Read on Bradwardine on the Liar

April 15, 2015

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Dedication: It is the greatest of pleasures to contribute this essay to my oldest philosophical colleague, co-author, and friend, Stephen Read, and to contribute it to this celebration of his work. Steve is an accomplished logician, shrewd philosopher, and medievalist *extraordinaire*. For the best part of 40 years now (gulp...), we have been cooperating—writing, reading each other's work, discussing, agreeing, disagreeing, attending conferences and organising reseach projects together. In the process, I have learned an enormous amount from him. In particular, he it was who introduced me in St Andrews to the fascinating world of medieval logic in the first year of my first philosophy position, and who has continued to show me its depths and sophistication ever since. Thank you, Steve. Dedicating this essay to him is a small and inadquate token of my gratitude.

1 The T-Schema and its Bradwardinization

In a number of recent papers, Stephen Read has revived a solution to the Liar paradox proposed by the 14th Century philosopher Thomas Bradwardine, phrasing it in modern terms and defending it.[1] A generally accepted principle concerning truth is the T-schema:

$T\langle A\rangle \leftrightarrow A$

[1] See, e.g., Read (2002), (2006), (2008a), (2008b).

where A is any non-indexical sentence, and $\langle A \rangle$ is its name. With standard techniques of self-reference, we can find a sentence, L, of the form $\neg T\langle L \rangle$, so that:

$$T\langle L \rangle \leftrightarrow \neg T\langle L \rangle$$

A modicum of logic then delivers the contradiction $T\langle L \rangle \wedge \neg T\langle L \rangle$. Read rejects the T-schema in favour of:

$$T\langle A \rangle \leftrightarrow \forall p(\langle A \rangle : p \rightarrow p)$$

where '$\langle A \rangle : p$' is to be understood as '$\langle A \rangle$ says that p'. Let us call this the Bradwardinized form of the T-schema. The thought here is that (an unambiguous) $\langle A \rangle$ may say many things, and for it to be true, *all* of them must hold. We may assume that $\langle A \rangle$ says that A, $\langle A \rangle : A$. Let C be the conjunction of all the other things that it says; then what the Bradwardinized form of the T-schema is telling us is that:

$$T\langle A \rangle \leftrightarrow (A \wedge C)$$

The left-to-right direction of the T-schema is clearly forthcoming; but not the right-to-left. Moreover, taking A to be L gives us:

$$T\langle L \rangle \leftrightarrow (\neg T\langle L \rangle \wedge C)$$

It quickly follows that $\neg T\langle L \rangle$, but then also that $\neg C$. The route back to $T\langle L \rangle$ is blocked.

Read's solution has been discussed at some length in the essays in Rahman, Tulenheimo, and Genot (2008), and solutions of this general kind are also discussed in Field (2007), ch. 7. In this paper, I want to table a couple of other points about the solution—both of them objections.

2 Denotation Paradoxes

First, if the solution is to be a robust one, it must apply to all paradoxes in the same family: same sort of paradox, same sort of solution.[2] How wide that family is, and, in particular, whether it extends to the set theoretic paradoxes,[3] may be moot. But it would be generally agreed that the family contains the paradoxes concerning satisfaction, denotation, and other semantic notions.

The naive S (satisfaction) schema is:

[2] See Priest (2002), 11.5.

[3] As argued in Priest (2002), Part 3.

$$\forall x(xS\langle A\rangle \leftrightarrow A_y(x))$$

where xSy means that x satisfies y,[4] A is any formula of one free variable, y, and $A_y(x)$ is the result of substituting x for all free occurrences of y (subject to preventing clashes with bound variables). Substituting $\neg ySy$ for A quickly leads to the Heterological paradox. The "Bradwardinized" version of the S-schema is, presumably:

$$\forall x(xS\langle A\rangle \leftrightarrow \forall P(\langle A\rangle : P \to Px))$$

where the second-order quantifiers range over properties, and '$\langle A\rangle : P$' means that 'A' expresses P. It is easy to check that the Heterological paradox is now solved in much the same way as the Liar.[5]

The paradoxes of denotation are, however, a very distinctive sub-family of the semantic paradoxes;[6] distinctive enough that solutions proposed to other paradoxes of self-reference do not necessarily carry over to them.[7] So it is here. These paradoxes depend, in the first instance, on the naive D (denotation) schema:

$$\forall x(\langle n\rangle Dx \leftrightarrow n = x)$$

where xDy means that x denotes y, and n is any name. The Bradwardized form of this, by analogy with the cases for truth and satisfaction, is:

$$\forall x(\langle n\rangle Dx \leftrightarrow \forall y(\langle n\rangle : y \to y = x))$$

It is less than clear how to read '$\langle n\rangle : y$' here. In the cases of truth and satisfaction, the paradoxes are blocked by supposing that sentences and predicates have "excess content". By analogy, we must suppose that names have "excess content" in a similar way. '$\langle n\rangle : y$' expresses the thought that y is part of the content of the name n, whatever we should take this to mean. I leave it to those sympathetic to the solution to make sense of this idea.[8]

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[4] Where this may be taken to entail that y is a formula.

