Chapter 24
Kripke's Thought-Paradox and the 5th Antinomy

Graham Priest

Abstract In 'A Puzzle about Time and Thought' Saul Kripke published a new paradox. The paradox is clearly a relative of Russell's paradox; but it deploys, as well as the notion of set, an intentional notion, *thought*. This ensures that it raises significantly different issues from Russell's paradox. Notably, the solution to Russell's paradox provided by ZF does not apply in any obvious way to this paradox.

In this paper I will first explain Kripke's paradox and compare it with another paradox which deploys the notion of thought. I will then show that it fits the Inclosure Schema, and so may be expected to have a solution which is the same as other inclosure paradoxes. Next, the paradox is stripped down to a much more acute form. Finally, in the light of this, some thoughts concerning possibilities for resolving the paradox are offered.

24.1 Introduction

In 'A Puzzle about Time and Thought'[1] Saul Kripke published a new paradox. The paradox is clearly a relative of Russell's paradox; but it deploys, as well as the notion of set, an intentional notion, *thought*. This ensures that it raises significantly different issues from Russell's paradox. Notably, the solution to Russell's paradox provided by ZF does not apply in any obvious way to this paradox.

In this paper I will first explain Kripke's paradox and compare it with another paradox which deploys the notion of thought. I will then show that it fits the Inclosure Schema, and so may be expected to have a solution which is the same as other inclosure paradoxes. Next, the paradox is striped down to a much more acute form. Finally, in the light of this, some thoughts concerning possibilities for resolving the paradox are offered.

[1] Kripke (2011).

G. Priest
Departments of Philosophy, The Graduate Center, City University of New York, New York, USA, and the University of Melbourne, Australia
e-mail: priest.graham@gmail.com

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24.2 Puzzles About Thought

A number of paradoxes of self-reference employ intensional notions. In what follows, it will be helpful to have in mind one particular one of these.[2]

Dedekind proved the existence of an infinite set as follows. Take any object you like—for the sake of illustration, let us say the empty set; then there is a thought of it; a thought of a thought of it; a thought of a thought of a thought of it; and so on. The resulting totality is infinite.[3] But we can iterate the process. Let s be the set of all these thoughts. Then there is a thought of s; a thought of a thought of s; and so on. We can iterate the process indefinitely. Indeed, we can simplify the process, at the same time as making it uniform. Suppose we index the stages by ordinals. Let us write tx for 'the thought of x'. Then we can define a sequence, f, as follows: $f(\alpha) = t\{f(\beta) : \beta < \alpha\}$. Each thought generated in this sequence is distinct—just as the ordinals are. Let Π be the totality of all thoughts generated in this way.[4] Then there is no such thing as $t\Pi$. If there were, it would be the next member of the series, which, *ex hypothesi*, there is not. But *there is* a thought of Π. Indeed, you have just had it.

In *Beyond the Limits of Thought*,[5] and with reference to Kant, I called this paradox the *5th Antinomy*.[6] The paradox is obviously, in some sense, an intentional version of Burali-Forti's paradox. However, it does no good to try to solve it, as one solves the Burali-Forti paradox in ZF, by saying that Π does not exist. Notoriously, there can be thoughts of non-existent objects, such as Pegasus and Sherlock Holmes.

Now to Kripke's paradox, which is as follows. Let k be the set of all times at which I am thinking of a set of times of which that time itself is not a member. There is a time at which I think of k. (There has just been such a time.) Let this be τ. Then by familiar reasoning, τ is both in and not in k.

