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MANY-VALUED MODAL LOGICS: A SIMPLE APPROACH

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1. Introduction.

1.1 In standard modal logics, the worlds are 2-valued in the following sense: there are 2 values (*true* and *false*) that a sentence may take at a world. Technically, however, there is no reason why this has to be the case. The worlds could be many-valued. This paper presents one simple approach to a major family of many-valued modal logics, together with an illustration of why this family is philosophically interesting.[1]

1.2 We will start with the general structure of a many-valued modal logic. To illustrate this, we will look briefly at modal logic based on Łukasiewicz continuum-valued logic.

1.3 We will then look at one particular many-valued modal logic in more detail, modal first-degree entailment (FDE). We will give a sound and complete proof theory for this (and for its special cases, K_3 and LP).

1.4 Modal many-valued logics engage with a number of philosophical issues. The final part of the paper will illustrate by looking at one such: the issue of future contingents.

2. General structure.

2.1 Semantically, a propositional many-valued logic is characterized by a structure $\langle \mathcal{V}, \mathcal{D}, \{f_c : c \in C\}\rangle$, where $\mathcal{V}$ is the set of semantic values, $\mathcal{D} \subseteq \mathcal{V}$ is the set of designated values, and for each connective, c, f_c is the truth function it denotes. An interpretation, ν, assigns values in $\mathcal{V}$ to propositional parameters; the values of all formulas can then be computed using the f_cs, and a valid inference is one that preserves designated values in every interpretation (see Priest, 2001, 7.2).

2.2 It is standard to assume that $\mathcal{V}$ comes with an ordering, $\leq$ (which may be a partial ordering). We will assume in what follows that this is so. We also assume that every subset of the values has a greatest lower bound (Glb) and least upper bound (Lub) in the ordering.

2.3 The language of a many-valued modal logic is the same as that of the many-valued logic, except that it is augmented by the monadic operators $\Box$ and $\Diamond$ in the usual way.

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[1] The earliest paper on a many-valued modal logic appears to have been Segerberg (1967), which specifies some 3-valued modal logics. More general approaches, somewhat different from the one presented here, were later provided by Thomason (1978), Morgan (1979), and Ostermann (1988). Semantically, the general approach here is closest to that of Fitting (1991, 1992), who generalizes to allow even the accessibility relation to be many-valued. Fuzzy modal logic is considered in Hájek (1999, 8.3). One first-order many-valued modal logic is investigated in Ostermann (1990).

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2.4 An interpretation for a many-valued logic modal is a structure $\langle W, R, S_L, \nu\rangle$, where W is a nonempty set of worlds, R is a binary accessibility relation on W, S_L is a structure for a many-valued logic, L, and for each propositional parameter, p, and world, w, ν assigns the parameter a value, $\nu_w(p)$, in $\mathcal{V}$. (In what follows, I will use $p, q, \ldots$ for propositional parameters and $A, B, \ldots$ for arbitrary sentences.)

2.5 The truth conditions for the many-valued connectives at a world simply deploy the functions f_c. Thus, if c is an n-place connective, $\nu_w(c(A_1, \ldots, A_n)) = f_c(\nu_w(A_1), \ldots, \nu_w(A_n))$. (So if c is conjunction, $\nu_w(A \wedge B) = f_\wedge(\nu_w(A), \nu_w(B))$.)

2.6 The natural generalization of the 2-valued truth conditions for the modal operators is as follows:[2]

$\nu_w(\Box A) = \text{Glb}\{\nu_{w'}(A)\colon wRw'\}$
$\nu_w(\Diamond A) = \text{Lub}\{\nu_{w'}(A)\colon wRw'\}$

2.7 Validity is naturally defined as follows. If the set of premises is Σ and the conclusion is A,

$\Sigma \models A$ iff for every interpretation, $\langle W, R, S_L, \nu\rangle$, and for every $w \in W$, whenever $\nu_w(B) \in \mathcal{D}$ for every $B \in \Sigma$, $\nu_w(A) \in \mathcal{D}$.

2.8 This gives the analog of the 2-valued modal logic K. Call it K_L. Stronger logics can be obtained by addition constraints on the accessibility relation, such as reflexivity (ρ), symmetry (σ), transitivity (τ), giving the logics $K_L\rho$, $K_L\sigma$, $K_L\rho\tau$, etc. (For the notation, see Priest, 2001, Ch. 3.)

3. Illustration: modal Łukasiewicz logic.
3.1 Section 2 gives the general structure of a many-valued modal logic. Let us illustrate with respect to the continuum-valued logic of Łukasiewicz, $Ł_\aleph$. The connectives of this are $\neg$, $\wedge$, $\vee$, and $\rightarrow$. $\mathcal{V}$ is the set of real numbers between 0 and 1, [0, 1]. The truth functions corresponding to the connectives are as follows:

$f_\neg(x) = 1 - x$
$f_\wedge(x, y) = \text{Min}(x, y)$
$f_\vee(x, y) = \text{Max}(x, y)$
$f_\rightarrow(x, y) = x \ominus y$

where Min in the minimum, Max is the maximum, and $x \ominus y = 1$ if $x \leq y$ and $1 - x + y$ otherwise. $\mathcal{D} = \{1\}$. (On all this, see Priest, 2001, 11.4.)

