own system, though. The symbol '=', and so '$\leq$', does duty for more than one thing. Thus, we find him writing things such as: $(a \leq b)(b \leq c) \leq (a \leq c)$. Here, if the main '$\leq$' is to be interpreted as subsethood, the things on either side of it must be sets. Hence, $a \leq c$, e.g., must be interpreted as $\overline{a} + c$. This is possible because $a \leq c$ iff $\overline{a} + c = 1$. However, the failure to draw this important conceptual distinction betokens an unfortunate confusion.

In Volume 2, and following Peirce, Schröder introduces a notion that may be thought of as quantification. He writes things such as $\sum\limits_i a_i$ to mean the (possibly infinite) sum of all things of the form a_i, where the i can take a value from some predetermined range. Similarly, he writes things such as $\prod\limits_i a_i$ to mean the (possibly infinite) product of all things of the form a_i. If one thinks of i as a free variable, this is some form of quantification. However, in virtue of the algebraic context in which Schröder is working, it arguably makes more sense to take $\sum$ and $\prod$ to be the infinitary generalisations of

+ and ×. If so, the notation is not so much a precursor of the notion of quantification, as that of languages where the formulas can be of infinite length, infinitary logic.[27]

The following is also worth noting. Modern presentations of algebras are axiomatic. That is, axioms concerning the algebra are laid down, and then theorems of the algebra are deduced. In a posthumously published essay, *Outline of the Algebra of Logic* (also edited by Müller),[28] he does offer something like a list of axioms; but in the *Lectures* the algebra is not developed axiomatically.

Volume 2 of the *Lectures* is devoted to the topic of the algebra of relations, developed by Peirce. If the objects in Volume 1 can be thought of as sets, the objects in Volume 2 can he thought of as relations (in modern understanding, sets of ordered pairs). Schröder introduces appropriate operations on these, such as converse, $\breve{a}$, and product, $a.b$ (in modern notation, $x\breve{a}y$ iff[29] yax; and $x(a.b)y$ iff $\exists z(xaz$ and $zby))$, and investigates their properties. It is certainly wrong to take the logic of relations to be unimportant for logic. In a certain sense, traditional logic recognises only monadic properties, not binary relations (or relations of higher arity). The recognition and incorporation of relations into the syntax of logic was a key feature in increasing the power of logic. However, Schröder's main concern in this volume is not so much with the application of the algebra of relations to logic, but to areas such as set theory. Hence, we may pass over this topic here.

There is no doubt that Schröder was an original thinker, and that he made important contributions to the nascent discipline of set theory, as it was being developed by the likes of Cantor and Dedekind. He certainly introduced novelties in logic as well, such as algorithms for operating on systems of equations. However, it must be said that both Boole and Peirce much more original in their thinking about the algebra of logic, and that Schröder's main contribution to this area was in the systematic exposition and polishing of others' thought.[30]

[27]On infinitary logic, see Bell (2012).

[28]*Abriss der Algebra der Logik*, Leipzig: Tuebner. Pt. 1, 1909; Pt. 2, 1910. This may also be found in 1966 version of the *Lectures*.

[29]Logicians' jargon for 'if and only if'.

[30]Discussions of Boole and Peirce can be found in Hailperin (2004) and Hilpinen (2004), respectively. Schröder is discussed in Pekhaus (2004). All three are discussed in Grattan-Guiness (2000), chs. 2 and 4.

8 Frege and *Begriffsschrift*

The same cannot be said of Friedrich Ludwig Gottlob Frege (1848-1925), who must count as one of the most original logicians in its history.