Let us spell out the paradox more formally. Let $\theta_t x$ be 'I am thinking about x at time t. Let T be the set of all times. Then $k = \{t \in T : \exists s \subseteq T(\theta_t s \wedge t \notin s)\}$. We are given that, at τ, k is the one and only thing I'm thinking about: $\theta_\tau x \leftrightarrow x = k$. Suppose that $\tau \notin k$; then, since $k \subseteq T$, $\exists s \subseteq T(\theta_\tau s \wedge \tau \notin s)$, namely, when s is k; that is, $\tau \in k$. So $\tau \in k$. But then $\exists s \subseteq T(\theta_\tau s \wedge \tau \notin s)$. And since the $\theta_\tau(s)$ entails that $s = k$, $\tau \notin k$.[7]

We can give an equivalent formulation of the paradox using definite descriptions. Let us use ι as a definite description operator, and E as an existence predicate (so that

[2] For a general discussion of intensional paradoxes, see Priest (1991).

[3] Dedekind (1888), Theorem 66. 'Thought' here means content, not act. Actual thoughts must give out after some finite time.

[4] Presumably, there are ordinal-many; but it doesn't really matter if the sequence peters out before the ordinals are exhausted.

[5] Priest (2002), hereafter, BLoT.

[6] BLoT, 6.9.

[7] I note that there is also a Curried version of the paradox. Let $k' = \{t \in T : \exists s \subseteq T(\theta_t s \wedge (t \in s \rightarrow \bot))\}$. Then reasoning in a natural way, one establises that $\tau \in k' \rightarrow (\tau \in k' \rightarrow \bot)$, and hence, by contraction, that $\tau \in k' \rightarrow \bot$. It follows that $\exists s \subseteq T(\theta_\tau s \wedge (\tau \in s \rightarrow \bot))$, that is, $\tau \in k'$; hence $\bot$.

$E(\iota x A)$ means $\exists y \forall x(A \leftrightarrow x = y)$. Then $k = \{t \in T : E(\iota x\theta_t x) \wedge t \notin \iota x\theta_t x\}$. Hence, $t \in k \leftrightarrow (E(\iota x\theta_t x) \wedge t \notin \iota x\theta_t x)$. We are now given as a premise that $k = \iota x\theta_\tau x$. *A fortiori*, $E(\iota x\theta_\tau x)$. Then $\tau \in k \leftrightarrow \tau \notin k$, and so we have a contradiction.

And just as with the 5th Antinomy, one cannot deploy the resources of standard set theory to solve it. In particular k is a subset of times (that is, we may suppose, a subset of the real numbers), and so is not an absolutely infinite set. One cannot, therefore, say that k does not exist on this ground, as one says of Russell's set in ZF.

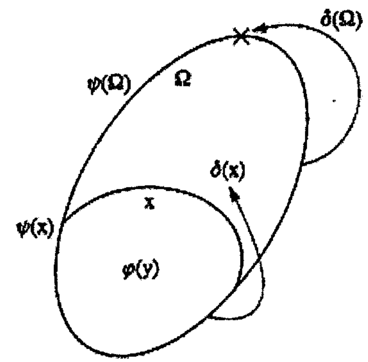
24.3 The Inclosure Schema

All the standard paradoxes of self-reference are *inclosure paradoxes*. Such paradoxes are ones that fit the Inclosure Schema. This it is which provides their underlying structure.[8] The Schema concerns an operator, δ, and conditions, $\varphi(y)$ and $\psi(x)$, such that, where $\Omega = \{y : \varphi(y)\}$, it appears to be the case that:

- Ω exists and $\psi(\Omega)$ (Existence)
- For all x such that $\psi(x)$ and $x \subseteq \Omega$:
  - $\delta(x) \notin x$ (Transcendence)
  - $\delta(x) \in \Omega$ (Closure)

Let is write $A \wedge \neg A$ as $A!$ The paradox arises when we take Ω itself for x. Then Transcendence and Closure give $\delta(\Omega) \in \Omega!$

The Scheme may be depicted as follows:



Applying δ to Ω gives an object that is both within and without Ω (depicted as on its boundary).

Kripke's paradox is an inclosure paradox too. In this:

- $\varphi(t)$ is $\exists s \subseteq T(\theta_t s \wedge t \notin s)$

[8] BLoT, 9.4, 11.0, 17.2.