3.2 In $K_{Ł_\aleph}$, the modal logic based on $Ł_\aleph$, if $\models A$ then $\models \Box A$, and $\models \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$. These are the characteristic modal properties of the 2-valued modal logic, K.

[2] Semantically, $\Box$ and $\Diamond$ are forms of (respectively) universal and particular quantifiers over worlds. The following truth conditions are the obvious analogues of the truth conditions for these quantifiers in many-valued logic (see, Priest, 2008, 21.3).

3.3 For the first of these, suppose that $\not\models \Box A$. Then, there is some interpretation and some world in that interpretation, w, such that $\nu_w(\Box A) \neq 1$. Thus, for some w' such that wRw', $\nu_{w'}(A) \neq 1$. Hence $\not\models A$.

3.4 For the second, it suffices to show that in any interpretation, $\nu_w(\Box(A \rightarrow B)) \leq \nu_w(\Box A \rightarrow \Box B)$, that is, $\text{Glb}\{\nu_{w'}(A) \ominus \nu_{w'}(B) : wRw'\} \leq \text{Glb}\{\nu_{w'}(A) : wRw'\} \ominus \text{Glb}\{\nu_{w'}(B) : wRw'\}$. Let $X = \{w' : wRw'\}$, and let a_x and b_x be $\nu_x(A)$ and $\nu_x(B)$, respectively. We need to show that

(*) $\text{Glb}\{a_x \ominus b_x : x \in X\} \leq \text{Glb}\{a_x : x \in X\} \ominus \text{Glb}\{b_x : x \in X\}$.

Suppose that $\text{Glb}\{a_x : x \in X\} \leq \text{Glb}\{b_x : x \in X\}$. Then, the right-hand side of (*) is 1 and we have the result. Conversely, suppose that $\text{Glb}\{a_x : x \in X\} > \text{Glb}\{b_x : x \in X\}$. Then for some $x \in X$, $a_x > b_x$. Let $X' = \{x \in X : a_x > b_x\}$. Then,

$$\begin{aligned}
\text{Glb}\{a_x \ominus b_x : x \in X\} &= \text{Glb}\{a_x \ominus b_x : x \in X'\} \\
&= \text{Glb}\{1 - a_x + b_x : x \in X'\} \\
&= 1 + \text{Glb}\{b_x - a_x : x \in X'\} \\
&= 1 + \text{Glb}\{b_x - a_x : x \in X\}.
\end{aligned}$$

Consequently, what needs to be shown is that

$$1 + \text{Glb}\{b_x - a_x : x \in X\} \quad \leq \quad 1 + \text{Glb}\{b_x : x \in X\} - \text{Glb}\{a_x : x \in X\}.$$

That is,

$$\text{Glb}\{b_x - a_x : x \in X\} \quad \leq \quad \text{Glb}\{b_x : x \in X\} - \text{Glb}\{a_x : x \in X\}.$$

We show this as follows.

For any $x \in X$,	$b_x - a_x \leq b_x - a_x$.
Hence,	$\text{Glb}\{b_x - a_x : x \in X\} \leq b_x - a_x$.
So	$\text{Glb}\{b_x - a_x : x \in X\} \leq b_x - \text{Glb}\{a_x : x \in X\}$.
That is,	$\text{Glb}\{b_x - a_x : x \in X\} + \text{Glb}\{a_x : x \in X\} \leq b_x$.
And so	$\text{Glb}\{b_x - a_x : x \in X\} + \text{Glb}\{a_x : x \in X\} \leq \text{Glb}\{b_x : x \in X\}$.
That is,	$\text{Glb}\{b_x - a_x : x \in X\} \leq \text{Glb}\{b_x : x \in X\} - \text{Glb}\{a_x : x \in X\}$.

3.5 In $K_{Ł_{\aleph}}$, none of the following hold:

$\models \Box A \rightarrow A$
$\models A \rightarrow \Box\Diamond A$
$\models \Box A \rightarrow \Box\Box A$

This follows from the fact that none of these is valid in K, and a K countermodel is (a special case of) a $K_{Ł_{\aleph}}$ countermodel (one where only the values 1 and 0 are taken by a formula).

3.6 However, the additions of the constraints ρ, σ, and τ suffice, respectively, to make the 3 hold. I continue to write a_w for $\nu_w(A)$.

- For the first, if wRw, $\nu_w(\Box A) = \text{Glb}\{a_{w'} : wRw'\} \leq a_w$, as required.
- For the second, suppose that wRw'. If R is symmetric, $a_w \leq \text{Lub}\{a_{w''} : w'Rw''\} = \nu_{w'}(\Diamond A)$. So $a_w \leq \text{Glb}\{\nu_{w'}(\Diamond A) : wRw'\}$, that is, $a_w \leq \nu_w(\Box\Diamond A)$, as required.