[5] Thus, we may suppose that $\langle A\rangle : \lambda yA$. If the other properties expressed by $\langle A\rangle$ are $\{P_i : i \in I\}$, let C be $\bigwedge_{i\in I} P_i y$. Then we have: $xS\langle A\rangle \leftrightarrow (\lambda yA(x) \wedge \lambda yC(x))$, so $xS\langle A\rangle \leftrightarrow (A_y(x) \wedge C_y(x))$. The left-to-right direction of the S-schema is forthcoming, but not the right-to-left. Now taking $\neg ySy$ for A gives $xS\langle\neg ySy\rangle \leftrightarrow \neg xSx \wedge C_y(x)$. Let t be $\langle\neg ySy\rangle$. Substituting this for x, we have $tSt \leftrightarrow \neg tSt \wedge C_y(t)$. $\neg tSt$ follows, but so does $\neg C_y(t)$, and the path back to tSt is blocked.

[6] First, they use descriptions essentially. Second, the argument for each of the contradictory conjuncts, C and $\neg C$, does not go by way of establishing $C \leftrightarrow \neg C$: an independent argument is given for each horn or the contradiction.

[7] See Priest (2006a).

[8] Actually, there is a way of avoiding the issue. We can define $\langle n\rangle Dx$ as $xS\langle v = n\rangle$. The Bradwardinised S-schema then gives $\forall x(\langle n\rangle Dx \leftrightarrow \forall P(\langle v = n\rangle : P \to Px)$. It is then the condition $v = n$ that has "excess content". But it seems rather arbitrary to insist that D *must* be a defined notion.

Given that $\langle n \rangle : n$, it follows from the Bradwardinized D-schema that $\langle n \rangle \, Dx \to x = n$. But we don't have the converse. In particular we can not get $\langle n \rangle \, Dn$ from $n = n$. Now, if one consults a formal proof of the simplest of the denotational paradoxes, Berry's, then one can see that the two claims about denotation required for the proof are that for a certain descriptive term, t, (i) $\langle t \rangle \, Dx \to x = t$ and (ii) $\langle t \rangle \, Dt$.[9] One may therefore use the proof as a *reductio* to establish that $\neg \langle t \rangle \, Dt$.

I confess that this strikes me as something of a *reductio* of the Bradwardine line itself. A sentence with a binary relation, aRb holds just if the objects denoted by 'a' and 'b' stand in the relation (expressed by) R. How could it be that the name 't' does not denote the object denoted by 't'? Maybe some funny business is going on if 't' has more than one denotation. But the left-to-right half of the D-schema entails that if a term has a denotation, it is unique.[10] Or again, maybe something strange is going on if 't' has no denotation. But the proof of the Berry paradox shows, using just the left-to-right half of the D-schema, that the condition in the description t is satisfied by a unique object; so that the term does have a denotation. t may be taken to be the description $\iota y A$, where A is 'y is the least natural number that is not denoted by a name (description) of less than 100 words'. Standard considerations (employing, I note again, only the left-to-right part of the D-schema) establish that $\exists ! y A$.[11] By the very way that descriptions work:

(*) $\exists ! y A \to \forall x (A_y(x) \to \langle \iota y A \rangle \, Dx)$

It follows that $\exists x \, \langle \iota y A \rangle \, Dx$. For good measure, given that:

(**) $\exists ! y A \to A_y(\iota y A)$

we have $\langle \iota y A \rangle \, D \iota y A$. That is, $\langle t \rangle \, Dt$, and we are back with an explicit contradiction.

The only move here would seem to be to deny (*). Even though there is a unique thing satisfying A, '$\iota y A$' does not denote it; '$\iota y A$' has no denotation. How, then, are descriptions supposed to work? The basic principle of

[9] See Priest (2006b), 1.8. The second of these can be inferred from the D-schema and $t = t$. This is the only place in which the right-to-left direction of the D-schema is employed.

[10] In truth, something already seems to have gone wrong at this stage. If the deotation of a term is unique, how can it have "excess content"?