- $\psi(x)$ is $\exists t \forall y(\theta_t y \leftrightarrow y = x)$
- $\delta(x)$ is $\varepsilon t \forall y(\theta_t y \leftrightarrow y = x)$

Given that $\psi(x)$, $\delta(x)$ is well-defined. That is, we have: $\forall y(\theta_{\delta(x)} y \leftrightarrow y = x)$.[9]

Checking the conditions of the Inclosure Schema: $\Omega = \{t \in T : \exists s \subseteq T(\theta_t s \wedge t \notin s)\}$. Being a subset of T, Ω exists, and we are given that $\psi(\Omega)$. (Existence). Suppose that $\psi(x)$ and $x \subseteq \Omega$. If $\delta(x) \in x$ then $\delta(x) \in \Omega$. That is, $\exists s \subseteq T(\theta_{\delta(x)} s \wedge \delta(x) \notin s)$. But since $\forall y(\theta_{\delta(x)} y \leftrightarrow y = x)$, this s is x. That is, $\delta(x) \notin x$. Hence, $\delta(x) \notin x$. (Transcendence). Thus, $\theta_{\delta(x)} x \wedge \delta(x) \notin x$. And since $x \subseteq \Omega \subseteq T$, we have $\exists s \subseteq T(\theta_{\delta(x)} s \wedge \delta(x) \notin s)$. That is, $\delta(x) \in \Omega$. (Closure.) The paradox is that $\delta(\Omega) \in \Omega$! In the presentation of Kripke's paradox above, $k = \Omega$ and $\tau = \delta(k)$.

The following tabulates the components of the Inclosure Schema for various paradoxes for comparison:

	$\varphi(y)$	$\psi(x)$	$\delta(x)$
Burali-Forti	y is an ordinal	$x = x$	$\mu z \forall y \in x\, z > x$
König	y is a definable ordinal	x is definable	$\mu z \forall y \in x\, z > x$
5th antinomy	$\exists \alpha\, y = f(\alpha)$	$x = x$	ιx
Russell	y is a set	$x = x$	$\{z \in x : z \notin z\}$
Kripke	$\exists s \subseteq T(\theta_y s \wedge y \notin s)$	$\exists t \forall y(\theta_t y \leftrightarrow y = x)$	$\varepsilon t \forall y(\theta_t y \leftrightarrow y = x)$

For good measure (and future reference) I have thrown in König's paradox. The number of definable ordinals is countable. There must therefore be a least. By definition, this is indefinable; but it is defined by the 'the least indefinable ordinal'.

24.4 Stripping Down the Paradox

Kripke's paradox is clearly modelled on Russell's. However, what we will now see is that it can be stripped-down and simplified in such a way as to be much more acute.

First, note that the contradiction is generated by a thought that is being had at time τ (now). That thoughts might also be had at other times is irrelevant. Hence, we can drop the subscript from $\theta_t x$, and just understand θx as: x is the one and only set being thought of (now). k then becomes: $\{t \in T : \exists s \subseteq T(\theta s \wedge t \notin s)\}$ (the set of all times which are not members of the set I am thinking about now). Let it be given, again, that I am thinking (now) about just this set: $\theta y \leftrightarrow y = k$. Let t be *any* time. Suppose that $t \notin k$. Then clearly $\exists s \subseteq T(\theta s \wedge t \notin s)$, namely, when s is k itself.

[9] ε is an indefinite description operator. Since times are not well-ordered, one cannot assume that any non-empty set of times has a first member. But thinkers being finite, if there are any times at which I think of just x, there is, presumably, a first. The indefinite description could therefore be traded in for a definite description.

That is, $t \in k$. So $t \in k$. But then $\exists s \subseteq T(\theta s \wedge t \notin s)$. And since $\theta(s)$ entails that $s = k$, $t \notin k$.