- For the third, suppose that wRw'. Since R is transitive, $\{w'' : w'Rw''\} \subseteq \{w'' : wRw''\}$. So $\{a_{w''} : w'Rw''\} \subseteq \{a_{w''} : wRw''\}$. Thus, $\text{Glb}\{a_{w''} : wRw''\} \leq \text{Glb}\{a_{w''} : w'Rw''\}$. So $\text{Glb}\{a_{w''} : wRw''\} \leq \text{Glb}\{\text{Glb}\{a_{w''} : w'Rw''\} : wRw'\}$, that is, $\nu_w(\Box A) \leq \nu_w\ (\Box\Box A)$, as required.

4. First-degree entailment.

4.1 Let us now look at 1 many-valued modal logic in more detail. The many-valued logic in question is FDE. The language for this has 3 connectives: $\wedge$, $\vee$, and $\neg$. (We can take $A \supset B$ to be a shorthand for $\neg A \vee B$.)

4.2 FDE is a 4-valued logic. $\mathcal{V} = \{t, f, b, n\}$—true (only), false (only), both, and neither. $\mathcal{D} = \{t, b\}$. The values are ordered as follows:

$$\begin{array}{ccccc} & & t & & \\ & \nearrow & & \nwarrow & \\ b & & & & n \\ & \nwarrow & & \nearrow & \\ & & f & & \end{array}$$

(The arrows point upwards.) $f_\wedge$ is the meet on this lattice; $f_\vee$ is the join; $f_\neg$ maps t to f, vice versa, and each of b and n to itself.

4.3 K_{FDE} is obtained by the general construction described. If we ignore the value n in the non-modal case (i.e., we insist that formulas take one of the values in $\{t, b, f\}$), we get the logic LP. In the modal case, we get K_{LP}. If we ignore the value b in the non-modal case, we get the logic K_3. In the modal case, we get K_{K_3}.

4.4 FDE can be formulated equivalently as a logic in which, instead of an evaluation, ν, there is a relation, ρ (not to be confused with the constraint on the accessibility relation), which relates a formula, A, to the values 1 (true) and 0 (false) as follows:

$\nu(A) = t$ iff $A\rho 1$ and it is not the case that $A\rho 0$
$\nu(A) = b$ iff $A\rho 1$ and $A\rho 0$
$\nu(A) = n$ iff it is not the case that $A\rho 1$ and it is not the case that $A\rho 0$
$\nu(A) = f$ iff it is not the case that $A\rho 1$, and $A\rho 0$

The appropriate truth/falsity conditions for the connectives are

$A \wedge B\rho 1$ iff $A\rho 1$ and $B\rho 1$
$A \wedge B\rho 0$ iff $A\rho 0$ or $B\rho 0$
$A \vee B\rho 1$ iff $A\rho 1$ or $B\rho 1$
$A \vee B\rho 0$ iff $A\rho 0$ and $B\rho 0$
$\neg A\rho 1$ iff $A\rho 0$
$\neg A\rho 0$ iff $A\rho 1$

Validity is defined in terms of the preservation of relating to 1. On all this, see Priest (2001, Ch. 8).

4.5 K_{FDE} can be formulated in the same way. The facts of 4.4 carry over with a subscript w to the νs and ρs. What of the truth/falsity conditions of the modal operators if FDE is formulated in this way? They may be given, in a very natural way, as follows:

$\Box A\rho_w 1$ iff for all w' such that wRw', $A\rho_{w'}1$
$\Box A\rho_w 0$ iff for some w' such that wRw', $A\rho_{w'}0$
$\Diamond A\rho_w 1$ iff for some w' such that wRw', $A\rho_{w'}1$
$\Diamond A\rho_w 0$ iff for all w' such that wRw', $A\rho_{w'}0$

4.6 The argument for this is as follows. Consider $\nu_w(\Box A)$, that is, Glb$\{\nu_{w'}(A)\colon wRw'\}$. This has 4 possible values.

- **t:** In this case, for all w' such that wRw', the value of $\nu_{w'}(A)$ is t. So for all w' such that wRw', $A\rho_{w'}1$ and it is not the case that $A\rho_{w'}0$. In this case, the truth/falsity conditions give that $\Box A\rho_w 1$ and it is not the case that $\Box A\rho_w 0$, as required.
- **b:** In this case, for all w' such that wRw', the value of $\nu_{w'}(A)$ is t or b and at least one is b. That is, for all w' such that wRw', $A\rho_{w'}1$ and for at least one such w', $A\rho_{w'}0$. In this case, the truth/falsity conditions give that $\Box A\rho_w 1$ and $\Box A\rho_w 0$, as required.
- **n:** In this case, for all w' such that wRw', the value of $\nu_{w'}(A)$ is t or n and at least one is n. That is, for all w' such that wRw', it is not the case that $A\rho_{w'}0$ and for at least one such w', it is not the case that $A\rho_{w'}1$. In this case, the truth/falsity conditions give that it is not the case that $\Box A\rho_w 1$ and it is not the case that $\Box A\rho_w 0$, as required.
- **f:** In this case, either there is some w' such that wRw' and $\nu_{w'}(A) = f$, or there are w' and w'', such that wRw' and wRw'', $\nu_{w'}(A) = b$ and $\nu_{w''}(A) = n$. In the first case, for all w' such that wRw', $A\rho_{w'}0$ and it is not the case that $A\rho_{w'}1$. In the second case, $A\rho_{w'}1$ and $A\rho_{w'}0$, and neither $A\rho_{w''}1$ nor $A\rho_{w''}0$. In either case, the truth/falsity conditions give that $\Box A\rho_w 0$ and it is not the case that $\Box A\rho_w 1$, as required.