The 19th Century was not only an epoch in which abstract algebra developed. It was also an epoch of increasing rigour in mathematics. In particular, a whole menagerie of kinds of number was known: natural numbers (0, 1, 2), rational numbers (1/2, 3/5), real numbers (π, $0.111\dot{1}$), complex numbers ($\sqrt{-1}$, $2+3i$) infinitesimals (used in the differential and integral calculus); but how exactly to understand these, and even how to operate with them exactly, was not really clear. (It is worth noting that the only branch of mathematics that had received an axiomatic treatments by this time, was geometry.) The 19th Century organised the zoo. Weierstrass and others showed how to do the calculus with appealing to infinitesimals; and they disappeared from the zoo entirely. Argand showed how complex numbers could be understood as pairs of real numbers. Weierstrass, Dedekind, and Cantor showed how real numbers could be seen as sets of rational numbers; and Tannery showed how rational numbers could be seen as sets of pairs of natural numbers.[31] So by the time we arrive at Frege, all the numbers could be seen as set-theoretic constructions out of the natural numbers. But what of the natural numbers themselves? Frege set out to show that they could be seen as constructions out of just sets, and, moreover, that set theory was simply part of logic.

To do this, he needed a language to express his ideas clearly, and, moreover, to draw inferences employing his concepts in a clear and rigorous way. Traditional logic was up to neither of these tasks. He had read Trendelenberg, Boole, and Lotze, but in none of them did he find what he needed. So he invented it, and called it Begriffsschrift. This was published in his *A Formula Language of Pure Thought Modelled upon the Formula Language of Arithmetic*[32]—known nowadays simply as the *Begriffsschrift*—a book that barely exceeds 100 pages in modern edition. Two subsequent books made the mathematical application of the language/logic that Frege envisaged; and a number of later essays articulated many of the philosophical ideas underpinning it. Three of the most important of these are 'Function and Concept', 'On Sense and Reference', and 'On Concept and Object'.[33]

[31]See Priest (1998).

[32]*Begriffsschrift, eine der arithmetischen nachgebildete Formalsprache des reinen Denkens*, Halle: L. Nebert, 1879. Translated as Bynum (1972).

[33]*Funktion und Begriff,* Jenna: H. Pohle, 1891, 'Über Begriff und Gegenstand', *Vierteljahrsschrift für wissenschaftliche Philosophie*, 16: 192-205 (1892). 'Über Sinn und Bedeutung', *Zeitschrift für Philosophie und philosophische Kritik*, 100: 25-50 (1892). All three

The sentences of Frege's formal language and their component parts were taken to have objective content, as for Lotze (and Bolzano). If A is a formula of the language, Frege writes $-A$ for its content. A vertical line indicates that a content is judged to be true. So $\vdash A$ means that the content of A is judged true. Psychology is thus separated from content right at the start. In 'Sense and Reference', and in an attempt to explain why, e.g., 'Hesperus is Hesperus' has a different content from 'Hesperus is Phospherus', even though 'Hesperus' and 'Phosphorus' refer to the same object, Frege comes to advocate a bicameral theory of content. Sentences and their parts have both a sense (Sinn) and a reference (Bedeutung). This does not play a role in the *Begriffsschrift*, however, content operating purely on the level of reference.

Sentence of the Begriffsschrift are constructed from basic (atomic) sentences. In a major break with the Aristotelian tradition, these are not necessarily of subject/predicate form. They are constituted by a verb phrase and the appropriate number of noun phrases, thus, e.g.: Sm, Ljm (which might express the claims, respectively, that Mary sings and that John loves Mary). (In the symbolism, and conventionally, the verb phrase is written at the start of the sentence.) The objective content of a noun phrase is the object it denotes. The objective content of a verb phrase, Frege calls a *concept*. This is a function in the mathematical sense. There are two special objects called *truth values*: the true, t, and the false, f. The content of a verb phrase is a function that maps the appropriate number of objects to one of these. Thus, the content of 'S' might be a function that maps an object to t iff that object is singing. And the content of 'L' might be a function that maps a pair of objects to t iff the first loves the second. The content of the whole sentence is the truth value you get when you apply the function which is the content of the verb phrase to the objects which are the contents of the noun phrases.