[11] Given that there is only a finite number of terms with the required number words, provided that each term has at most one denotation, the number of numbers denoted by them is finite. There must therefore be numbers that are not denoted, and so a least. For details, see Priest (2006b), 1.8.

descriptions, that $\exists! yA \rightarrow A_y(\iota yA)$, would itself seem to lose all rationale. At the very least, Read owes us a theory of definite descriptions. Moreover, if (**) fails, this on its own, is sufficient to avoid the Berry conclusion, since the principle is appealed to in the argument for it. The Bradwardinian machinery is otiose. Indeed, if t really does have no denotaton, $\exists x\, xDt$ no longer follows from $\forall x(\langle t\rangle\, Dx \leftrightarrow x = t)$, since we cannot instantiate the quantifier with t (and even if we could, depending on one's theory of non-denotation, $t = t$ might not be true). We do not have to give up the naive D-Schema at all.

3 Propositional Quantification

Let us now turn to the second problem: the Bradwardine solution to the Liar itself. This has to be phrased using propositional quantification. Some have found such quantification problematic, and reject the notion entirely. Clearly, Read is in no position to do this; let us grant its legitimacy, at least for the sake of argument. There is certainly a grammatical awkwardness in reading propositional quantification in English. Thus, '$\exists p\ p$' is something like 'some proposition is such that it'. One feels the need to stick 'is true' on the end. But propositional quantification dispenses with the need for this. And just because of that fact, propositional quantification allows us to express what normally requires the truth predicate. Indeed, the plain vanilla p says, in effect, that p is true. It appears this allows us to formulate the liar paradox without an explicit truth predicate.

Thus, consider a proposition, a, that is identical to its own negation; that is, a proposition of the form 'not me'. By the standards of propositional quantification, this is certainly meaningful. And since every proposition is equivalent to itself, a is equivalent to its own negation. So we are back with a contradicton.

The obvious reply here is to deny that 'a' refers to anything; that is, to say that a does not exist. But on what grounds can one do this? a is obviously self-referential ('not me'). But there would appear to be nothing problematic about self-reference as such. Merely consider: 'this proposition is being expressed by me now', 'this proposition can be expressed by a sentence with 11 words'. Neither is it a problem to construct a theory of ungrounded propositions to do justice to propositions of this kind. One such is given by Barwise and Etchemendy (1987). In their non-well-founded theory, given any boolean function of propositions, $F(x)$, there are solutions to the fixed-point equation $x = F(x)$.[12]

[12] One might wonder why this does not yield inconsistency in the form of the proposition

At the very least, the solution needs to be backed up by a theory of propositions, and not one specially rigged to give the result. One, can of course, use the paradox as a *reductio* of the claim that a exists. But this is cheap: if one builds any pre-condition into a principle employed in a paradoxical argument, one can use the argument as a *reductio* of this. Moreover, and again, if one has to resort to this move, it finesses the whole Bradwardinian machinery. One can endorse the plain unvarnished T-schema: this simply applies to propositions, and the liar sentence does not express a proposition.

And in the last instance, denying the existence of the paradoxical proposition does not work; as usual, strengthened paradoxes arise. By techniques of diagonalisation, one can certainly find a term, b, of the form $\exists q\, q = b \wedge \neg q$.[13] Now, suppose that b; that is, $\exists q(q = b \wedge \neg q)$. Then $\neg b$ by elementary quantifier rules and the substitutivity of identicals. Conversely, suppose that $\neg b$. Then $b = b \wedge \neg b$, and so $\exists q(q = b \wedge \neg q)$; that is, b.

And now, if one denies that 'b' refers to anything, that is, one asserts that b does not exist, we are back in trouble. For suppose that b does not exist, i.e., that $\neg\exists q\; q = b$. From this it follows that $\neg\exists q(q = b \wedge \neg q)$, which is $\neg b$. Moreover, it follows that $b = b \wedge \neg b$, and by existential quantification, $\exists q(q = b \wedge \neg q)$, that is, b. (Both of these steps might be taken to fail of b does not exist. But since $\neg b$ (is true), $\neg b$ must exist; so, then, must b.)

One might try to avoid this objection by ramification, though there is no reason to suppose that Bradwardine had any inclination in this direction (and, as we shall see, reasons why he should not have). Propositional quantifiers come in a hierarchy, indexed, say, by the natural numbers. All propositions have a rank, which is the greatest index of all bound variables in the formula specifying it;[14] and quantifiers range only over propositions of lower rank. Thus, our paradoxical proposition is a proposition, b, of the form $\exists q_n(q_n = b \wedge \neg q_n)$, for some n. From b we can now infer $\neg b$; so $\neg b$. When we go in the reverse direction, we can get to $b = b \wedge \neg b$. But since

such that $x = \neg x$. The answer is that such objects are not guaranteed to satisfy excluded middle, $x \vee \neg x$, in a situation (and, for that matter, they may also satisfy $x \wedge \neg x$ in a situation). The internal logic of propositions is First Degree Entailment. This might suggest moving from a classical to a non-classical logic to accommodate non-well-founded propositions. But if one is to do this, one might just as well have done it with the pristine T-schema.