The case for $\Diamond$ is similar and is left as an exercise.

4.7 In the context of the relational semantics, LP is obtained by requiring that, for all p, either $p\rho 1$ or $p\rho 0$ (see Priest, 2001, 8.4.9). The same is true with the appropriate subscript w on ρ for K_{LP}.

4.8 In the context of the relational semantics, K_3 is obtained by requiring that, for all p, not both $p\rho 1$ and $p\rho 0$ (see Priest, 2001, 8.4.6). The same is true with the appropriate subscript w on ρ for K_{K_3}.

4.9 If we add both conditions in the nonmodal case, we get classical logic. In the modal case, we get the classical modal logic K.

4.10 All the many-valued modal logic may be extended by adding the constraints on the accessibility relation ρ, σ, and τ to give $K_{\mathrm{FDE}}\rho$, $K_{\mathrm{LP}}\rho\tau$, $K_{K_3}\sigma$, etc.

4.11 Note that K_{FDE} and K_{K_3} and all their normal extensions have no logical truths. To see this, just consider the interpretation with one world, w, such that wRw, and for all p, neither $p\rho_w 1$ nor $p\rho_w 0$. An easy induction shows the same to be true for all formulas.

4.12 Note also that interpretations for any logic in the family we are considering are monotonic, in the following sense. Let $\mathcal{I}_1 \preceq \mathcal{I}_2$ iff the 2 interpretations have the same worlds and accessibility relation, and, in addition, for all propositional parameters, p, and all worlds, w:

if $p\rho_{1w}1$ then $p\rho_{2w}1$
if $p\rho_{1w}0$ then $p\rho_{2w}0$

where ρ_1 and ρ_2 are the evaluation relations of $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively. If $\mathcal{I}_1 \preceq \mathcal{I}_2$, the conditions just displayed obtain for an arbitrary formula, A. The proof is by a simple induction.

4.13 A corollary is that $\models_K A$ iff $\models_{K_{\mathrm{LP}}} A$ (and similarly for $K\rho$ and $K_{\mathrm{LP}}\rho$, etc.). From right to left, the result is straightforward, since any classical interpretation is an LP interpretation. For the converse, suppose that $\not\models_{K_{\mathrm{LP}}} A$. Then there is an interpretation, $\mathcal{I}_2$, such that A does not hold at some world, w_0, in $\mathcal{I}_2$ $\big($i.e., it is not the case that $A\rho_{w_0}1\big)$. Let $\mathcal{I}_1$ be any classical interpretation obtained from $\mathcal{I}_2$ simply by resolving contradictory propositional parameters one way or the other. That is, when $p\rho_{2w}1$ and $p\rho_{2w}0$, only one of these holds for ρ_{1w}. Then $\mathcal{I}_1 \preceq \mathcal{I}_2$. By monotonicity, A does not hold at w_0 in $\mathcal{I}_1$ and $\mathcal{I}_1$ is an interpretation for K.

5. Tableaux.

5.1 We may obtain tableau systems that are sound and complete with respect to the systems we have been looking at by modifying the tableau system for FDE in the same way that the tableau system for classical propositional logic is modified to obtain those for the modal logics K, $K\rho$, etc.[3]

5.2 Thus, for K_{FDE}, tableau lines are of the form $A, +i$, $A, -i$, or irj. The first indicates that A holds at world i (i.e., relates to 1); the second that A fails at world i (i.e., does not relate to 1); the third indicates that world i relates to world j. We start with a line of the form $B, +0$ for every premise, B, and a line of the form $A, -0$, where A is the conclusion. A branch of the tableau closes if it contains lines of the form $A, +i$ and $A, -i$. The tableau is closed if all branches close.

5.3 The rules for the extensional connectives are as follows (see Priest, 2001, 8.3.4):

$$\begin{array}{ccc} A \wedge B, +i & \qquad & A \wedge B, -i \\ \downarrow & & \swarrow \quad \searrow \\ A, +i & & A, -i \quad B, -i \\ B, +i & & \end{array}$$

$$\begin{array}{ccc} A \vee B, +i & \qquad & A \vee B, -i \\ \swarrow \quad \searrow & & \downarrow \\ A, +i \quad B, +i & & A, -i \\ & & B, -i \end{array}$$

$$\begin{array}{ccccc} \neg(A \vee B), \pm i & \qquad & \neg(A \wedge B), \pm i & \qquad & \neg\neg A, \pm i \\ \downarrow & & \downarrow & & \downarrow \\ \neg A \wedge \neg B, \pm i & & \neg A \vee \neg B, \pm i & & A, \pm i \end{array}$$

The $\pm$ can be disambiguated uniformly as either $+$ or $-$.

[3] Tableaux of a somewhat more complicated kind are given for some modal many-valued logics in Fitting (1995) and Sakalauskaite (2002).