Next, note that the fact that it is a set of times which is being thought about, is irrelevant: it could be people (as Kripe indicates), numbers, or any other kind of thing. Indeed, it could be a very small set—one which is either ϕ (0) or $\{\phi\}$ (1). Let us redefine k in this way, as follows: $\{n < 1 : \exists s \subseteq 1(\theta s \wedge n \notin s)\}$. Given, again, that $\theta x \leftrightarrow x = k$, we have:

(*) $0 \in k \leftrightarrow (\theta k \wedge 0 \notin k)$

and then $0 \in k$ and $0 \notin k$. Hence either $0 \in 0$ or $0 \notin 1$.

I note that the stripped-down version of the paradox still fits the Inclosure Schema: $\phi(y)$ is $\exists s \subseteq 1(\theta s \wedge y \notin s)$; $\psi(x)$ is $\forall y(\theta y \leftrightarrow y = x)$; $\delta(x) = 0$. The details are easy to check.

24.5 Solving the Paradox

In this final section, I will make some comments on solving Kripke's paradox, and in particular, its stripped-down form. As we have seen, these paradoxes are inclosure paradoxes. Since all inclosure paradoxes are of the same form, they require the same sort of solution (the Principle of Uniform Solution).[10] This imposes tight constraints on acceptable solutions.

Let us start with the two possible solutions which Kripke himself mentions (though he does not commit himself to either of these). The first is some form of ramification. Specifically, in his version of the paradox (similar comments can be applied to the stripped-down version) the predicate θ_t is ramified into a hierarchy of predicates θ_t^n. The definition of k then becomes $\{t \in T : \exists s \subseteq T(\theta_t^n s \wedge t \notin s)\}$. Since k is specified by a predicate of order n, any thought of it must be of order $n + 1$. Hence we have only $\theta_t^{n+1} x \leftrightarrow x = k$, and the argument is broken. Ramification has many and well known problems, which it is out of place to discuss here.[11] Here I note only that ramification will not solve the 5th Antinomy. The construction in this "blows the top" off all ramification.

The second possible solution which Kripke notes is to use a non-classical logic of the kind described in Kripke (1975). This breaks the paradox by rejecting the Law of Excluded Middle (LEM). Arguably, the most sophisticated extant account of this kind is that given by Field (2008). Again, this is not the place to discuss the problems of this kind of approach in detail.[12] I note here, again, only that the approach does not seem to be able to handle the 5th Antinomy, since this does not employ the LEM.[13]

[10] This is argued in BLoT, Chap. 11.

[11] See, e.g, BLoT, Chaps. 9 and 10.

[12] See Priest (2006), Chap. 1, and Priest (2010a).

[13] Or other paradoxes that do not employ the LEM, such as Berry's and König's paradoxes. See Priest (2010a), Sect. 6. Note that the 5th Antinomy does not even deploy the least-number principle, as these do.

Another possible solution is to deny the empirical premise of the paradoxes: that there was a time when I was thinking of the one thing, k. This is the line taken by Prior (1961) in another paradox involving thought. This is a version of the Liar paradox concerning a thought of the form 'Every thought being had in Room 7 at the present moment is false', where that is the only thought being had in Room 7 at that time. Prior avers that, much as one might suppose otherwise, no such unique thought is being had. This strikes me as a move of desperation. When a mathematician says 'Think of the set of prime numbers. I will show you that it is infinite', it is clear that their utterance gets us to think of the set of prime numbers. So it is with k. (k does not even have to exist, note, to be thought of.) Could it be that when I think of k I think of something *else* as well? This can hardly be the case: it is of the nature of intentionality to be directed at a single target. Moreover, the assumption of uniqueness plays no role in the 5th Antinomy, so this move is of no avail.[14] In any case, and again, the whole line obtains no purchase with respect to the set theoretic paradoxes, such as Russell's and Burali-Forti's.