[13]This actually requires the language to contain the notion of identity between propositions. (As in Bloom and Suszko (1971) and (1972).) But this hardly seems problematic: if we can quantify over propositions, we can, presumably, talk about their identity. Diagonalisation is normally applied to formulas, not terms; but it can be applied to terms just as well. See Priest (1997).

[14]Actually, even at this stage there is a problem, since there is no particular reason as to why only one formula should specificy the proposition. But let that pass.

b is a proposition of rank n, we cannot move to $\exists q_n(q_n = b \wedge \neg q_n)$, only to $\exists q_{n+1}(q_{n+1} = b \wedge \neg q_{n+1})$.

Ramification brings familiar problems, however.[15] For a start, ramification seems far too strong. It rules out the possibility of saying perfectly intelligible things.[16] The natural expression of the law of excluded middle with propositional quantification is: $\forall p(p \vee \neg p)$ (where that very proposition is within the scope of the quantifier). After ramification, there is no proposition that expresses this: the closest one can get is $\forall p_n(p_n \vee \neg p_n)$, for some particular n; this is obviously weaker. One might try to invoke the device of typical ambiguity, so that the formula is to be taken as asserted for all values of n. But an appeal to typical ambiguity works only when we are dealing with a universally quantified sentence. Thus, the natural way to express the dialetheic thesis, and say that some contradictory sentence is true, is $\exists p(p \wedge \neg p)$, and this is not equivalent to a typically ambiguous assertion of $\exists p_n(p_n \wedge \neg p_n)$, which asserts the existence of many more dialetheias—one of each type.[17]

And in the case of the Bradwardine solution, ramification is particularly disasterous. If we are going to have to resort to some kind of ramification, we might just as well have resorted to it in the first place, ramifying the truth predicate, *à la* Tarski. The net effect is the same, and is much simpler, dispensing, as it does, with propositional quantification altogether. Finally, Bradwardine's account of truth cannot even be formulated with ramified quantifiers. The nearest we can get is:

$$T\langle A\rangle \leftrightarrow \forall p_n(\langle A\rangle : p_n \rightarrow p_n)$$

for some rank, n, but which? It might be thought that this would be determined by A. Unfortunately, it cannot. According to Read's account, *saying that* is closed under entailment. But A will entail propositions of arbitrarily high rank; say, for arbitrarily high m, $A \vee \forall p_m(p_m \vee \neg p_m)$. Thus, no n is going to be adequate in a statement of the theory of truth.[18]

[15]See Priest (2006b), 1.5, and Priest (2002), ch. 10.

[16]Including, most notably, the theory itself. For to say, as I did, that all propositions have some rank is exactly to quantify over all propositions, which is exactly what, according to the theory, cannot, grammatically, be done. The theory is then self-refuting.

[17]There is also something rather disingenuous with typical ambiguity, even in the universal case. What is *meant* by a typically ambiguous assertion of $\forall p_n A(p_n)$, is exactly $\forall n \forall p_n A(p_n)$. The effect of ramification is to disallow ourselves the ability to say what we mean.

[18]Nor does appealing to substitutional quantification resolve any of these problems.

4 Conclusion

Bradwardine's solution to the liar paradox is a very clever one, both by the standards of the great period of medieval logic and by contemporary standards; and Read's revival and careful contemporary defence of it are much to be welcomed. But what we seem to learn, in the end, is that the solution, though not subject to some of the perhaps more obvious objections, is sunk by essentially the same kinds of considerations that sink all consistent solutions to the paradoxes, such as their inability to handle other paradoxes of the same kind, and extended paradoxes.

What Bradwardine would have said about these matters, we can, of course, only speculate.[19]

References

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[19] Versions of this paper were given to the Philosophy Department, University of Connecticut, September 2009, the Logic Group, Melbourne University, May 2011, and at the conference *Paradoxes of Truth and Denotation*, LOGOS Group, University of Barcelona, June, 2011. My thanks go to the members of those audiences for helpful comments, and especially to Lloyd Humberstone, Tomas Kowalski, Greg Restall, Dave Ripley, and Sven Rosencrantz. Last, but by no means least, thanks go to Steve Read himself for his comments on and discussions of previous versions of the paper.

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