5.4 The rules for the modal operators are as follows:

$$
\begin{array}{cccc}
\Box A, +i & \Box A, -i & \Diamond A, +i & \Diamond A, -j \\
irj & \downarrow & \downarrow & irj \\
\downarrow & irj & irj & \downarrow \\
A, +j & A, -j & A, +j & A, -j
\end{array}
$$

In the middle 2 rules, j is new to the branch. In the other 2, the rule is applied whenever something of the form irj is on the branch. In addition, we have the 'commuting rules':

$$
\begin{array}{cc}
\neg\Box A, \underline{+}i & \neg\Diamond A, \underline{+}i \\
\downarrow & \downarrow \\
\Diamond\neg A, \underline{+}j & \Box\neg A, \underline{+}i
\end{array}
$$

5.5 Here are tableaux to show that $\Box A \wedge \neg\Box B \vdash_{K_{\mathrm{FDE}}} \Diamond(A \wedge \neg B)$ and $\Box(p \supset q)$, $\Diamond p \nvdash_{K_{\mathrm{FDE}}} \Diamond q$:

$$
\begin{array}{c}
\Box A \wedge \neg\Box B, +0 \\
\Diamond(A \wedge \neg B), -0 \\
\Box A, +0 \\
\neg\Box B, +0 \\
\Diamond\neg B, +0 \\
0r1 \\
\neg B, +1 \\
A, +1 \\
A \wedge \neg B, -1 \\
\swarrow \quad \searrow \\
A, -1 \quad \neg B, -1 \\
\times \qquad\quad \times
\end{array}
$$

$$
\begin{array}{c}
\Box(p \supset q), +0 \\
\Diamond p, +0 \\
\Diamond q, -0 \\
0r1 \\
p, +1 \\
q, -1 \\
p \supset q, +1 \\
\swarrow \quad \searrow \\
\neg p, +1 \quad q, +1 \\
\qquad\qquad\quad \times
\end{array}
$$

5.6 To read off a countermodel from an open branch, $\mathcal{B}$, of a tableau, we let $W = \{w_i$: a line of the form $A, \underline{+}i$ occurs in $\mathcal{B}\}$; $w_i R w_j$ iff irj occurs on $\mathcal{B}$; for every propositional parameter, p, $p\rho_{w_i}1$ iff $p, +i$ is on $\mathcal{B}$; $p\rho_{w_i}0$ iff $\neg p, +i$ is on $\mathcal{B}$. Thus, in the countermodel determined by the open branch of the last tableau, $W = \{w_0, w_1\}$, $w_0 R w_1$ (and no other R relations hold) $p\rho_{w_1}1$, and $p\rho_{w_1}0$ (and no other ρ relationships hold). In a diagram:

$$\begin{array}{ccc} w_0 & \rightarrow & w_1 \\ & & p+ \\ & & \neg p+ \\ & & q- \end{array}$$

Since p holds at w_1, $\Diamond p$ holds at w_0. Since $\neg p$ holds at w_1, $p \supset q$ holds at w_1, so $\Box(p \supset q)$ holds at w_0. But q fails at w_1; hence $\Diamond q$ fails at w_0.

6. Variations.

6.1 For K_{K_3}, we add the extra closure rule:

$$\begin{array}{c} A, +i \\ \neg A, +i \\ \times \end{array}$$

6.2 For K_{LP}, we add the extra closure rule:

$$\begin{array}{c} A, -i \\ \neg A, -i \\ \times \end{array}$$

6.3 To get the systems corresponding to the semantic conditions ρ, σ, and τ, we add the rules:

$$\begin{array}{ccc} . & irj & irj \\ \downarrow & \downarrow & jrk \\ iri & jri & \downarrow \\ & & irk \end{array}$$

respectively.

6.4 Here are tableaux to show that $\Box A \vdash_{K_{K_3}\tau} \Box\Box A$ and $\Box p \nvdash_{K_{\mathrm{LP}}\rho} \Box\Box p$.

$$\begin{array}{c} \Box A, +0 \\ \Box\Box A, -0 \\ 0r1 \\ \Box A, -1 \\ 1r2 \\ A, -2 \\ 0r2 \\ A, +2 \\ \times \end{array}$$

$$\Box p, +0$$
$$\Box\Box p, -0$$
$$0r0$$
$$0r1,\ 1r1$$
$$\Box p, -1$$
$$1r2,\ 2r2$$
$$p, -2$$

6.5 Countermodels are read off from open branches as in 5.6, except that for K_{LP} and its extensions, $p\rho_{w_i}1$ iff $p, -i$ is *not* on the branch and $p\rho_{w_i}0$ iff $\neg p, -i$ is *not* on the branch. Thus, the countermodel given by the last tableau may be depicted by

$$\begin{array}{ccccc} \circlearrowright & & \circlearrowright & & \circlearrowright \\ w_0 & \rightarrow & w_1 & \rightarrow & w_2 \\ p+ & & p+ & & p- \\ \neg p+ & & \neg p+ & & \neg p+ \end{array}$$

Since p holds at w_1, $\Box p$ holds at w_0. Since p fails at w_2, $\Box p$ fails at w_1 and $\Box\Box p$ fails at w_0.