Finally, to dialethic solutions. In BLoT, Part 3, I argued that all inclosure paradoxes require such a solution. There had better, therefore, be such a solution to Kripke's paradox and its stripped-down form. A dialetheic solution to the paradoxes involves accepting a contradiction delivered by the paradoxical argument. In many cases, this may be the explicit paradoxical conclusion, but it need not be. Sometimes the argument may deliver only a disjunction of contradictions.[15] In the case of Kripke's paradox, the most obvious dialetheic thought is that the time τ is both in and not in k. Assuming that we identify times with the real numbers, this shows us that a certain set of real numbers is contradictory. Perhaps that is palatable. But in the stripped-down version of the paradox, the corresponding thought is that a small finite set—one which has either no members of whose only member is the empty set—is contradictory. This would appear to be much harder to swallow.

At this point, we need break the discussion into two cases. One may formulate the axioms of naive paraconsistent set theory using either a detachable conditional or a material non-detachable conditional.[16] Suppose that we adopt the first of these possibilities, and take the comprehension principle in a completely unrestricted form, where the set being specified can occur in the specification itself (building in the possibility of a certain fixed point).[17] Then every non-empty set, a, has an inconsistent subset. Thus we have: $x \in c \leftrightarrow (x \in a \wedge x \notin c)$. Clearly, $c \subseteq a$. And given that

[14] For a more general critique of Prior, see Priest (1991).

[15] Thus, in the case of the sorites paradox, the argument forces the conclusion that at least *one* of the objects in the sorites progression is contradictory. See Priest (2010b). More generally, see BLoT, p. 130, fn. 7.

[16] Both possibilities are considered in Priest (2006), Chap. 18.

[17] This form of the principle is found in Routley (1977) and Weber (2010). It is known to be very powerful, but non-trivial.

$z \in a$, we can infer that $z \in k$ and $z \notin k$. It may not, then, be surprising that k has an inconsistent subset (itself).[18]

However, I think a more plausible solution involves taking the second option in formulating paraconsistent set theory, where the conditional is material. In that case, the biconditional (*) delivers only $0 \in k! \vee \neg\theta k$; and then the empirical premise gives us $0 \in k! \vee \theta k!$. We can put the contradictory blame on the second disjunct: k both is and is not being thought about. This is exactly the case in the 5th Antinomy, of course. There is no thought of the totality, Π; *a fortiori* it cannot be thought about; but, by inspection, one can.

There are other precedents. Consider König's paradox. This concerns definability, not thought; but these are not that different: thinking about something can be taken to be bringing before the mind an appropriate noun-phrase which refers to it.[19] Moreover, the Denotation schema involved in König's paradox, 'n' defines $x \leftrightarrow x = n$, is the analogue of the condition $\theta(x) \leftrightarrow x = k$ involved in Kripke's paradox. And given that confluence, the paradoxes have similar shape. In both paradoxes, that a certain object *cannot* be thought about or named is established by a theoretical argument; and that it *can* be thought about or named is established by "direct inspection".

Kripke's paradox and the second dialetheic solution therefore seem in good company.[20]

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[18] It is important to note that in this version of paraconsistent set theory, sets that are extensionally equivalent, in a certain sense, may be distinct. Thus, one may be able to show that there is nothing satisfying either $\varphi(x)$ or $\psi(x)$. But this does not suffice to establish that $\forall x(\varphi(x) \leftrightarrow \psi(x))$, and so that $\{x : \varphi(x)\} = \{x : \psi(x)\}$. In particular, then, the set I have labelled '1' need not be the same as the von Neumann ordinal 1.

[19] The connection between thinking and referring is noted in BLoT, 4.8.

[20] Versions of this paper were given at a one day workshop of the Melbourne Logic Group in August 2011, a conference on Kripke's *Philosophical Troubles*, at the Graduate Center, CUNY, in September 2011, and to the Logic Group at the University of Indiana, Bloomington, in October 2011. Thanks go to the people in those audiences for their thoughts, and particularly to Colin Caret, Lloyd Humberstone, and, especially, Zach Weber.

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