The rest of the sentences in the Begriffsschrift are generated from the atomic sentences by applying various grammatical constructions, which can be iterated recursively. The first kind of construction comprises connectives: $\neg$ (it is not the case that), $\supset$ (if ... then ..[34]), $\wedge$ (and), $\vee$ (or). (Some of these can be defined in terms of others; Frege takes $\neg$ and $\supset$ as basic.) Traditional logic (though not Medieval logic) recognises only two of these ($\vee$ and $\supset$) and does not iterate them. But Frege, following the algebraists, and mindful of what mathematicians need to express, was well aware that if makes perfectly good sense to say things of the form $A \supset (B \supset C)$.

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essays are translated into English in Geach and Black (1952).

[34]Though it was orthodox to read the symbol in this way in the first half of the 20th Century, this is highly problematic. Frege, is very careful not to read it like this. His gloss is more like: it is not the case that (... and not ...).

The objective content of a connective is a function, and the content of a sentence formed by a connective applied to some sentences is obtained by applying the function which is the content of the connective to the truth values which are the contents of the sentences. Thus, the content of $\neg$ is a function which maps t to f and vice versa. The content of $\supset$ is a function that maps the pair $\langle a, b\rangle$ to f iff a is t and b is f; other pairs of truth functions get mapped to the value f.

The other kind of grammatical construction involved in generating complex sentences comprises quantifiers. This constitutes another, and perhaps the most significant, break from traditional logic. For Aristotle, quantifier phrases such as 'some man' and 'no woman' are of the same grammatical kind as noun phrases such as 'John' and 'Mary'. But once relations enter the picture, this leads to problems. Thus, 'every man loves some woman' is ambiguous, depending on whether it means 'every man loves some woman or other' (maybe his mother), or it means that there is some woman whom every man loves (same woman in each case, maybe the Virgin Mary). How to account for this ambiguity?

Given any sentence, $A(n)$, containing a noun phrase, n, we can replace this with a variable, x, to obtain $A(x)$. We can then prefix this with a quantifier phrase $\forall x$, $\exists x$ (all x are such that, some x is such that). $\forall x A(x)$ is true (i.e., has the content t) just if whatever object we were to take x to refer to, $A(x)$ would be true. Similarly, $\exists x A(x)$ is true just if there is some object we can take x to refer to which would make $A(x)$ true. The ambiguity noted is then explained by the order in which the quantifier phrases are applied. Thus, the difference is that between $\forall x \exists y Rxy$ and $\exists y \forall x Rxy$.

It is worth noting that this sort of ambiguity had played havoc in mathematics in the period leading up to Frege. A (real-valued) function, f, is continuous (smooth) if *for every ε, however small, some δ is such that* if you make the difference between x and y less than δ, the difference between $f(x)$ and $f(y)$ will be less than ε. Note (as the italics show) that this is one of those ambiguous sentences containing a universal and a particular quantifier. The ambiguity corresponds to the difference between (plain) continuity and uniform continuity.[35] These two notions have somewhat different mathematical properties, and mathematicians had been befuddled by the difference, though they had got it straight by Frege's time. It may well be that Frege realised the need for his analysis of quantifiers by reflecting on this kind of situation.

The sort of quantifiers I have been talking about so far are first-order,

[35] The precise details are a bit more complicated than this, but this is right enough for the present context.

where we quantify over objects. The Begriffsschrift also has second-order quantifiers, where we quantify over concepts. Given any sentence, $A(N)$, containing a verb phrase, N (let us suppose that this is monadic, to keep things simple), we can replace this with a different kind of variable, X, to obtain $A(X)$. We can then prefix this with a quantifier phrase $\forall X$ or $\exists X$. $\forall X A(X)$ is true just if whatever concept we take X to refer to, $A(X)$ is true; and $\exists X A(X)$ is true just if there is some concept we can take X to refer to which makes $A(X)$ true.