6.6 The tableau systems for all the logics we have considered are sound and complete. This is proved in the technical appendix, Section 9.

7. Future contingents.

7.1 Many-valued modal logics engage with a number of philosophical controversies. Let me illustrate with respect to Aristotle's argument concerning future contingents. In *De Interpretatione*, Ch. 9, Aristotle argued famously that if contingent statements about the future were now either true or false, fatalism would follow. He therefore denied that contingent statements about the future are true or false.

7.2 The argument that the law of excluded middle entails fatalism is worth quoting in detail:[4]

> ... if a thing is white now, it was true before to say that it would be white, so that of anything that has taken place, it was always true to say 'it is' or 'it will be'. But if it was always true to say that a thing is or will be, it is not possible that it should not be or not come to be, and when a thing cannot not come to be, it is impossible that it should not come to be, and when it is impossible that it should not come to be, it must come to be. All then, that is about to be must of necessity take place. It results from this that nothing is uncertain or fortuitous, for if it were fortuitous it would not be necessary.

7.3 One way to read the passage is as follows. Let q be any statement about a future contingent event. Let T_q be the statement that it is (or was) true that q. Then $\Box(T_q \rightarrow q)$.

[4] *De Int*. 18^b10-16. Translation from Vol. 1 of Ross (1928).

Hence $T_q \to \Box q$. And since $\Box q$ is not true, neither is T_q. A similar argument can be run for $\neg q$. So neither T_q nor $T_{\neg q}$ holds. Read in this way, the reasoning contains a modal fallacy (passing from $\Box(A \to B)$ to $(A \to \Box B)$). Many commentators have read the passage thus (e.g., Priest, 2001, 7.9).

7.4 But this may not do Aristotle justice. It is clear that he thinks that the past and present are fixed (unchangeable, now inevitable). So if s is a statement about the past or present, $s \to \Box s$. Hence, $T_q \to \Box T_q$, and since $\Box(T_q \to q)$, so that $\Box T_q \to \Box q$, it follows that $T_q \to \Box q$. There is no fallacy here.

7.5 In fact, we can simplify the argument. Neither T_q nor the conditional is playing an essential role. We may run the argument as follows. If q were true, this would be a present fact, and so fixed; that is, it would be necessarily true, that is, $q \models \Box q$. Similarly, if it were false, it would be necessarily false: $\neg q \models \Box\neg q$. Since neither $\Box q$ nor $\Box\neg q$ holds, neither q nor $\neg q$ holds.

7.6 To do justice to Aristotle's argument, we must take seriously the thought that some things might be neither true nor false. Since Aristotle does not countenance violations of the 'Law of Non-Contradiction', an appropriate logic is K_{K_3}—or one of its normal extensions—not K_{FDE} or K_{LP}.

7.7 Think of the accessibility statement wRw' as meaning that w' may be obtained from w by some number (possibly zero) of further things happening. Clearly, R is reflexive and transitive. According to Aristotle, once something is true/false, it stays so. We may capture the idea by the heredity conditions: for every propositional parameter, p, and world, w:

If $p\rho_w 1$ and wRw', $p\rho_{w'}1$
If $p\rho_w 0$ and wRw', $p\rho_{w'}0$

Call this the *Persistence Constraint.* The displayed conditions follow for all unmodalized formulas, as may be shown by an easy induction.

7.8 They do not hold for modalized formulas, however; nor would one expect them to. Let s be the sentence 'It rains in St Andrews on 1/1/2100'. $\Diamond s$ and $\Diamond\neg s$ are both true. But there is a possible future (indeed, a probable one!) in which s is true, and so $\Box s$ is true, and $\Diamond\neg s$ is false.

7.9 Call $K_3\rho\tau$ augmented by the Persistence Constraint, $\mathfrak{A}$ (for Aristotle). In this logic, $p \models \Box p$ and $\neg p \models \Box\neg p$. Aristotle's argument therefore works. But, of course, in $\mathfrak{A}$, $p \vee \neg p$ may fail to be true. Here is a simple countermodel (I omit the arrows of reflexivity):

$$\begin{array}{cccccc} & & & w_1 & p+ \\ & & \nearrow & & \\ \begin{array}{l} p- \\ \neg p- \end{array} & w_0 & & & \\ & & \searrow & & \\ & & & w_2 & \neg p+ \end{array}$$

Aristotle is vindicated.[5]

[5] One may object: the Persistence Constraint should hold only for those things that are *genuinely* about the present (w). (A sentence can be gramatically present but essentially about the

7.10 Matters are a little more difficult than this, however, since later in the same chapter Aristotle says:[6]

> A sea fight must either take place tomorrow, or not; but it is not necessary that it should take place tomorrow, neither is it necessary that it should not take place, yet it is necessary that it either should or should not take place tomorrow.

He is saying that, for the appropriate p, we have neither $\Box p$ nor $\Box\neg p$. We still have $\Box(p \vee \neg p)$, however. As may be checked, $\Box(p \vee \neg p)$ is not valid in A.

7.11 The matter may be remedied by modifying the truth conditions for $\Box$. Though neither p nor $\neg p$ may be true at a world, w, it is natural to suppose on the Aristotelian picture that the truth value of p will eventually be decided. We may therefore view things 'from the end of time', when everything undetermined has been resolved. Call a world *complete* if every propositional parameter is either true or false. A natural way of giving the truth conditions for $\Box$ is as follows:

$\Box A\rho_w 1$ iff for all complete w' such that wRw', $A\rho_{w'}1$
$\Box A\rho_w 0$ iff for some complete w' such that wRw', $A\rho_{w'}0$

The truth/falsity conditions for $\Diamond$ are the same with 'all' and 'some' interchanged. $\Box A$ may be thought of as expressing the idea that A is inevitable. It is not difficult to show that for any complete world, w, persistence holds for all formulas. It follows that at such a world, A is true iff $\Box A$ is and that all formulas are either true or false.

7.12 It is not difficult to check that with the revised truth/falsity conditions for $\Box$, $p \models \Box p$, $\neg p \models \Box\neg p$ (so Aristotle's argument still works), $\models \Box(p \vee \neg p)$, but not $\models \Box p \vee \Box\neg p$. For the first of these, if p is true at w, then, by the Persistence Constraint, p holds at any complete world accessed by w. Hence, $\Box p$ is true at w. The argument for the second is similar. For the third, in any complete world accessed by w, either p or $\neg p$ holds. Hence, $p \vee \neg p$ holds and $\Box(p \vee \neg p)$ is true at w. (Indeed, the same holds for an arbitrary formula, A.) For the last, consider the interpretation of 7.9. We may suppose that all the parameters other than p also take a classical value at w_1 and w_2, and hence that these worlds are complete. Neither $\Box p$ nor $\Box\neg p$ is true at w_0.[7]

8. Conclusion.

8.1 Many-valued modal logics are relevant to many other philosophical debates. I give just one example.

8.2 It is natural to ask what happens to issues about essentialism in the context of vagueness. Can vague predicates express essential properties? Can vague objects, assuming there

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future—such as the sentence "'it will rain" is true'.) Enforcing the Persistence Constraint for those p that are covertly about the future in this way all may therefore be thought to be question begging.

[6] 19^a30-32.

[7] What one loses on this account is, of course, the validity of the inference from $\Box A$ to A, even though the accessability relation is reflexive. The inference is guaranteed to preserve truth only at complete worlds.

to be some, have essential properties? To investigate such questions, one clearly needs a modal logic.

8.3 But, it is often argued, a logic of vagueness is many-valued: it is either some continuum-valued logic or it is some 3-valued logic with or without a supervaluation technique. If this is so, the investigation of such questions requires a many-valued modal logic.

8.4 In fact, since the matter involves predication and identity, what is required is a first-order many-valued modal logic. The construction and investigation of such logics are appropriate for another occasion.

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10. Technical appendix: soundness and completeness proofs.

10.1 In this section, we prove soundness and completeness for all the tableau systems mentioned in the paper. These simply amalgamate the proofs of Priest (2001, Chs. 2, 3, and 8).

10.2. Definition. *Let $\mathcal{I} = \langle W, R, S_{\mathrm{FDE}}, \rho \rangle$ be any K_{FDE} interpretation and $\mathcal{B}$ any branch of a tableau. Then $\mathcal{I}$ is faithful to $\mathcal{B}$ iff there is a map, f, from the natural numbers to W such that:*

If $A, +i$ is on $\mathcal{B}$, $A\rho_{f(i)}1$ in $\mathcal{I}$
If $A, -i$ is on $\mathcal{B}$, then it is not the case that $A\rho_{f(i)}1$ in $\mathcal{I}$
If irj is on $\mathcal{B}$, $f(i)Rf(j)$

10.3. Soundness Lemma for K_{FDE} *Let $\mathcal{B}$ be any branch of a tableau and $\mathcal{I}$ any interpretation. If $\mathcal{I}$ is faithful to $\mathcal{B}$ and a tableau rule is applied, then it produces at least one extension, $\mathcal{B}'$, such that $\mathcal{I}$ is faithful to $\mathcal{B}'$.*

Proof. We merely have to check the rules, one by one. The rules for the extensional connectives are straightforward and left as exercises (see Priest, 2001, 8.7.3). Here are the cases for the rules for $\Box$. Those for $\Diamond$ are similar. The rules in question are

$$\begin{array}{ccc} \Box A, +i & \Box A, -i & \neg\Box A, \underline{+}i \\ irj & \downarrow & \downarrow \\ \downarrow & irj & \Diamond\neg A, \underline{+}i \\ A, +j & A, -j & \end{array}$$

For the first, suppose that f shows $\mathcal{I}$ to be faithful to a branch containing the premises. Then $\Box A$ holds at $f(i)$ and $f(i)Rf(j)$. Hence, A is true at $f(j)$, as required. For the second, suppose that f shows $\mathcal{I}$ to be faithful to a branch containing the premise. Then $\Box A$ fails at $f(i)$. There must therefore be a w such that $f(i)Rw$ and A fails at w. Let f' be the same as f except that $f'(j) = w$. Then f' shows $\mathcal{I}$ to be faithful to $\mathcal{B}$. For the third, here is the case for $+$. That for $-$ is similar. Suppose that f shows $\mathcal{I}$ to be faithful to a branch containing the premise. Then $\neg\Box A$ is true at $f(i)$. So for some w such that $f(i)Rw$, A is false at w. So $\neg A$ is true at w, and $\Diamond\neg A$ holds at $f(i)$. □