A word on notation. I have, in discussing Frege, as for the other thinkers I have discussed, used contemporary notation. Frege's actual notation in the *Begriffsschrift*, though, is highly unusual. (An example is given in Fig. 1.) In particular, it is two-dimensional. Thus he writes $A \supset B$ as a horizontal line with B at the right-hand end of it; descending from the horizontal, there is a capital 'L' shape, with B at the bottom right-hand end of it. This is hard for most people with a standard training in mathematics to read, and it did not catch on. (The notation currently in use derives essentially from that developed by the Italian mathematician Giuseppe Peano.)

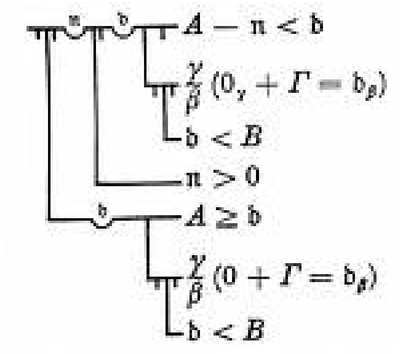

Fig. 1. An example of Begriffsschrift

So much for the language of the Begriffsschrift. In virtue of its contents there will be some sentences that are true whatever the noun phrases and verb phrases in them refer to. These are the logical truths. Frege provided an axiom system for these. He specified a number of axioms, and rules of inference for inferring one sentence from another.[36] For example, one axiom was $A \supset (B \supset A)$, and *modus ponens* was a rule of inference: from A and $A \supset B$ infer B. Frege was quite clear that axioms and rules of inference

[36] Strictly speaking, both axioms and rules were schemas. Ignore this if you do not know what it means.

are different kinds of things. Logicians, even of the stature of Russell, however, standardly confused them until Hilbert and his school systematised the theory of formal systems in the 1920s.

Remarkably, it later turned out that Frege's axiom system was complete. That is, if we ignore formulas with second order quantifiers, everything that is logically valid can be proved in Frege's axiom system. The result was proved by Gödel in the 1920s. Frege had no way of addressing this question, though, or even of framing it properly, since it depends on a notion of validity developed only in the early 20th century, essentially by Tarski (but pretty much that of Bolzano). A corollary of another of the significant results proved in the 1930s Gödel established that once second order quantifiers are on board, no axiom system can do this completely. All this, however, belongs to the logical history of the 20th century.

As the 19th Century itself was coming to an end, we find Schröder and Frege debating which of them could best claim to have inherited Leibniz' logical mantle.[37] The answer, in the end, is, it seems to me, mainly of interest to Leibniz scholars: the facts about what each of them had acheived are clear enough.[38]

9 Conclusion

History rarely runs smoothly. As Hegel observed, the cunning of reason has strange ways of its own. Frege's major project, to show that the truths about whole numbers, and hence about all sorts of numbers, were part of logic, crashed and burned spectacularly, due to the discovery of what has come to be known as Russell's paradox. But nothing of this bore on the success of the *Begriffsschrift* in its own right. What was supposed to be but a means to an end turned out to be perhaps the most significant development in two milenia of logic. And even here: Frege's work was transmitted into the 20th Century by Russell and Wittgenstein, and their overlay served to obscure it. It was not until the middle of the 20th Century that Frege's acheivements came to be generally appreciated.

Of course, nothing comes from nothing. And the developments Frege produced would have been impossible without all that had gone before, including the work of Leibniz, the turmoil in logic post-Hegel, the work of the algebraists, and developments in 19th Century mathematics. By the end of

[37]See Peckhaus (2004), 9.4.

[38]Discussions of Frege's logic can be found in Sullivan (2004), Zalta (2012), and Grattan-Guiness (2000), ch. 4. The later chapters of last of these also discusses developments in early 20th century logic.

the century, however, the third great phase in the development of Western logic was well set in place. The 19th Century had witnessed a fundamental rupture in logical history; and German thought had played a major role in this.

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