10.4. Soundness Theorem for K_{FDE} *For finite* Σ, *if* $\Sigma \vdash A$ *then* $\Sigma \vDash A$.

Proof. This follows from the Soundness Lemma in the usual way. □

10.5. Definition. *Given an open branch, $\mathcal{B}$, of a tableau for FDE, the interpretation $\mathcal{I}$ induced by $\mathcal{B}$ is the structure where $W = \{w_i\colon\ i$ occurs on $\mathcal{B}\}$; $w_i R w_j$ iff irj occurs on $\mathcal{B}$; for every propositional parameter, p, $p\rho_{w_i}1$ iff $p, +i$ is on $\mathcal{B}$; $p\rho_{w_i}0$ iff $\neg p, +i$ is on $\mathcal{B}$.*

10.6. Completeness Lemma for K_{FDE} *Let $\mathcal{B}$ be a complete open branch of a tableau (i.e., one where every rule that can be applied has been applied). Then:*

If $A, +i$ is on $\mathcal{B}$, $A\rho_{w_i}1$
If $A, -i$ is on $\mathcal{B}$, it is not the case that $A\rho_{w_i}1$
If $\neg A, +i$ is on $\mathcal{B}$, $A\rho_{w_i}0$
If $\neg A, -i$ is on $\mathcal{B}$, it is not the case that $A\rho_{w_i}0$

Proof. This is proved by recursion on A. It is true by definition (and the fact that $\mathcal{B}$ is open) when A is atomic. The induction cases for the extensional connectives are straightforward and left as exercises (see Priest, 2001, 8.7.6). Here are the cases for $\Box$. The cases for $\Diamond$ are similar.

Suppose that $\Box B, +i$ is on $\mathcal{B}$. Then for every w_i such that $w_i R w_j$, $B, +j$ is on $\mathcal{B}$. By induction hypothesis, B is true at w_j. Hence, $\Box B$ is true at w_i.

Suppose that $\Box B, -i$ is on $\mathcal{B}$. Then for some j such that $w_i R w_j$, $B, -j$ is on $\mathcal{B}$. By induction hypothesis, B is not true at w_j. Hence, $\Box B$ is not true at w_i.

Suppose that $\neg\Box B, +i$ is on $\mathcal{B}$. Then $\Diamond\neg B, +i$ is on $\mathcal{B}$. So for some w_i such that $w_i R w_j$, $\neg B, +j$ is on $\mathcal{B}$. By induction hypothesis, B is false at w_j. Hence, $\Box B$ is false at w_i.

Finally, suppose that $\neg\Box B, -i$ is on $\mathcal{B}$. Then $\Diamond\neg B, -i$ is on $\mathcal{B}$. Hence, for all j such that $w_i R w_j$, $\neg B, -j$ is on $\mathcal{B}$. By induction hypothesis, B is not false at w_j. So $\Box B$ is not false at w_i. □

10.7. Completeness Theorem for K_{FDE} *For finite* Σ, if $\Sigma \vDash A$ *then* $\Sigma \vdash A$.

Proof. This follows from the Completeness Lemma in the usual way. □

10.8. Soundness and Completeness Theorem for K_{K_3} **and** K_{LP} *The tableau systems for K_{K_3} and K_{LP} are sound and complete.*

Proof. The proof for K_{K_3} is exactly the same as for K_{FDE}. In the Completeness Lemma, we merely have to check that the induced interpretation is an interpretation for K_{K_3}. This follows from the fact that the K_3 closure rule is in operation. The proof for K_{LP} is the same, except that in the induced interpretation, ρ is defined slightly differently: for every propositional parameter, p, $p\rho_{w_i}1$ iff $p, -i$ is not on $\mathcal{B}$; $p\rho_{w_i}0$ iff $\neg p, -i$ is not on $\mathcal{B}$. This is an interpretation for K_{LP} because of the LP closure rule. The new definition makes the basis case of the Completeness Lemma slightly different. If $p, +i$ is on $\mathcal{B}$, then, by closure, $p, -i$ is not on $\mathcal{B}$. So $p\rho_{w_i}1$. If $p, -i$ is on $\mathcal{B}$, it is not the case that $p\rho_{w_i}1$. The cases for 0 are similar. □

10.9. Soundness and Completeness Theorems for The Extensions of These Logics Obtained by Adding Constraints on the Accessibility Relation *The addition of the rules for ρ, σ, and τ are sound and complete with respect to the corresponding semantics.*

Proof. In the Soundness Lemma, we merely have to check the cases for the additional rules. In the Completeness Lemma, we have to check that the induced interpretation is appropriate. This is all straightforward. (Details can be found in Priest, 2001, 3.7.1–3.7.4.) □

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